

Mathematical Notes for E&M

Gradient, Divergence, and Curl

In these notes I explain the differential operators *gradient*, *divergence*, and curl (also known as *rotor*), the relations between them, the integral identities involving these operators, and their role in electrostatics.

DEFINITIONS:

- **Gradient** of a scalar field $S(x, y, z)$ is a vector field

$$\mathbf{grad} S \equiv \nabla S \quad \text{with components} \quad \left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z} \right). \quad (1)$$

Note formal vector structure of a product of a vector ∇ with a scalar S :

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \implies \nabla S = \left(\frac{\partial}{\partial x} S, \frac{\partial}{\partial y} S, \frac{\partial}{\partial z} S \right). \quad (2)$$

- **Divergence** of a vector field $\mathbf{A}(x, y, z)$ is a scalar field

$$\mathbf{div} \mathbf{A} \equiv \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \quad (3)$$

Note formal structure of a scalar product of a vector ∇ and a vector \mathbf{A} :

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z. \quad (4)$$

- **Curl** or **rotor** of a vector field \mathbf{A} is a vector field

$$\mathbf{curl} \mathbf{A} \equiv \mathbf{rot} \mathbf{A} \equiv \nabla \times \mathbf{A} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right). \quad (5)$$

Note formal structure of a vector product of ∇ and \mathbf{A} :

$$\begin{aligned} (\nabla \times \mathbf{A})_x &= \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y, \\ (\nabla \times \mathbf{A})_y &= \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z, \\ (\nabla \times \mathbf{A})_z &= \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x. \end{aligned} \quad (6)$$

IDENTITIES:

- *A gradient has zero curl:*

$$\nabla \times (\nabla S) \equiv 0. \quad (7)$$

Mnemonics: $\nabla \times (\nabla S) = (\nabla \times \nabla)S = 0$ because $\nabla \times \nabla = 0$ as a cross product of a vector ∇ with itself..

Formal proof:

$$\left[\nabla \times (\nabla S) \right]_x = \frac{\partial}{\partial y} (\nabla S)_z - \frac{\partial}{\partial z} (\nabla S)_y = \frac{\partial}{\partial y} \frac{\partial S}{\partial z} - \frac{\partial}{\partial z} \frac{\partial S}{\partial y} = 0 \quad (8)$$

because the $\partial/\partial y$ and $\partial/\partial z$ partial derivatives can be taken in any order without changing the result. Likewise, the y and z components of $\nabla \times (\nabla S)$ vanish for similar reasons.

- ★ In particular, in electrostatics $\nabla \mathbf{E} = 0$.

- *A curl has zero divergence:*

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0. \quad (9)$$

Mnemonics: $\nabla \cdot (\nabla \times \mathbf{A}) = (\nabla \times \nabla) \cdot \mathbf{A} = 0$ because $\nabla \times \nabla = 0$.

Formal proof:

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{A}) &= \frac{\partial}{\partial x} (\nabla \times \mathbf{A})_x + \frac{\partial}{\partial y} (\nabla \times \mathbf{A})_y + \frac{\partial}{\partial z} (\nabla \times \mathbf{A})_z \\ &= \frac{\partial}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ &= \left(\frac{\partial}{\partial x} \frac{\partial A_z}{\partial y} - \frac{\partial}{\partial y} \frac{\partial A_z}{\partial x} \right) + \left(\frac{\partial}{\partial y} \frac{\partial A_x}{\partial z} - \frac{\partial}{\partial z} \frac{\partial A_x}{\partial y} \right) \\ &\quad + \left(\frac{\partial}{\partial z} \frac{\partial A_y}{\partial x} - \frac{\partial}{\partial x} \frac{\partial A_y}{\partial z} \right) \\ &= 0 + 0 + 0 = 0. \end{aligned} \quad (10)$$

LEIBNIZ RULES:

$$\nabla(SP) = S(\nabla P) + P(\nabla S), \quad (11)$$

$$\nabla \cdot (S\mathbf{A}) = (\nabla S) \cdot \mathbf{A} + S(\nabla \cdot \mathbf{A}), \quad (12)$$

$$\nabla \times (S\mathbf{A}) = (\nabla S) \times \mathbf{A} + S(\nabla \times \mathbf{A}), \quad (13)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}. \quad (14)$$

INTEGRALS AND THEOREMS

For ordinary functions of one variable, there is **Newton's Theorem** about integrals of derivatives: *for any function $f(x)$ and its derivative $f'(x) = \frac{df}{dx}$,*

$$\int_a^b f'(x) dx = f(b) - f(a). \quad (15)$$

The **gradient theorem** generalizes Newton's theorem to integrals of gradients over curved lines in 3D space: *for any path P from point A to point B and any scalar field $S(x, y, z)$,*

$$\int_{P:A \rightarrow B} (\nabla S) \cdot d\mathbf{r} = \int_A^B dS(\mathbf{r}) = S(B) - S(A). \quad (16)$$

For example, in Electrostatics $\mathbf{E} = -\nabla V$, and therefore for any path P from point A to point B ,

$$\int_{P:A \rightarrow B} \mathbf{E} \cdot d\mathbf{r} = -V(B) + V(A). \quad (17)$$

There are also theorems concerning volume integrals of divergences and surface integrals of curls. Let me state these theorems without proofs; hopefully, you should learn the proofs in a Math class.

GAUSS THEOREM AND GAUSS LAW

Another very important theorem for the electrostatics and the electromagnetism is the Gauss's divergence theorem which relates the flux of a vector field through a surface and the volume integral of the field's divergence.

Gauss's Theorem (also known as **Ostrogradsky's theorem** or **divergence theorem**):
Let \mathcal{V} be a volume of space and let \mathcal{S} be its boundary, i.e., the complete surface of \mathcal{V} surrounding \mathcal{V} on all sides. Then, for any differentiable vector field $\mathbf{A}(x, y, z)$, the flux of \mathbf{A} through \mathcal{S} equals to the volume integral of the divergence $\nabla \cdot \mathbf{A}$ over \mathcal{V} ,

$$\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{A}) d^3\text{Volume} = \iint_{\mathcal{S}} \mathbf{A} \cdot d^2\mathbf{Area}. \quad (18)$$

Note: in these integrals, the infinitesimal volume $d^3\text{Volume} = dx dy dz$ acts as a scalar, but the infinitesimal area $d^2\mathbf{Area}$ acts as a vector; its direction is normal to the surface \mathcal{S} , from the inside of \mathcal{V} to the outside.

This theorem allows us to rewrite the **Gauss Law in differential form**:

$$\nabla \cdot \mathbf{E}(x, y, z) = 4\pi k \times \rho(x, y, z) \quad (19)$$

where ρ is the volume density of the electric charge and k is the Coulomb constants. In Gaussian units

$$\nabla \cdot \mathbf{E} = 4\pi \times \rho \quad (20)$$

while in MKSA units

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \times \rho. \quad (21)$$

Note: the Gauss Law — in the integral form or in the differential form — applies not just in electrostatics but also in electrodynamics. Indeed, the Maxwell's equations of electrodynamics include the Gauss Law (19).

To derive eq. (19), let's start with the integral form of the Gauss Law which relates the net flux of the electric field \mathbf{E} through any closed surface \mathcal{S} to the net electric charge in the volume \mathcal{V} enclosed within \mathcal{S} ,

$$\Phi_E[\mathcal{S}] \equiv \iint_{\mathcal{S}} \mathbf{E} \cdot d^2\mathbf{Area} = 4\pi k \times Q_{\text{net}}[\text{inside } \mathcal{V}]. \quad (22)$$

By Gauss's divergence theorem, the flux of \mathbf{E} equals to the volume integral of the divergence $\nabla \cdot \mathbf{E}$, hence

$$\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{E}) d^3\text{Volume} = \iint_{\mathcal{S}} \mathbf{E} \cdot d^2\mathbf{Area} = 4\pi k \times Q_{\text{net}}[\text{inside } \mathcal{V}]. \quad (23)$$

For a continuous charge distribution with volume density $\rho(x, y, z)$,

$$Q_{\text{net}}[\text{inside } \mathcal{V}] = \iiint_{\mathcal{V}} \rho d^3\text{Volume}, \quad (24)$$

hence

$$\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{E}) d^3\text{Volume} = \iiint_{\mathcal{V}} 4\pi k \times \rho d^3\text{Volume}. \quad (25)$$

Note: both sides of this equation are integrals over the same volume \mathcal{V} , and the equality must hold for any such volume. Consequently, the integrands on both sides must be identically equal at all (x, y, z) , thus

$$\nabla \cdot \mathbf{E}(x, y, z) = 4\pi k \times \rho(x, y, z). \quad (26)$$

Quod erat demonstrandum.

ELECTROSTATIC ENERGY

In class I have derived the electrostatic potential energy of a system of several charged conductors. If each conductor has net charge Q_i and potential V_i (which is constant over the whole conductor), then the net electrostatic energy is

$$U = \frac{1}{2} \sum_i V_i \times Q_i. \quad (27)$$

To generalize this formula to charged insulators where the electric charge is distributed throughout the insulator's volume and the potential is not locally constant, we simply replace the sum over conductors to the volume integral

$$U = \frac{1}{2} \iiint V(\mathbf{r}) \times \rho(r) d^3\mathbf{r} \quad (28)$$

where $\rho(\mathbf{r})$ is the electric charge density and

$$d^3\mathbf{r} = d^3\text{Volume} = dx dy dz. \quad (29)$$

Formally, the integral in eq. (28) is over the whole 3D space, but we may reduce the integration domain to the volume actually occupied by the electric charges since everywhere else $\rho = 0$.

In light of the Gauss Law (19), we may rewrite eq. (28) for the electrostatic energy in terms of the electric field \mathbf{E} and the potential V :

$$\rho = \frac{\nabla \cdot \mathbf{E}}{4\pi k} \implies U = \frac{1}{8\pi k} \iiint V(\nabla \cdot \mathbf{E}) d^3\mathbf{r}. \quad (30)$$

Now let's take this integral by parts. By one of the Leibniz rules,

$$\nabla \cdot (V\mathbf{E}) = V(\nabla \cdot \mathbf{E}) + (\nabla V) \cdot \mathbf{E} = V(\nabla \cdot \mathbf{E}) - \mathbf{E}^2, \quad (31)$$

hence by Gauss's theorem

$$\begin{aligned} \iiint_{\mathcal{V}} V(\nabla \cdot \mathbf{E}) d^3\mathbf{r} &= \iiint_{\mathcal{V}} \mathbf{E}^2 d^3\mathbf{r} + \iiint_{\mathcal{V}} \nabla \cdot (V\mathbf{E}) d^3\mathbf{r} \\ &= \iiint_{\mathcal{V}} \mathbf{E}^2 d^3\mathbf{r} + \iint_S V\mathbf{E} \cdot d^2\mathbf{Area}. \end{aligned} \quad (32)$$

In this formula \mathcal{V} can be any volume while \mathcal{S} is its complete surface. In the context of equations (28) and (30), we want the volume \mathcal{V} to contain all the electric charges, but it's OK if it also contains some empty space. For the moment, let \mathcal{V} be a very large sphere of radius R , so \mathcal{S} is a very large sphere. Consequently, the surface-integral term in eq. (32) becomes

$$\iint_{\mathcal{S}} V \mathbf{E} \cdot d^2 \mathbf{Area} = \iint V(\mathbf{r}) \times E_{\text{rad}}(\mathbf{r}) \times R^2 d^2 \Omega \quad (33)$$

where $d^2 \Omega$ is the infinitesimal solid angle.

Now let's take the infinite radius limit, $R \rightarrow \infty$. As long as the net charge of the system is finite, at very large distances from it the electric field decreases like $E \propto R^{-2}$ while the potential decreases like $V \propto R^{-1}$. Consequently,

$$V(R) \times E_{\text{rad}}(R) \times R^2 \propto \frac{1}{R} \xrightarrow{R \rightarrow \infty} 0, \quad (34)$$

and the surface integral (33) approaches zero. At the same time, the volume integrals on the left and right sides of eq. (32) become integrals over the whole space, thus

$$\iiint_{\text{whole space}} V(\nabla \cdot \mathbf{E}) d^3 \mathbf{r} = \iiint_{\text{whole space}} \mathbf{E}^2 d^3 \mathbf{r}. \quad (35)$$

Consequently, the net electrostatic energy of the system may be written as

$$U = \frac{1}{8\pi k} \iiint_{\text{whole space}} \mathbf{E}^2 d^3 \mathbf{r}. \quad (36)$$

In MKSA units this formula becomes

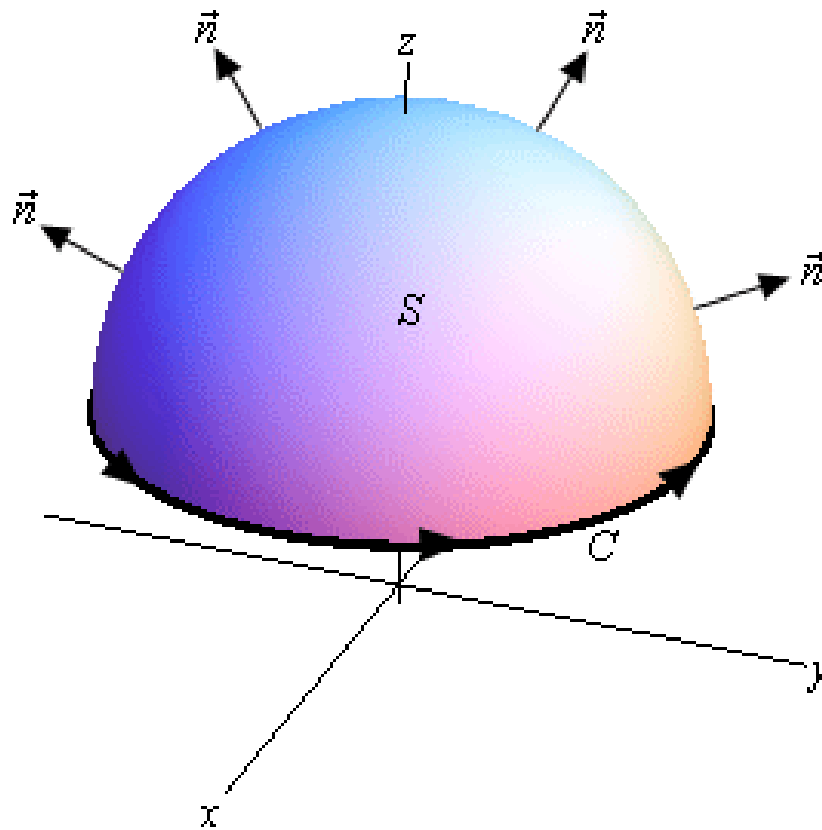
$$U = \frac{\epsilon_0}{2} \iiint_{\text{whole space}} \mathbf{E}^2 d^3 \mathbf{r} \quad (37)$$

while in Gaussian units it becomes

$$U = \frac{1}{8\pi} \iiint_{\text{whole space}} \mathbf{E}^2 d^3 \mathbf{r}. \quad (38)$$

OTHER USEFUL THEOREMS

Stokes's theorem (also known as the **curl theorem**): *Let C be a closed loop in 3D space and let S be a surface spanning C as shown on the picture below:*



Note: S should not have any holes, or any boundaries other than C . Let \mathbf{A} be a vector field and $\mathbf{B} = \nabla \times \mathbf{A}$ its curl. Then the flux of the curl through S equals to the loop integral of \mathbf{A} over C ,

$$\iint_S (\nabla \times \mathbf{A}) \cdot d^2 \mathbf{Area} = \oint_C \mathbf{A} \cdot d\mathbf{r}. \quad (39)$$

The Stokes's theorem will be very useful when we study the magnetic field.

Poincaré Lemma is a rather general theorem in differential topology. For the present purposes, let me state it for two particularly simple — and particularly important — cases.

- *If a vector field $\mathbf{A}(x, y, z)$ has a curl $\nabla \times \mathbf{A}$ which vanishes everywhere in space, then \mathbf{A} is a gradient of some scalar field,*

$$\begin{aligned} &\text{if } \nabla \times \mathbf{A}(x, y, z) \equiv 0 \quad \forall (x, y, z), \\ &\text{then } \exists \text{ scalar field } S(x, y, z) \text{ such that} \\ &\mathbf{A}(x, y, z) \equiv \nabla S(x, y, z). \end{aligned} \tag{40}$$

For examples, in electrostatics (but not in electrodynamics) the electric field \mathbf{E} has zero curl everywhere in space, so it's a gradient of a scalar field $-V(x, y, z)$,

$$\nabla \times \mathbf{E}(x, y, z) \equiv 0 \quad \forall (x, y, z) \implies \exists V(x, y, z) \text{ for which } \mathbf{E} \equiv -\nabla V. \tag{41}$$

- *If a vector field $\mathbf{B}(x, y, z)$ has a divergence $\nabla \cdot \mathbf{B}$ which vanishes everywhere in space, then \mathbf{B} is a curl of another vector field $\mathbf{A}(x, y, z)$,*

$$\begin{aligned} &\text{if } \nabla \cdot \mathbf{B}(x, y, z) \equiv 0 \quad \forall (x, y, z), \\ &\text{then } \exists \text{ vector field } \mathbf{A}(x, y, z) \text{ such that} \\ &\mathbf{B}(x, y, z) \equiv \nabla \times \mathbf{A}(x, y, z). \end{aligned} \tag{42}$$

In particular, there are no magnetic charges, so the magnetic field $\mathbf{B}(x, y, z)$ has zero divergence everywhere in space. Consequently, it can be written as a curl of the *vector potential* $\mathbf{A}(x, y, z)$,

$$\nabla \cdot \mathbf{B}(x, y, z) \equiv 0 \quad \forall (x, y, z) \implies \exists \mathbf{A}(x, y, z) \text{ for which } \mathbf{B} \equiv \nabla \times \mathbf{A}. \tag{43}$$

LAPLACE OPERATOR:

The *Laplace operator* or the *Laplacian* is a second-order differential operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (44)$$

From the rotational point of view, the Laplacian is a scalar, so when it acts on a scalar field it makes a scalar, when it acts on a vector field it makes a vector, *etc.* For scalar fields, the Laplacian acts as a divergence of the gradient,

$$\Delta S = \nabla \cdot (\nabla S). \quad (45)$$

In particular, the Laplacian of the electrostatic potential $V(x, y, z)$ is related by this formula to the divergence of the electric field and hence by Gauss Law to the electric charge density:

$$\nabla V = -\mathbf{E} \implies \Delta V = \nabla \cdot (\nabla V) = -\nabla \cdot \mathbf{E} = -4\pi k \times \rho. \quad (46)$$

This relation is known as the *Poisson equation*.