Lecture 25
GEN_ENG 205-2: Engineering Analysis 2
Winter Quarter 2018
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Chapters 13: §13.3 Straight-Line Motion when the Acceleration Depends on Velocity or Position[1]

Acknowledgements

Portions of these lecture notes are taken from those of Prof. Jeff Thomas.

Acceleration Specified as a Function of Velocity

Suppose that acceleration is given as a function of velocity, rather than time. For example, the drag force on an object is given by

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

where

- F_D is the drag force
- ρ is the density of the fluid
- v is the velocity (speed) of the object relative to the fluid
- A is the cross-sectional area of the object
- C_D is the so-called drag coefficient (a dimensionless number).

[1] Bedford, A., & Fowler, W. (2008). *Engineering Mechanics: Statics and Dynamics* (5th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.

If drag is the only force acting (e.g., lateral motion of a projectile), then

$$ma = F_D = \frac{1}{2}\rho v^2 C_D A \quad \Rightarrow \quad a = \left[\frac{\rho C_D A}{2m}\right] v^2$$

Since $a = \frac{dv}{dt}$, one finds

$$\frac{dv}{dt} = a(v) \qquad (\#)$$

How do we integrate to find velocity from acceleration, which depends on *v*? We cannot directly integrate.

$$\begin{aligned} v &= v_0 + \int_{t_0}^{t} a\,dt \\ &= v_0 + \int_{t_0}^{t} \xcancel{a(v)\,dt} \end{aligned}$$

Eq. (#) is in fact a <u>differential equation</u>.

In this case, we can use the technique of <u>separation of variables</u>, putting terms involving *v* to one side of the equation and terms involving *t* to the other side:

$$\frac{dv}{a(v)} = dt$$

Now, we can integrate

$$\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{t_0}^{t} dt = t - t_0$$

Example: Drag force (and only drag force).

If $a = cv^2$, where c is a constant, then

$$\begin{aligned}
\int_{v_0}^{v} \frac{dv}{a(v)} &= \int_{v_0}^{v} \frac{dv}{cv^2} \\
&= \frac{1}{c}\int_{v_0}^{v} v^{-2} dv \\
&= -\frac{1}{c} v^{-1}\Big|_{v_0}^{v} \\
&= -\frac{1}{c}\left(v^{-1} - v_0^{-1}\right) \\
&= \frac{1}{c}\left(\frac{1}{v_0} - \frac{1}{v}\right)
\end{aligned}$$

Setting this (LHS) to $t - t_0$ (the RHS) and solving for v gives

$$v = \frac{v_0}{1 - cv_0\left(t - t_0\right)}$$

To then determine position as a function of time, we could, in principle, solve for velocity as a function of time, as above, and then integrate.

Alternatively, we can directly obtain a relationship between velocity and position using the chain rule

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} \quad \text{or} \quad \boxed{a = \frac{dv}{ds}v} \quad (*)$$

This expression is used frequently, and it useful when acceleration or velocity is given as a function of position rather than time.

Considering $a = a(v)$, and using the same separation of variables technique as before, one finds

$$\frac{v}{a(v)}dv = ds$$

$$\int_{v_0}^{v} \frac{v}{a(v)}dv = \int_{s_0}^{s} ds = s - s_0$$

Example: Drag force (continued) with $a = cv^2$.

$$\begin{aligned}
\int_{v_0}^{v} \frac{v}{a(v)}dv &= \int_{v_0}^{v} \frac{v}{cv^2}dv \\
&= \frac{1}{c}\int_{v_0}^{v} \frac{1}{v}dv \\
&= \frac{1}{c}\ln v\Big|_{v_0}^{v} \\
&= \frac{1}{c}\left(\ln v - \ln v_0\right) \\
&= \frac{1}{c}\ln\left(\frac{v}{v_0}\right)
\end{aligned}$$

Using the result above, we find

$$\frac{1}{c}\ln\left(\frac{v}{v_0}\right)=s-s_0$$

Since we have velocity as a function of time, we can substitute directly to find position as a function of time.

$$\begin{aligned}s&=\frac{1}{c}\ln\left(\frac{v}{v_0}\right)+s_0\\&=\frac{1}{c}\ln\left[\frac{1}{1-cv_0\left(t-t_0\right)}\right]+s_0\\&=-\frac{1}{c}\ln\left[1-cv_0\left(t-t_0\right)\right]+s_0\end{aligned}$$

This is the same expression as you would obtain by integrating the expression we derived earlier:

$$v=\frac{v_0}{1-cv_0\left(t-t_0\right)}$$

$$\begin{aligned}s&=\int vdt\\&=\int\frac{v_0}{1-cv_0\left(t-t_0\right)}dt\\&=-\frac{1}{c}\ln\left[1-cv\left(t-t_0\right)\right]+C\end{aligned}$$

Acceleration Specified as a Function of Position

Suppose that acceleration is given as a function of position, rather than time or, as in the previous section, velocity. Examples where this would be the case arise due to gravitational or spring loading.

As before, we cannot integrate directly to find velocity or position

$$v = v_0 + \int_{t_0}^{t} a\,dt$$

$$= v_0 + \int_{t_0}^{t} \cancel{a(s)\,dt}$$

We also cannot use separation of variables, since the equation contains three variables:

$$\frac{dv}{dt} = a(s)$$

$$\frac{dv}{a(s)} = dt$$

$$\cancel{\int \frac{dv}{a(s)}} = \int dt$$

We can, however, use the result derived from the chain rule, recalling $\boxed{a = \frac{dv}{ds}v}$:

$$a(s) = \frac{dv}{ds}v$$

$$v\,dv = a(s)\,ds$$

$$\int_{v_0}^{v} v dv = \int_{s_0}^{s} a(s) ds$$

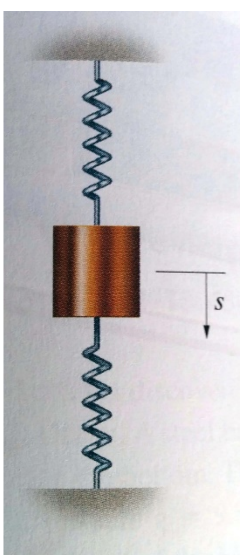
This will yield an expression for the <u>velocity in terms of position</u>. We could then use separation of variables to solve for position as a function of time:

$$v(s) = \frac{ds}{dt}$$

$$\frac{ds}{v(s)} = dt$$

$$\int_{s_0}^{s} \frac{ds}{v(s)} = \int_{t_0}^{t} dt$$

Example: Problem 13.60



The mass is released from rest with the springs unstretched. Its downward acceleration is $a = 32.2 - 50s$ (ft/s^2), where s is the position of the mass measured from the position in which it is released.

(a) How far does the mass fall?

(b) What is the maximum velocity of the mass as it falls?

Strategy: Do not worry about springs[2], since we are given the motion. We need to integrate the acceleration, which is given as a function of position, i.e., $a(s)=\frac{dv}{dt}$. We need to use the chain rule: $\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=\frac{dv}{ds}v$. We can then use separation of variables and integrate to determine velocity as a function of position. Let us first see where this gets us.

Chain rule:

$$\begin{aligned}\frac{dv}{dt}&=v\frac{dv}{ds}\\&=32.2-50s\end{aligned}$$

Separation of variables:

$$v\,dv=(32.2-50s)\,ds$$

$$\int v\,dv=\int(32.2-50s)\,ds$$

$$\frac{1}{2}v^2=32.2s-25s^2+C$$

Now, the mass is at rest when it is released, so $v=0$ when $s=0$:

$$0=0+C \quad\Rightarrow\quad C=0$$

$$v=\sqrt{64.4s-50s^2}\ \ (\#)$$

[2] But it helps to give a physical interpretation and enable the trick based on looking at the condition $v = 0$.

Now, we could use separation of variables again with $v(s)=\dfrac{ds}{dt}$ to determine position as a function of time:

$$\frac{ds}{v(s)}=dt$$

$$\int\frac{ds}{v(s)}=\int dt$$

$$\int\frac{ds}{\sqrt{64.4s-50s^2}}=t+C_2$$

This looks horrible, and it is! We must have enough information already to solve.

We know that the position is maximized when the velocity is zero[3], so in fact we can solve directly using Eq. (#):

$$v=0=\sqrt{64.4s-50s^2} \quad\Rightarrow\quad 64.4s-50s^2=0$$

$$64.4s-50s^2=s(64.4-50s)=0$$

Two solutions: $s=0$ (start) and $s=\dfrac{64.4}{50}=1.288\text{ ft}$ (maximum)

Part (a): $\boxed{s=1.29\text{ ft}}$

[3] This can be concluded from mathematics (optimization) or, in this case, from the physical nature of the problem.

To solve Part (b), we use the same trick regarding maximization and minimization. Namely, we note that the velocity is maximal (or minimal) when $\frac{dv}{dt} = a = 0$.

$$a = 32.2 - 50s = 0 \quad \Rightarrow \qquad s = 0.644 \text{ ft}$$

Observe that this is halfway through the drop!

To determine the velocity, we plug this solution into our expression for the velocity, Eq. (#):

$$\begin{aligned} v_{\text{max}} &= v(0.644) \\ &= \sqrt{64.4(0.644) - 50(0.644)^2} \\ &= 4.55 \ \frac{\text{ft}}{\text{s}} \end{aligned}$$

Part (b): $\boxed{v_{\text{max}} = 4.55 \ \frac{\text{ft}}{\text{s}}}$