Lecture 27
GEN_ENG 205-2: Engineering Analysis 2
Winter Quarter 2018
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Chapters 13: §13.5 Angular Motion[1]

Acknowledgements

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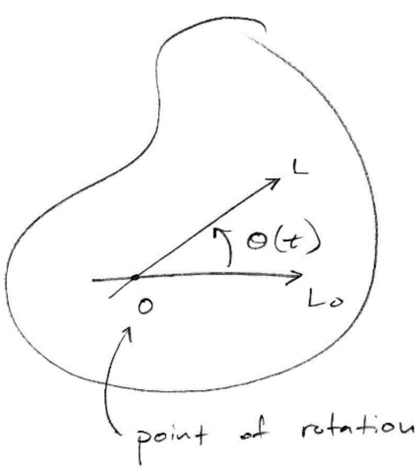
Angular Motion

A rigid body motion (for objects that do not deform internally) generally consists of translation and rotation. One can have pure translation, pure rotation, or a combination of translation and rotation[2].

We now consider an object that undergoes pure rotation. In some ways, this is an analogue to straight-line motion, except we need to replace the distance $s(t)$ with the angle $\theta(t)$.



Any line on a body undergoing pure rotation by angle θ also rotates by angle θ.

[1] Bedford, A., & Fowler, W. (2008). *Engineering Mechanics: Statics and Dynamics* (5th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.
[2] This type of motion is dealt with generally in Chapter 17.

The angular position is of line L relative to the reference line L_0 is given by $\theta(t)$.

The angular velocity of L is defined by

$$\omega = \frac{d\theta}{dt}$$

The angular acceleration of L is defined by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Dimensions:

$[\theta] = \varnothing$ (e.g., rad)

$[\omega] = \frac{1}{T}$ (e.g., rad/s, deg/s, rpm)

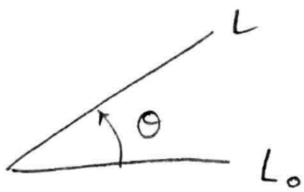
$[\alpha] = \frac{1}{T^2}$ (e.g., rad/s^2)

We can analyze problems in the same way as for straight-line motion:

$$s(t) \to \theta(t)$$
$$v(t) \to \omega(t)$$
$$a(t) \to \alpha(t)$$

Answers can be expressed in degrees or radians, but always use radians for calculations!

<u>Example</u>: Problem 13.95



Given:

- The angular acceleration of line L relative to line L_0 is $\alpha = 2.5 - 1.2t \left(\frac{\text{rad}}{\text{s}^2}\right)$.
- The angular velocity of L relative to L_0 at time t = 0, and at angular position $\theta = 0$, is $\omega = 5\ \frac{\text{rad}}{\text{s}}$.

Determine θ and ω at t = 3 s.

Strategy: Integrate!

$$\begin{aligned}\omega(t) &= \int \alpha(t)\,dt \\ &= \int (2.5 - 1.2t)\,dt \\ &= 2.5t - \frac{1}{2}(1.2t^2) + C_1\end{aligned}$$

$$\omega(0) = C_1 = 5\ \frac{\text{rad}}{\text{s}}$$

$$\boxed{\omega(t) = 5 + 2.5t - 0.6t^2} \quad \text{(rad/s)}$$

$$\begin{aligned}\theta(t) &= \int \omega(t)\,dt \\ &= \int (5 + 2.5t - 0.6t^2)\,dt \\ &= 5t + \frac{1}{2}(2.5t^2) - \frac{1}{3}(0.6t^3) + C_2\end{aligned}$$

$$\omega(0) = C_2 = 0$$

$$\boxed{\theta(t) = 5t + 1.25t^2 - 0.2t^3} \quad \text{(rad)}$$

At $t = 3\text{s}$:

$$\boxed{\begin{aligned}\theta(3) &= 20.9 \text{ rad} \\ \omega(3) &= 7.10 \ \frac{\text{rad}}{\text{s}}\end{aligned}}$$

<u>Rotating Unit Vector</u>

Recall that, if unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are fixed in a Cartesian reference frame, then

$$\begin{aligned}\vec{v}(t) &= \frac{d\vec{r}(t)}{dt} \\ &= \frac{d}{dt}\left[x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}\right] \\ &= \frac{d}{dt}\left[x(t)\hat{i}\right] + \frac{d}{dt}\left[y(t)\hat{j}\right] + \frac{d}{dt}\left[z(t)\hat{k}\right] \\ &= \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}\end{aligned}$$

What if the unit vectors used to describe the motion <u>rotate</u> as the point move?

In particular, let us write

$$\vec{r}(t) = s(t)\hat{e}(t)$$

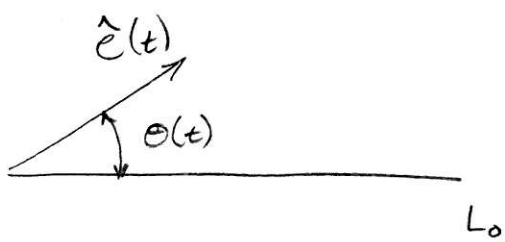
Now, the velocity is

$$\begin{aligned}\vec{v}(t) &= \frac{d\vec{r}(t)}{dt} \\ &= \frac{d}{dt}\left[s(t)\hat{e}(t)\right] \\ &= \frac{ds(t)}{dt}\hat{e}(t) + s(t)\frac{d\hat{e}(t)}{dt}\end{aligned}$$

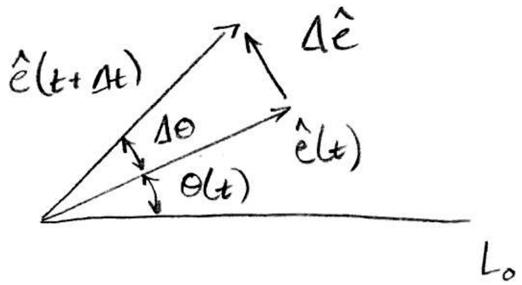
What is $\frac{d\hat{e}(t)}{dt}$?

$$\frac{d\hat{e}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\hat{e}(t+\Delta t) - \hat{e}(t)}{\Delta t}$$

Consider a particular instant in time:



Now, consider the change due to small increment of time Δt:



The triangle is isosceles, so by the cosine rule:

$$\begin{aligned}\left|\Delta\hat{e}\right| &= 2\left|\hat{e}\right|\sin\left(\frac{\Delta\theta}{2}\right)\\ &= 2\sin\left(\frac{\Delta\theta}{2}\right)\end{aligned}$$

Introduce a unit vector $\hat{n}$ that points in the direction of $\Delta\hat{e}$ as a matter of convenience:

$$\begin{aligned}\Delta\hat{e} &= \left|\Delta\hat{e}\right|\hat{n}\\ &= 2\sin\left(\frac{\Delta\theta}{2}\right)\hat{n}\end{aligned}$$

Now we find

$$\begin{aligned}\frac{d\hat{e}(t)}{dt} &= \lim_{\Delta t\to 0}\frac{\hat{e}(t+\Delta t)-\hat{e}(t)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\Delta\hat{e}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{2\sin(\Delta\theta/2)}{\Delta t}\hat{n}\end{aligned}$$

We can rewrite this as

$$\begin{aligned}\frac{d\hat{e}(t)}{dt} &= \lim_{\Delta t\to 0}\frac{2\sin(\Delta\theta/2)}{\Delta t}\hat{n}\\ &= \lim_{\Delta t\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\frac{\Delta\theta}{\Delta t}\hat{n}\end{aligned}$$

In the limit, we have

$$\frac{\sin\left(\Delta\theta/2\right)}{\Delta\theta/2}=1$$

and

$$\frac{\Delta\theta}{\Delta t}=\frac{d\theta}{dt}$$

Also, in the limit $\hat{n}$ becomes a unit vector perpendicular to $\hat{e}(t)$.

$$\boxed{\frac{d\hat{e}}{dt}=\frac{d\theta}{dt}\hat{n}=\omega\hat{n}}$$

where $\hat{n}$ is a unit vector that is perpendicular to $\hat{e}$ and points in the positive θ direction.