Lecture 28
GEN_ENG 205-2: Engineering Analysis 2
Winter Quarter 2018
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Chapters 13: §13.6 Curvilinear Motion—Normal and Tangential Components[1]

Acknowledgements

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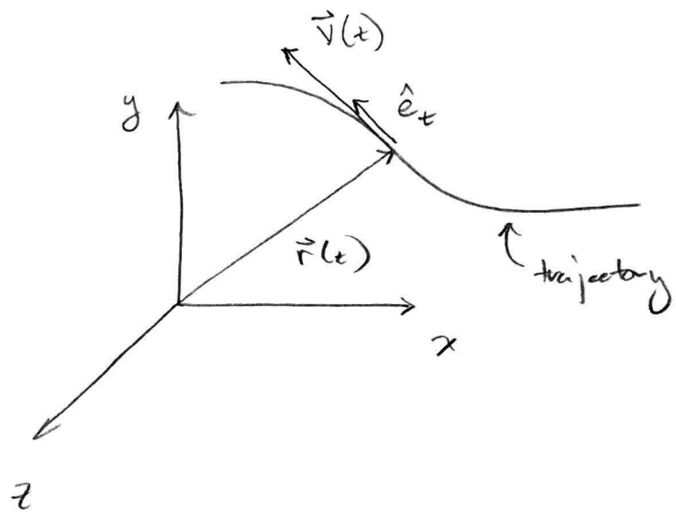
Curvilinear Motion—Normal and Tangential Components

Recall that velocity is tangent to the path of motion (trajectory). We can express it as

$$\vec{v}(t) = v(t)\hat{e}_t$$

where $\hat{e}_t$ is the unit vector tangent to the path. Clearly, this varies from point to point, so $\hat{e}_t(t)$ is a function of time.



[1] Bedford, A., & Fowler, W. (2008). *Engineering Mechanics: Statics and Dynamics* (5th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.

Can we calculate acceleration directly from this expression of velocity, $\vec{v}(t)=v(t)\hat{e}_t(t)$?

Yes, but we need to consider the fact that the unit vector $\hat{e}_t$ also changes in time[2].

$$\begin{aligned}\vec{a}(t)&=\frac{d\vec{v}(t)}{dt}\\&=\frac{d}{dt}\left[v(t)\hat{e}_t(t)\right]\\&=\frac{dv(t)}{dt}\hat{e}_t(t)+v(t)\frac{d\hat{e}_t(t)}{dt}\end{aligned}$$

The second term is zero for straight-line motion.

What is $\dfrac{d\hat{e}_t(t)}{dt}$?

Note the parallel to the result we derived in the previous lecture:

$$\boxed{\frac{d\hat{e}}{dt}=\frac{d\theta}{dt}\hat{n}=\omega\hat{n}}$$

We will now derive an expression similar to this for $\hat{e}_t$.

Consider the 2D problem. In Cartesian components, we can write

$$\hat{e}_t(t)=e_x(t)\hat{i}+e_y(t)\hat{j}=\cos\left[\theta(t)\right]\hat{i}+\sin\left[\theta(t)\right]\hat{j}$$

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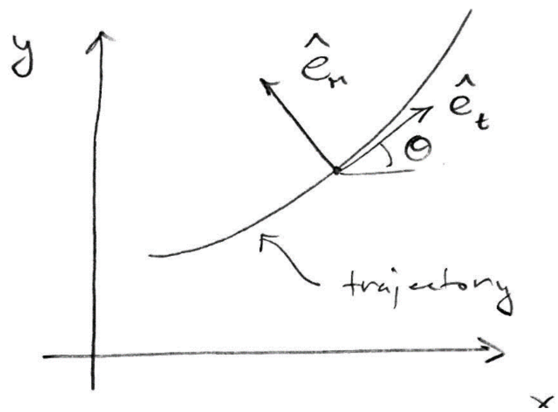
[2] Recall the *product rule*: $\left[f(x)g(x)\right]'=f'(x)g(x)+f(x)g'(x)$

where $\theta(t)$ is the <u>path angle</u>, which also varies from point to point.



$$\frac{d\hat{e}_t(t)}{dt} = \frac{d\hat{e}_t(t)}{d\theta}\frac{d\theta(t)}{dt}$$
$$= \left(-\sin\left[\theta(t)\right]\hat{i} + \cos\left[\theta(t)\right]\hat{j}\right)\frac{d\theta(t)}{dt}$$

What is $-\sin\left[\theta(t)\right]\hat{i} + \cos\left[\theta(t)\right]\hat{j}$?

It is perpendicular (normal) to $\hat{e}_t(t) = \cos\left[\theta(t)\right]\hat{i} + \sin\left[\theta(t)\right]\hat{j}$. How do we show it?

Use the dot product: $\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)\cdot\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) = -\sin\theta\cos\theta + \cos\theta\sin\theta = 0$

It is the unit vector normal to the path, which we will denote by $\hat{e}_n$.

$$\boxed{\frac{d\hat{e}_t(t)}{dt} = \hat{e}_n(t)\frac{d\theta(t)}{dt}} \quad \text{or} \quad \boxed{\frac{d\hat{e}_t(t)}{dt} = \hat{e}_n(t)\omega(t)}$$

We can then also write

$$\vec{a}(t) = \frac{dv(t)}{dt}\hat{e}_t(t) + v(t)\frac{d\theta(t)}{dt}\hat{e}_n(t)$$
$$= a_t(t)\hat{e}_t(t) + a_n(t)\hat{e}_n(t)$$

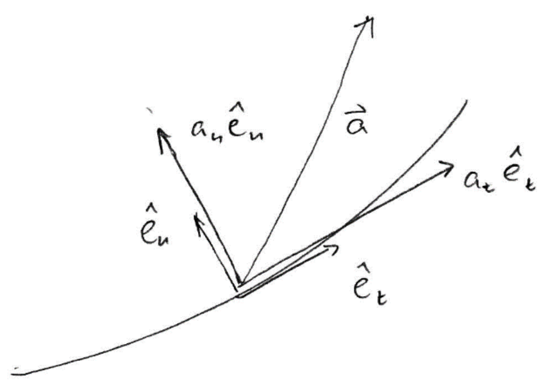
or

$$\boxed{\vec{a}(t) = a_t(t)\hat{e}_t(t) + a_n(t)\hat{e}_n(t)}$$

where

$\boxed{a_t(t) = \frac{dv(t)}{dt}}$ tangential acceleration

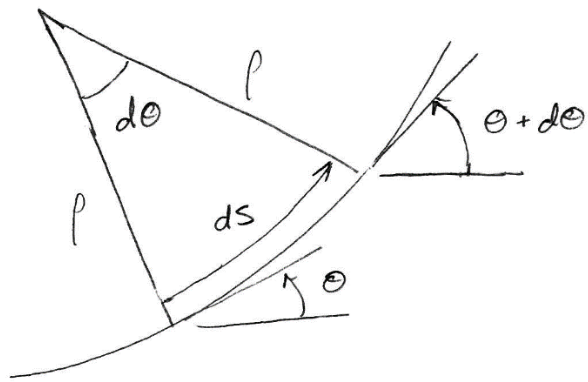
$\boxed{a_n(t) = v(t)\frac{d\theta(t)}{dt}}$ normal acceleration



Normal acceleration is always present when traveling along a curved path[3].

[3] The simplest example is that of a weight spun on a string. The string applies a *centripetal force* to the weight to continuously change its trajectory.

The normal acceleration can be related to the curvature of the path at a given point (or time).



From geometry, we can write

$$ds = \rho\, d\theta$$

where ρ is the instantaneous radius of curvature of the path.

Dividing the expression above by *dt* we find[4]:

$$\frac{ds}{dt} = \rho \frac{d\theta}{dt}$$

or, upon rearranging

$$\frac{d\theta}{dt} = \frac{1}{\rho}\frac{ds}{dt} = \frac{v}{\rho}$$

[4] This is not rigorous, but it works. Incidentally, this is what differentiates engineers from mathematicians.

Remember, these are functions of time:

$$\frac{d\theta(t)}{dt} = \frac{v(t)}{\rho(t)}$$

Substituting this into the expression for normal acceleration, $a_n(t) = v(t)\frac{d\theta(t)}{dt}$, we find

$$a_n(t) = v(t)\frac{d\theta(t)}{dt} = \frac{\left[v(t)\right]^2}{\rho(t)}$$

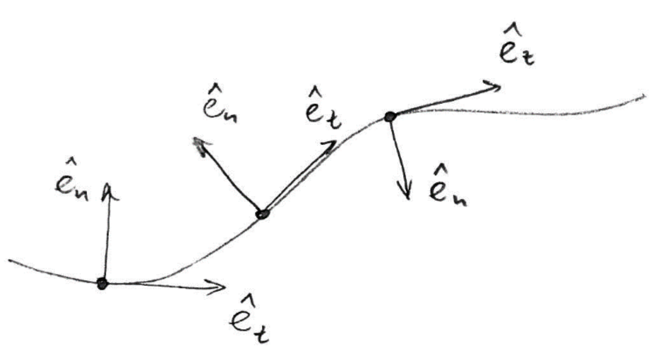
or, omitting the argument *t*,

$$\boxed{a_n = \frac{v^2}{\rho}}$$

$$\boxed{\vec{a} = \frac{dv}{dt}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n}$$

Note that we again have two scalar functions, $v(t)$ and $\rho(t)$.

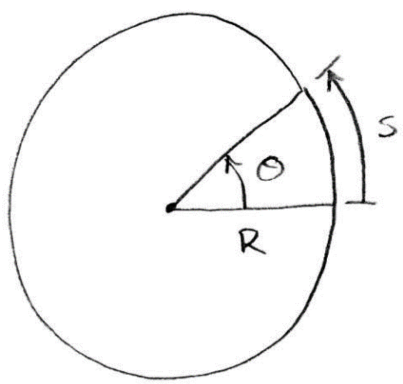
When using this expression, $\hat{e}_n$ <u>must point toward the concave side of the path</u>.



The expression that we just derived is a useful alternative to defining acceleration in a fixed (e.g., Cartesian) reference frame. It is particularly useful for assessing acceleration at a particular moment in time, when $\hat{e}_t$ and $\hat{e}_n$ are known. It also leads us naturally into circular motion, which we will now consider.

Circular motion

If the path is circular, ρ is simply the radius of the circle defining the path, $R = \rho$.



The distance travelled along the circular path is

$$s(t) = R\theta(t)$$

Note that s and θ are interchangeable, and the meaning of θ is slightly different compared to pure rotation.

The velocity (speed) along the path is

$$\begin{aligned} v(t) &= \frac{ds(t)}{dt} \\ &= R\frac{d\theta(t)}{dt} \\ &= R\omega(t) \end{aligned}$$

where $\omega(t)$ is the angular velocity.

The <u>tangential component of the acceleration</u> is given by differentiating again:

$$\begin{aligned} a_t(t) &= \frac{dv(t)}{dt} \\ &= R\frac{d\omega(t)}{dt} \\ &= R\alpha(t) \end{aligned}$$

where $\alpha(t)$ is the angular acceleration.

The <u>normal component of the acceleration</u> is given using the previously derived expression with $R = \rho$:

$$a_n(t) = \frac{\left[v(t)\right]^2}{R}$$

If speed is constant, $v(t) = v_0$, then

$$\boxed{\begin{aligned} a_t &= 0 \\ a_n &= \frac{v_0^2}{R} \end{aligned}}$$

This expression for a_n is important and well worth remembering!

In the formulas for circular motion, θ, ω, and α take on a different (but similar) meaning compared to pure rotation:

Pure rotation	Circular motion
$\theta(t)$	$s(t) = R\theta(t)$
$\omega(t)$	$v(t) = R\omega(t)$
$\alpha(t)$	$\begin{cases} a_t(t) = R\alpha(t) \\ a_n(t) = \dfrac{[v(t)]^2}{R} \end{cases}$