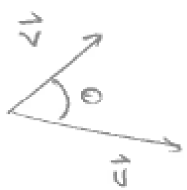
Lecture 3 and 4
GEN_ENG 205-2: Engineering Analysis 2
Winter Quarter 2018
Prof. James P. Hambleton
§2.4 Dot Products; §2.5 Cross Products[1]

Acknowledgements

Portions of these lecture notes are taken from those of Prof. Jeff Thomas.

Dot Products



$$\boxed{\vec{U}\cdot\vec{V}=\left|\vec{U}\right|\left|\vec{V}\right|\cos\theta}$$

Commutative and associative (order does not matter)

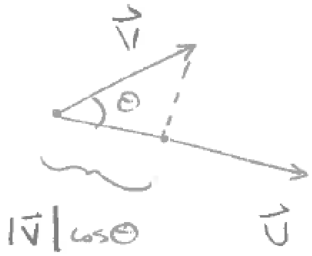
$$\vec{U}\cdot\vec{V}=\vec{V}\cdot\vec{U}$$

$$a\left(\vec{U}\cdot\vec{V}\right)=\left(a\vec{U}\right)\cdot\vec{V}=\vec{U}\cdot\left(a\vec{V}\right)$$

$$\vec{U}\cdot\left(\vec{V}+\vec{W}\right)=\vec{U}\cdot\vec{V}+\vec{U}\cdot\vec{W}$$

[1] Bedford, A., & Fowler, W. (2008). *Engineering Mechanics: Statics and Dynamics* (5th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.

Dot product as a projection



$$\left|\vec{V}\right|\cos\theta = \frac{\vec{U}\cdot\vec{V}}{\left|\vec{U}\right|}$$ projection of $\vec{V}$ onto $\vec{U}$

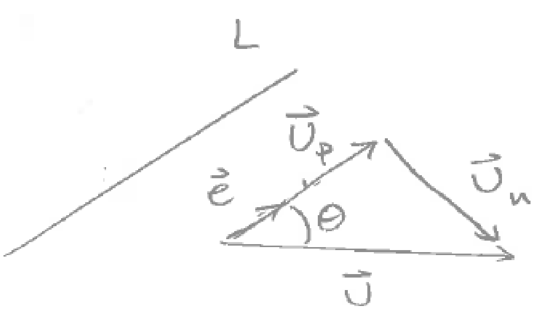
$\theta = \frac{\pi}{2}$: $\left|\vec{V}\right|\cos\theta = 0 \Rightarrow \vec{U}\cdot\vec{V} = 0$ vectors are perpendicular!

$\theta = 0$: $\left|\vec{V}\right|\cos\theta = \left|\vec{V}\right| \Rightarrow \vec{U}\cdot\vec{V} = \left|\vec{U}\right|\left|\vec{V}\right|$ not generally useful but we will use this…

Dot product of a vector with itself

$$\vec{U}\cdot\vec{U} = \left|\vec{U}\right|\left|\vec{U}\right|\cos 0 = \left|\vec{U}\right|^2 \Leftrightarrow \left|\vec{U}\right| = \sqrt{\vec{U}\cdot\vec{U}} = \sqrt{U_x^2 + U_y^2 + U_z^2}$$

<u>Vector components parallel and normal to a line</u>



$$\left|\vec{U}_P\right| = \left|\vec{U}\right|\cos\theta$$

Let $\hat{e}$ be a unit vector parallel to line L

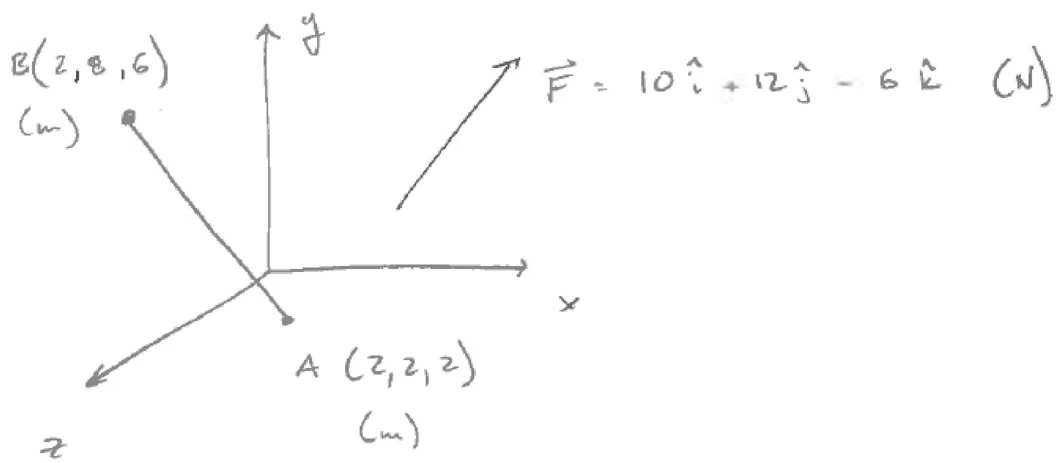
$$\hat{e}\cdot\vec{U} = |\vec{e}||\vec{U}|\cos\theta = |\vec{U}|\cos\theta = |\vec{U}_P|$$

Also note $\vec{U}_P = |\vec{U}_P|\hat{e}$

Combine previous two results to write $\boxed{\vec{U}_P = \left(\hat{e}\cdot\vec{U}\right)\hat{e}}$, the parallel vector component.

Upon considering $\vec{U} = \vec{U}_P + \vec{U}_N$, we find $\boxed{\vec{U}_N = \vec{U} - \vec{U}_P}$, the normal vector component.

Example



Given the vector $\vec{F} = 10\hat{i} + 12\hat{j} - 6\hat{k}$ (units of N) and points A and B as shown above, find the components of $\vec{F}$ parallel and perpendicular to a line through AB.

Solution:

Step 1: Find the unit vector along *AB*.

$$\vec{r}_{AB} = (2-2)\hat{i} + (8-2)\hat{j} + (6-2)\hat{k} = 6\hat{j} + 4\hat{k} \quad \text{(units of m)}$$

$$\vec{e}_{AB} = \frac{\vec{r}_{AB}}{\left|\vec{r}_{AB}\right|} = \frac{6\hat{j} + 4\hat{k}}{\sqrt{6^2 + 4^2}} = 0.832\hat{j} + 0.555\hat{k} \quad \text{(dimensionless)}$$

Step 2: Find the parallel component.

$$\vec{F}_P = \left(\hat{e}_{AB} \cdot \vec{F}\right)\hat{e}_{AB}$$

$$\hat{e}_{AB} \cdot \vec{F} = (0)(10\text{ N}) + (0.832)(12\text{ N}) + (0.555)(-6\text{ N}) = 6.654$$

$$\vec{F}_P = (6.654)\left(0.832\hat{j} + 0.555\hat{k}\right)$$

$$\boxed{\vec{F}_P = 5.54\hat{j} + 3.69\hat{k}} \quad \text{(units of N)}$$

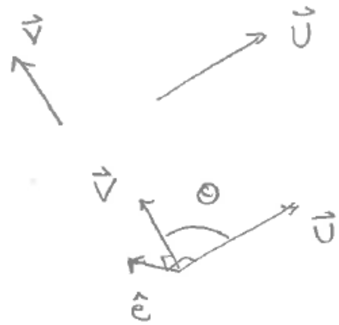
Step 3: Find the perpendicular (normal) component.

$$\begin{aligned}\vec{F}_N &= \vec{F} - \vec{F}_P \\ &= (10-0)\hat{i} + (12-5.54)\hat{j} + (-6-3.69)\hat{k}\end{aligned}$$

$$\boxed{\vec{F}_N = 10\hat{i} + 6.46\hat{j} - 9.69\hat{k}} \quad \text{(units of N)}$$

Question: What would be a good check?

Cross Products



$\vec{U}\times\vec{V}=\left|\vec{U}\right|\left|\vec{V}\right|\sin\theta\hat{e}$

Unit vector $\hat{e}$ is defined be perpendicular to both $\vec{U}$ and $\vec{V}$, according to the right-hand rule.

ENDED LECTURE HERE

Note $\vec{U}\times\vec{V}$ is a vector!

$\vec{U}\times\vec{V}=-\vec{V}\times\vec{U}$

The cross product is *not* commutative!

However, it is associative w.r.t. scalar multiplication and distributive w.r.t. vector addition:

$$a\left(\vec{U}\times\vec{V}\right)=\left(a\vec{U}\right)\times\vec{V}=\vec{U}\times\left(a\vec{V}\right)$$
$$\vec{U}\times\left(\vec{V}+\vec{W}\right)=\vec{U}\times\vec{V}+\vec{U}\times\vec{W}$$

Cross products in terms of components

$$\vec{U}\times\vec{V}=\left(U_x\hat{i}+U_y\hat{j}+U_z\hat{k}\right)\times\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right)$$
$$=U_xV_x\hat{i}\times\hat{i}+U_xV_y\hat{i}\times\hat{j}+U_xV_z\hat{i}\times\hat{k}$$
$$+U_yV_x\hat{j}\times\hat{i}+U_yV_y\hat{j}\times\hat{j}+U_yV_z\hat{j}\times\hat{k}$$
$$+U_zV_x\hat{k}\times\hat{i}+U_zV_y\hat{k}\times\hat{j}+U_zV_z\hat{k}\times\hat{k}$$
$$=\quad 0 \quad +U_xV_y\hat{k}\ +U_xV_z\left(-\hat{j}\right)$$
$$+U_yV_x\left(-\hat{k}\right)\quad +0\quad +U_yV_z\hat{i}$$
$$+U_zV_x\left(\hat{j}\right)\ +U_zV_y\left(-\hat{i}\right)\quad +0$$

$$\boxed{\vec{U}\times\vec{V}=\left(U_yV_z-U_zV_y\right)\hat{i}-\left(U_xV_z-U_zV_x\right)\hat{j}+\left(U_xV_y-U_yV_x\right)\hat{k}}$$

This is easiest to remember by writing it as a determinant:

$$\boxed{\vec{U}\times\vec{V}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ U_x & U_y & U_z\\ V_x & V_y & V_z\end{vmatrix}}$$

Mixed (scalar) triple product

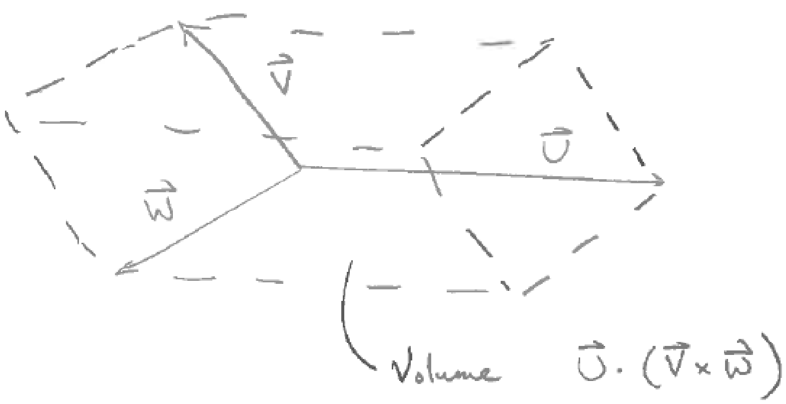
$$\vec{U}\cdot\left(\vec{V}\times\vec{W}\right)=U_x\left(V_yW_z-V_zW_y\right)-U_y\left(V_xW_z-V_zW_x\right)+U_z\left(V_xW_y-V_yW_x\right)$$
$$=\begin{vmatrix}U_x & U_y & U_z\\ V_x & V_y & V_z\\ W_x & W_y & W_z\end{vmatrix}$$

Note that the order matters!

$$\vec{U}\cdot\left(\vec{V}\times\vec{W}\right)=-\vec{W}\cdot\left(\vec{V}\times\vec{U}\right) \text{ (one swap)}$$

$$\vec{U}\cdot\left(\vec{V}\times\vec{W}\right)=\vec{W}\cdot\left(\vec{U}\times\vec{V}\right) \text{ (two swaps)}$$

If $\vec{U}$, $\vec{V}$, and $\vec{W}$ form a right-handed system, the mixed triple product gives the volume of the parallelpiped.



Go through Examples 2.15[2] and 2.16[3] in the textbook on your own.

WENT THROUGH EXAMPLE 2.15 IN CLASS, AND THEN ONLY STARTED ON FORCES(NEXT SECTION IN BOOK), ~5 MINUTES.

[2] Example of using the cross product to find the minimum distance between a point and a line.
[3] Example of assessing components of a force normal to a plane.