Lecture 24
GEN_ENG 205-2: Engineering Analysis 2
Winter Quarter 2018
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Chapters 13: §13.1 Position, Velocity, and Acceleration, §13.2 Straight-Line Motion[1]

Acknowledgements

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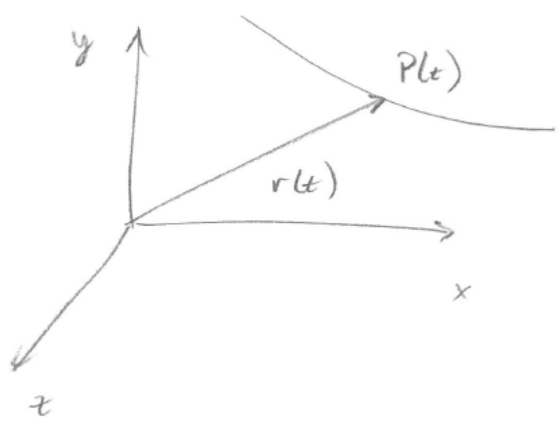
Position, Velocity, and Acceleration

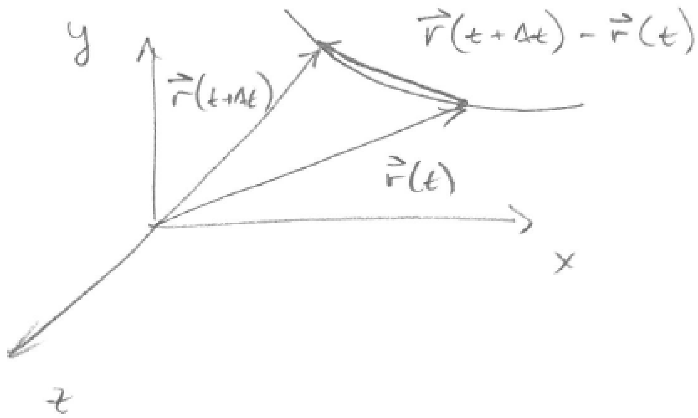
Previously, we considered forces and equilibrium, or kinetics. To move to dynamics, we now need to consider the motion of objects, or the kinematics[2].

Consider the motion of a particle, or point P, through three-dimensional space. We first need a coordinate system, or reference frame, to define the motion. The location of point P as a function of time[3] is then specified through the position vector $\vec{r}(t)$.



[1] Bedford, A., & Fowler, W. (2008). *Engineering Mechanics: Statics and Dynamics* (5th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.
[2] In this class we will consider objects. However, this can be generalized to the motion of deformable bodies.
[3] Make sure to explain notation.

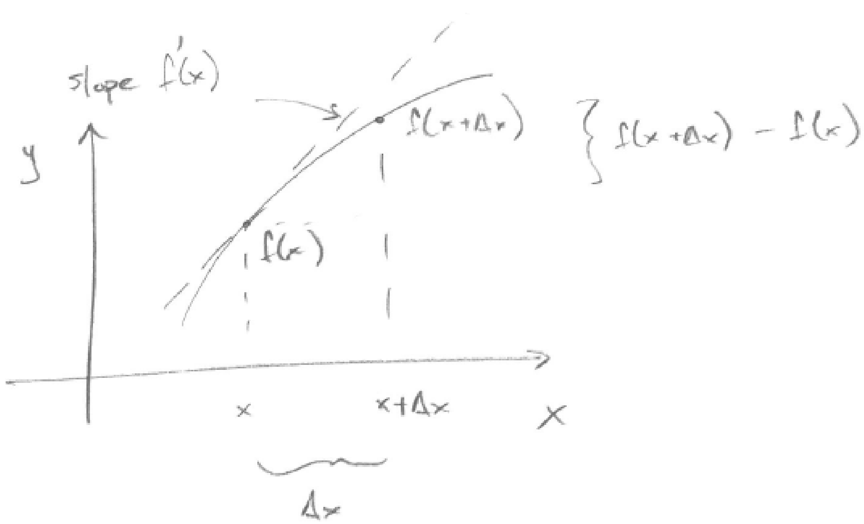
Consider a small increment of time Δt:



Notice that the displacement vector $\vec{r}\left(t+\Delta t\right)-\vec{r}\left(t\right)$ gives the *change in position* and becomes tangent to the curve as Δt becomes small.

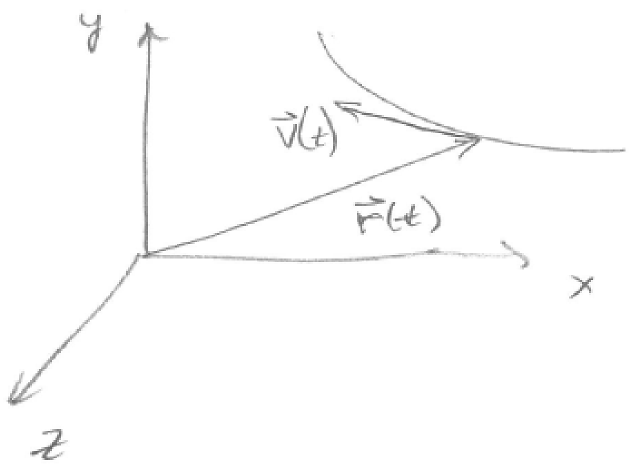
Recall the definition of a derivative for a scalar function $f(x)$:

$$f'\left(x\right)=\frac{d}{dx}f\left(x\right)=\lim_{\Delta x\to 0}\frac{f\left(x+\Delta x\right)-f\left(x\right)}{\Delta x}$$



In our case, we can divide the displacement by Δt (a scalar!) and take the limit to find the velocity vector $\vec{v}(t)$:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$



We typically omit the independent variable t (for simpler notation) to write

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt}}$$

One can imagine plotting the curve traced by the velocity vector $\vec{v}(t)$, and we can then define the acceleration vector, which gives the *change in velocity*, as

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

Omitting the independent variable *t*, we can write simply

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt}}$$

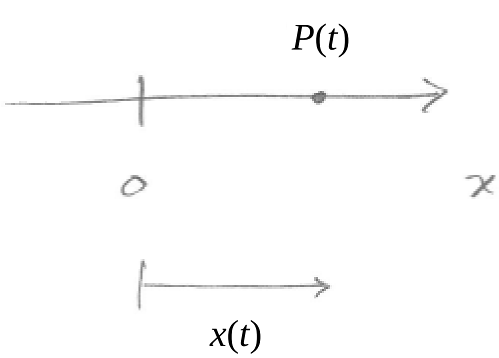
Acceleration is the second derivative of position

$$\boxed{\vec{a} = \frac{d^2\vec{r}}{dt^2}}$$

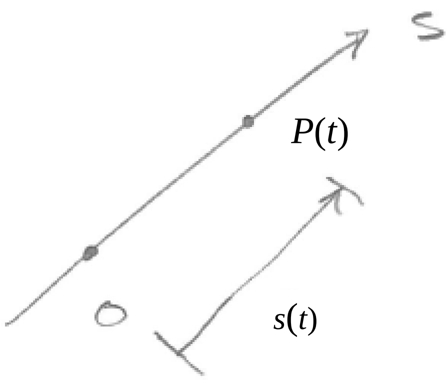
We can of course also integrate to find (1) position from velocity, (2) velocity from acceleration, and (3) by integrating twice, position from acceleration.

<u>Straight-Line Motion</u>

Suppose, for example, that we only move along the *x*-axis.

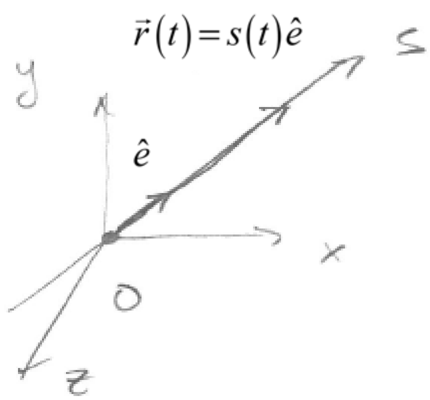


We can generalize this to any line:



The line is defined in terms of a unit vector $\hat{e}$, so we can write

$$\vec{r}(t) = s(t)\hat{e}$$



Again, we may omit the independent variable *t* for simplicity to write

$$\vec{r} = s\hat{e}$$

Then, the velocity vector is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(s\hat{e}) = \frac{ds}{dt}\hat{e}$$

The velocity of the point *P* along the straight line is therefore simply

$$\boxed{v = \frac{ds}{dt}}$$

Similarly, the acceleration vector is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\hat{e}) = \frac{dv}{dt}\hat{e}$$

And the acceleration of the point *P* along the line is

$$\boxed{a = \frac{dv}{dt}}$$

To relation acceleration $a(t)$ to position $s(t)$, we substitute to find

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

$$\boxed{a = \frac{d^2s}{dt^2}}$$

Since everything moves along the line, it is sufficient to consider the scalars $s(t)$, $v(t)$, and $a(t)$. Note that the units are $[s]=L$, $[v]=L/T$, and $[a]=L/T^2$.

We can differentiate and integrate to move between position, velocity, and acceleration.

Example:

The position of the point on an object is given by $s(t)=t^3+3t^2-7$, where $[s]=m$ and $[t]=s$. Determine the velocity and acceleration as functions of time.

Strategy: Differentiate!

$$\begin{aligned} v(t) &= \frac{ds(t)}{dt} \\ &= \frac{d}{dt}\left(t^3+3t^2-7\right) \\ &= 3t^2+(2)(3t) \\ &= 3t^2+6t \end{aligned}$$

$$\begin{aligned} a(t) &= \frac{dv(t)}{dt} \\ &= \frac{d}{dt}\left(3t^2+6t\right) \\ &= (2)(3t)+6 \\ &= 6t+6 \end{aligned}$$

$\boxed{v(t)=3t^2+6t}$ (units of m/s)

$\boxed{a(t)=6t+6}$ (units of m/s^2)

<u>Example</u>: (An important and hopefully familiar example.)

Assume that acceleration is constant, and that position and velocity at time $t = 0$ are known. Determine expressions for position and velocity.

First, write what we are given mathematically:

$$a(t) = a_0, \quad v(0) = v_0, \quad s(0) = s_0$$

Now, integrate to find velocity:

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int a_0\,dt \\ &= a_0 t + C \end{aligned}$$

$$v(0) = v_0 \quad \Rightarrow \quad C = v_0$$

$$\boxed{v(t) = a_0 t + v_0}$$

Integrate again to find position:

$$\begin{aligned} s(t) &= \int v(t)\,dt \\ &= \int (a_0 t + v_0)\,dt \\ &= \frac{1}{2} a_0 t^2 + v_0 t + D \end{aligned}$$

$$s(0) = s_0 \quad \Rightarrow \quad D = s_0$$

$$\boxed{s(t) = \frac{1}{2}a_0t^2 + v_0t + s_0}$$

Generally integration is much more difficult that differentiation, as antiderivatives can be difficult to determine and we need to deal with the constants of integration, as above.

Appendix A in the textbook gives antiderivatives for a number of common integrals.

We can generally write

$$v(t) = v_0 + \int_{t_0}^{t} a(t)\,dt$$

where v_0 is the velocity at time t_0.

The change in velocity, $v(t) - v_0$, from time t_0 to t can be viewed as the area under the acceleration curve.

We can also generally write

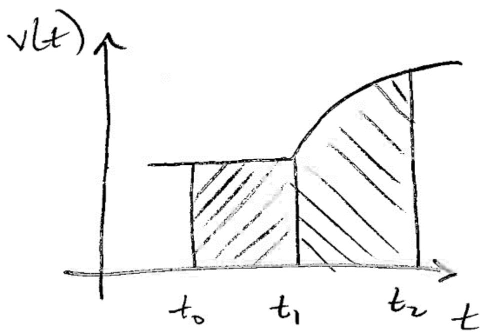
$$s(t) = s_0 + \int_{t_0}^{t} v(t)\,dt$$

where s_0 is the velocity at time t_0.

The <u>change in position</u>, $s(t)-s_0$, from time t_0 to t can be viewed as the area under the velocity curve.

<u>Non-smooth functions</u>

Often the motion is defined over intervals. In these cases, it is useful to graph the function:



Here, the velocity is defined over two time intervals, $t_0 \le t \le t_1$ and $t_1 \le t \le t_2$.

The velocity in this case is discontinuous. To integrate to find the position, we need to split the integral:

$$
\begin{aligned}
s(t) &= \int_{t_0}^{t_2} v(t)\,dt + s_0 \\
&= \int_{t_0}^{t_1} v(t)\,dt + \int_{t_1}^{t_2} v(t)\,dt + s_0
\end{aligned}
$$