On the Uniqueness of Equilibrated Dynamic Traffic Flow Pattern in Unidirectional Networks

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TRAFFIC CONTROL IN A TOWN WITH TWO RING ROADS AND A MACROSCOPIC FUNDAMENTAL DIAGRAM

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Introduction
Uniqueness of DUE

• DUE: dynamic user equilibrium
  – Problems with no departure time choice are considered here. Departure times are externally given and fixed. Only route choices are considered.

• In DUE, every vehicle is assigned to a route whose actual arrival time is earliest among all valid routes connecting vehicle’s origin and destination.
Uniqueness of DUE

• Uniqueness is still a main issue of DUE.
• Thus far, it is known that the following two types of network structure guarantees uniqueness:
  – Single-bottleneck-per-route network
    • Mounce (2007)
  – Single-OD-pair network
    • Mounce and Smith (2007 at 17th ISTTT)
Uniqueness of DUE

Example of single-bottleneck-per-route network
Non-Uniqueness of DUE

- There exists a counterexample to uniqueness of DUE in a loopy network structure.

OD pairs in this network are:
- OD-1: Node A to Node D
- OD-2: Node C to Node B

Iryo (2011)
• Given a certain demand profile, there exists a non-convex solution set in which link travel time pattern is not unique.
  • Numbers indicate maximum link travel times in minutes.
Condition for uniqueness of DUE?

• These studies suggest that restricting network structure is a key to prove uniqueness.

• The concept of ‘unidirectional network’ is proposed in this study to prove it.
  
  • Uniqueness of link travel times is to be proven.

The proposed proof is inspired by the time-decomposition technique (Kuwahara and Akamatsu, 1993, at 12th ISTTT), which has been also utilised in other studies such as Akamatsu (2000), Akamatsu (2001), Akamatsu and Heydecker (2003a, b), Waller and Ziliaskopoulos (2006), Iryo (2010), and Iryo (2011b)
Problem setting of DUE assignment

• Non-atomic vehicles are considered
  – The flow ratio of vehicles departing from any origin to any destination is given by a function of time, which is Lipschitz continuous.

• Point-queue bottleneck model is used.

• Drivers do not choose departure times.
  – Travel cost consists of travel time only.

• Wardrop’s first principle is adopted.
Outline of the proof:

1. Unidirectional network
2. Node-potential function
3. Temporal order of vehicles
4. Partial loading
5. Proving uniqueness
   — for a certain duration of time,
   — then for entire duration of study time

PLUS:
Yet other non-unique case to show why the concept of unidirectional network is important.
1. Unidirectional network
Definition of unidirectional network

• Condition 1:
In a unidirectional network, any pair of two origins is directly or indirectly connected.

– If origins o1 and o2 are directly connected (denoted by o1≈o2), there exists a destination that is reachable from them.
– If o1≈o2 and o2 ≈o3, o1 and o3 are indirectly connected (denoted by o1~o3).

• Moreover, o4~o5 and o5~o6 implies o4~o6.
Definition of unidirectional network

\[ o_1 \approx o_2 \]

\[ o_2 \approx o_3 \]

\[ o_1 \sim o_3 \]

\[ o_1 \sim o_3 \]
Definition of unidirectional network

• Condition 2:
  – Consider any arbitrary time-dependent travel time profile to any link.
    • Ratio of temporal change is not less than -1, owing to the FIFO discipline.
Definition of unidirectional network

• Condition 2:
  – Pick up any two arbitrary routes from any origin to any destination that pass through a certain node at the same time.
    • A spatial-temporal route is considered in this context.
  – Then, if they share the same node en route, they pass through it at the same time as well.

$t_1 = s_1$  
$t_2 = s_2$
Example of unidirectional network

Vehicles from O1 to D2 have two routes, while others (O1 to D1 and O2 to D2) have one route only.

Condition 1 is satisfied in this network.
Example of unidirectional network

Case A: Upper route is shortest

This link is not used
Example of unidirectional network

**Case A:**
Upper route is shortest

**Polytree:**
Condition 2 is satisfied.
Example of unidirectional network

Case B: Lower route is shortest

This link is not used
Example of unidirectional network

Polytree:
Condition 2 is satisfied.

Case B:
Lower route is shortest
Example of unidirectional network

Case C: Both routes are shortest

Travel time between A and B are the same

Condition 2 is satisfied
Example of non-unidirectional network

1 mins

O1

O2

OK

NG

D2

D1

0

1

2

3

4

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2. Node-potential function
Node-potential function

• Theorem 1:
  – In any unidirectional network, a time at which any shortest route pass through a node can be always denoted by a node-potential function.
  – The value of the node-potential function is determined by node and reference time.
    • A reference node is set to determine the reference time.
    • Node-potential function is defined only at nodes included by a shortest route.
Example of node-potential function

Reference node

01

Reference time

O1

\[ x = \text{Value of node-potential function} \]
Reference time and order of vehicles

• Reference time can be used to determine the time stamp of each vehicle running in the network via its shortest route.
Reference time and order of vehicles

Reference node

Reference time

Reference time of a vehicle departing from O2 at time 1 is 0

x: Value of node-potential function
Reference time and order of vehicles

Reference node

O1

Reference time

A

B

D2

O2

X

Y

D1

Reference time of a vehicle departing from O2 at time 2 is 1

x: Value of node-potential function

1

2

3

4

5

6 mins

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3. Temporal order of vehicles
Temporal order of vehicles

• Theorem 2:

  – Consider any two vehicles, V1 and V2, running via their shortest routes.

  – [Ref. Time V1] < [Ref. Time V2] implies that V1 runs through any node earlier than V2, and vice versa.

• The positive-flow assumption is necessary to prove this theorem, while it can be relaxed later (see paper for details).
Temporal order of vehicles

- Theorem 2 sets the order relationship between any vehicles by the order of their reference times.

Set of (non-atomic) vehicles associated with reference time between $t$ and $t + \Delta t$
4. Partial loading
• In the full loading, route choices of all vehicles are determined so as to satisfy Wardrop’s first principle.
  – It corresponding to an equilibrium solution.
  – Route choices are expressed by (proportion of) route traffic volume
Partial loading

- Identify all vehicles whose reference time is not greater than a certain time $t_p$ in an equilibrium solution of a full loading.
Partial loading

- Identify all vehicles whose reference time is not greater than a certain time $t_p$ in an equilibrium solution of a full loading.
- Remove all vehicles,
Partial loading

• Identify all vehicles whose reference time is not greater than a certain time $t_p$ in an equilibrium solution of a full loading.

• Remove all vehicles, and load all identified vehicles again using drivers’ route choices in the full loading.

• This process is called partial loading.
Partial loading

• Theorem 3:
  – Wardrop’s first principle is also satisfied in the partial loading.
  – For all vehicles loaded in the partial loading, values of node-potential function in the full loading are the same of those in the partial loading.
5. Proving uniqueness
Uniqueness for a certain time duration

• Theorem 4:
  – Consider two equilibrium solutions and denote vehicles’ route choices $X^*$ and $X^{**}$.  
  – Denote the node-potential function by
    • $p ([\text{Node}],[\text{RefTime}]; X^*)$ or
    • $p ([\text{Node}],[\text{RefTime}]; X^{**})$
Uniqueness for a certain time duration

• Theorem 4:
  – Then, for any reference time \( t \),
    \[
    p(i, t; X^*) = p(i, t; X^{**}) \implies p(i, t+\Delta t; X^*) = p(i, t+\Delta t; X^{**})
    \]
    between \( t \) and \( t+\Delta t \).

  – \( \Delta t \) is assumed to be very small and needs to be not greater than a certain positive value.

To be explained later
Owing to Theorems 2 and 3, loading of vehicles in this duration only depends on $p(i, t; X^*)$ or $p(i, t; X^{**})$.
- All vehicles are loaded at the end of existing queues.
Loading for a short time duration

• Loading of vehicles between $t$ and $t + \Delta t$ can be described by a linear equation system (inspired by Akamatsu (2000)), hence uniqueness of the solution is proven when $p(i, t; X^*) = p(i, t; X^{**})$.

— see Appendix of the paper for details.

• Important note: the above-mentioned technique can be used only when:
  1. congested link is always congested,
  2. no link is newly included by a shortest path.

These conditions limit $\Delta t$ to the certain value.
Uniqueness for the entire time duration

• Rough idea to prove this (Theorem 5):

Empty load
(link travel times are uniquely determined!)
Uniqueness for the entire time duration

- Rough idea to prove this (Theorem 5):

Unique node-potential function

\[ t \quad t + \Delta t \]
Uniqueness for the entire time duration

• Rough idea to prove this (Theorem 5):

Unique node-potential function

\[ t \quad t + \Delta t \quad \text{RefTime} \]
Uniqueness for the entire time duration

• Rough idea to prove this (Theorem 5):

Unique node-potential function

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Uniqueness for the entire time duration

• Rough idea to prove this (Theorem 5):

Unique node-potential function
Uniqueness for the entire time duration

• Issue:
  – Max. value of $\Delta t$ is limited to a certain value.
  – Hence, $t + \Delta t$ has to be smaller than by a certain finite value, denoted by $t_{TH}$.
  – The procedure is only applicable till $t_{TH}$.

• How to resolve it?
  – Prove uniqueness for $t + \Delta t \rightarrow t_{TH}$.
  – Because link travel time is Lipschitz continuous, it also proves uniqueness of node-potential function at $t_{TH}$.
  – Initiate calculation again from $t_{TH}$ to accomplish the proof for the entire study time.
Yet other non-unique case...
Non-unique case

• Conditions for the unidirectional network is rather strict.
• It is worth to show a counterexample to uniqueness in a network which seems to be unidirectional but does not meet to its conditions.
OD pairs: 1 to 2 and 3 to 4
Volume of flow: 2 per unit time, which lasts forever.
Numbers on links: free-flow travel times
Non-unique case

Route $r_{A1}$

Route $r_{B2}$

Route $r_{A2}$

Route $r_{B1}$
Non-unique case

Solution 1:
$r_{A1}$ and $r_{B1}$ are used
Non-unique case

Solution 1: rA2 and rB2 are used

Route travel time of $r_{A1}$ and $r_{B1}$

Route travel time of $r_{A2}$ and $r_{B2}$

Route $r_{A1}$

Route $r_{A2}$

Route $r_{B1}$

Route $r_{B2}$
Conclusion and future directions

• Uniqueness of DUE in a unidirectional network is proven.
• While the definition of the unidirectional property in this study is rather strict, the counterexample implies that this strict definition is vital to guarantee the uniqueness.
• The finite duration of the demand pattern should also be a key to proof uniqueness.
  – According to Mounce (2007), any single-bottleneck-per-route network should have a unique DUE solution if the demand duration is finite.
  – On the other hand, the counterexample in this study assumes demand that lasts forever.
References (1)

Non-Uniqueness of DUE
Uniqueness for a certain time duration

Full load

Partial load

Solution $X^*$

$p(i, t; X^*)$
Uniqueness for a certain time duration

Full load

Partial load

Solution $X^{**}$

Full $X^{**}$

Partial $X^{**}$

$t$

$p(i, t; X^{**}) = p(i, t; X^*)$