NOTES ON HYSTERESIS

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Based on “Aggregate Demand and the Dynamics of Unemployment” by Edouard Schaal and Mathieu Taschereau-Dumouchel

1. Model

- Workers produce a continuum of intermediate inputs
- All agents are risk neutral and discount future at rate $\beta$
- Continuum of workers $j \in [0, 1]$
- If worker $j$ is employed, produces good $j$

$$Y_j = Ae^z$$

where $z$ stochastic (see below)
- Unemployed workers enjoy leisure $b$
- To be employed workers need to be matched with firms (below on matching)
- Differentiated goods produced by each worker enter production function of final good

$$Y = \left( \int_{j \in E} Y_j^{\frac{\sigma - 1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma - 1}}$$

where $E$ set of employed workers
- Inverse demand function for good $j$

$$P_j = P \left( \frac{Y_j}{Y} \right)^{-\frac{1}{\sigma}}$$

- So revenue of firm employing worker $j$ is

$$P_j Y_j = P \left( \frac{Y_j}{Y} \right)^{-\frac{1}{\sigma}} Y_j = P (Ae^z)^{1 - \frac{1}{\sigma}} Y^\frac{1}{\sigma}$$

- With $1 - u$ employed workers total output is

$$Y = Ae^z (1 - u)^{\frac{\sigma}{\sigma - 1}}$$

- So real revenues per firm are

$$\rho (u) = \frac{P_j}{P} Y_j = Ae^z (1 - u)^{\frac{1}{\sigma - 1}}$$

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• Now to the search and matching part
• Unemployed workers meet vacancies at rate

\[ m(u, v) \]

where \( u \) is mass of unemployed and \( v \) mass of vacancies
• Assume CRS
• Probability of finding a vacancy for unemployed worker

\[ p \left( \frac{v}{u} \right) \equiv \frac{m(u, v)}{u} = m \left( 1, \frac{v}{u} \right) \]
• Probability of finding a worker for a vacancy

\[ q \left( \frac{v}{u} \right) \equiv \frac{m(u, v)}{v} = m \left( \frac{1}{v/u}, 1 \right) \]

• Value functions
• Value of a job

\[ J_t = \rho_t - w_t + \beta (1 - \delta) E_t [J_{t+1}] \]
• Value of a employment

\[ W_t = w_t + \beta E_t [(1 - \delta) W_{t+1} + \delta U_{t+1}] \]
• Value of unemployment

\[ U_t = b + \beta E_t [p_t W_{t+1} + (1 - p_t) U_{t+1}] \]

• Nash bargaining, wages set so that \( \gamma \) fraction of surplus goes to workers

\[ J_t = (1 - \gamma) S_t \]

where \( S_t \) is the total surplus of a match

\[ S_t \equiv J_t + W_t - U_t \]

• Total surplus dynamics

\[ S_t = \rho_t - b + \beta E_t [(1 - \delta) [J_{t+1} + W_{t+1}] + \delta U_{t+1} - p_t W_{t+1} - (1 - p_t) U_{t+1}] \]

\[ = \rho_t - b + \beta E_t [(1 - \delta) [J_{t+1} + W_{t+1} - U_{t+1}] - p_t (W_{t+1} - U_{t+1})] \]

\[ S_t = \rho_t - b + \beta E_t [(1 - \delta) S_{t+1} - p_t \gamma S_{t+1}] \]

• Given dynamics for \( \rho \) and \( \theta \) we can recover dynamics of \( S \) from last equation
• Job creation
• Create vacancy if

\[ \kappa \leq \beta q_t E_t J_{t+1} = \beta q_t (1 - \gamma) E_t S_{t+1} \]
• Heterogeneity in $\kappa$
• Mass $M$ of potential entrants with $\kappa$ distributed with CDF $F(\kappa)$
• So
  \[
  \theta_t = \frac{v_t}{u_t} = \frac{MF(\hat{\kappa}_t)}{u_t}
  \]
  where
  \[
  \hat{\kappa}_t = (1 - \gamma) \beta q(\theta_t) E_t S_{t+1}
  \]
• Given $E_t S_{t+1}$ this equation gives us $\hat{\kappa}_t$
• Unemployment dynamics
  \[
  u_{t+1} = (1 - p(\theta_t)) u_t + \delta (1 - u_t)
  \]

2. Dynamics
• Let’s remove the $z$ shock
• Markov equilibrium with unique state variable $u$
• Functional equation in $S$
  \[
  S(u) = \rho(u) - b + \beta [(1 - \delta) S(u') - p(\theta) \gamma S(u')]
  \]
  \[
  u' = (1 - p(\theta)) u + \delta (1 - u)
  \]
  \[
  \kappa = (1 - \gamma) q(\theta) \beta S(u')
  \]
  \[
  \theta u = MF(\kappa)
  \]
• Notice that if $\rho(u) = \rho$ and no heterogeneity in job creation, then we can show that there is a unique equilibrium in which $S(u)$ is flat (Mortensen and Nagypal)
• In this case there is a form of block recursivity: the value of a filled job can be solved without knowing the state $u$ as solution to
  \[
  S = \rho - b + \beta \left[(1 - \delta) S - \frac{\gamma}{1 - \gamma} \kappa\right]
  \]
  then constant $\theta$ can be found from
  \[
  \kappa = (1 - \gamma) q(\theta) S
  \]
and unemployment dynamics from
  \[
  u' = (1 - p(\theta)) u + \delta (1 - u)
  \]
• With general $\rho(u)$ this does not hold and we have
  \[
  \theta = \Theta(u)
  \]
• Solving the functional equation
• Conjecture $S(.)$; for given $u$ find $\theta$ that solves

$$\theta u = MF ((1 - \gamma) q (\theta) \beta S ((1 - p (\theta)) u + \delta (1 - u)))$$

(if there are multiple solutions... pick one)

• Let distribution $\kappa$ have bounds $\underline{\kappa}, \overline{\kappa}$

• Then, if at $\theta = M/u$

$$(1 - \gamma) q (\theta) \beta S ((1 - p (\theta)) u + \delta (1 - u)) > \overline{\kappa}$$

then

$$\theta = M/u$$

if

$$(1 - \gamma) \beta S (u + \delta (1 - u)) < \underline{\kappa}$$

then

$$\theta = 0$$

• Use Cobb-Douglas matching function so

$$p (\theta) = \min\{\eta \theta^\alpha, \theta, 1\}$$

$$1 - p (\theta) u + \delta (1 - u) < 1$$

$$q (\theta) = p (\theta) / \theta$$

• Update

$$S (u) = \rho (u) - b + \beta [(1 - \delta) S (u') - p (\theta) \gamma S (u')]$$

• In matlab folder a code that follows the algorithm above

• Easy to add $z$ shocks

3. A CONTINUOUS TIME VERSION

• The continuous time version of the model gives us two ODEs for $u, S$

$$\dot{u} = \delta (1 - u) - p (\theta) u$$

$$rS = \rho (u) - b - (\delta + \gamma p (\theta)) S + \dot{S}$$

• Where

$$p (\theta) = \eta \theta^\alpha$$

$$\rho (u) = A (1 - u)^{1/\sigma}$$

• Consider the case with a fixed $\kappa$

• Then free entry condition is

$$\kappa = (1 - \gamma) q (\theta) S$$
where
\[ q(\theta) = \eta \theta^{\alpha-1} \]
gives a relation between \( S \) and \( \theta \)

- Let’s derive the latter relation explicitly and substitute in \( p(\theta) \) to obtain the finding rate as a function of the surplus \( S \)

\[ p = f(S) \equiv \xi S^{1-\alpha} \]

where
\[ \xi = \eta \frac{1}{1-\alpha} \left( \frac{1-\gamma}{\kappa} \right)^{1-\alpha} \]

- Then the ODEs become

\[ \dot{u} = \delta (1-u) - f(S) u \]
\[ \dot{S} = (r + \delta + \gamma f(S)) S - \rho(u) + b \]

- Steady state conditions

\[ \delta (1-u) = f(S) u \]
\[ S = \frac{\rho(u) - b}{r + \delta + \gamma f(S)} \]

3.1. **Numerical analysis.**

- Choose all parameters except \( A, b \)
- Calibrate to 2 steady states \( u_L, u_H \) as follows
- Choose \( u_L, u_H \)
- Get values of \( S_L, S_H \) from

\[ S = \left( \frac{\delta 1-u}{\xi - u} \right)^{1-\alpha} \]

- Choose \( A \) and \( b \) so following the equation holds for both

\[ (r + \delta + \gamma f(S)) S = A (1-u)^{\frac{1}{\sigma-1}} - b \]

- That is, set

\[ A = \frac{(r + \delta + \gamma f(S_H)) S_H - (r + \delta + \gamma f(S_L)) S_L}{(1-u_H)^{\frac{1}{\sigma-1}} - (1-u_L)^{\frac{1}{\sigma-1}}} \]
\[ b = A (1-u_L)^{\frac{1}{\sigma-1}} - (r + \delta + \gamma f(S_L)) S_L \]

- Then analyze dynamic properties of the ODEs, solving backward starting near SS
- The following is the phase diagram for an example

- In this example there are multiple equilibria, because for \( u \) in some interval two equilibrium paths are possible
In fact, the dynamics near the high $u$ equilibrium are a sink-spiral, that is, the linearized system has two complex conjugate eigenvalues with a positive real part (see below a quick review of ODEs).

This means that the model actually features indeterminacy as for $u$ in some range there is a continuum of $S$ where you can start the equilibrium dynamics.

The following exercise asks you to look for dynamics more like the ones emphasized in the paper: no multiple equilibria, but multiple steady states, so the initial condition for $u$ matters for long run dynamics.

Figure 1. Two steady states and equilibrium dynamics
• Maybe it can be done, maybe you need to add heterogeneity in $\kappa$ as in the full blown model in the paper

**Exercise 1.** Can you find parameters such that the economy has two ss: one of them a saddle and the other one a source and not a spiral?

### 4. Quick review of dynamic systems

- If $$\dot{x} = h(x)$$

  and $x$ is vector, let it have ss at 0 (normalization)

- Linearize near 0 $$\dot{x} = Dh \cdot x$$

- Local saddle if $Dh$ has two real eigenvalues of opposite sign

- Let $$Dh = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

  and notice that characteristic polynomial is

  $$(h_{11} - \lambda)(h_{22} - \lambda) - h_{12}h_{21} = \lambda^2 - (h_{11} + h_{22})\lambda + h_{11}h_{22} - h_{21}h_{12}$$

- This quadratic has two real roots of opposite sign iff $$h_{11}h_{22} - h_{21}h_{12} < 0$$

- This restriction can be expressed as a restriction on the relative slope of the loci $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$

- The direction of the inequality depends on the signs of $h_{12}$ and $h_{22}$

- In our case $h_{22} > 0$ and $h_{12} < 0$ so we get

  $$\frac{h_{11}}{h_{12}} > \frac{h_{21}}{h_{22}}$$

  or

  $$\frac{h_{11}}{h_{12}} < -\frac{h_{21}}{h_{22}}$$

  so the locus $\dot{x}_1 = 0$ must have smaller derivative than the locus $\dot{x}_2 = 0$

- If a steady state is not a saddle it can be:
  - a source (all eigenvalues have positive real part, if $h_{11} + h_{22} > 0$)
  - or a sink (all eigenvalues have negative real part, if $h_{11} + h_{22} < 0$)

- In both cases it can be a spiral if eigenvalues have imaginary parts, that is, if

  $$(h_{11} + h_{22})^2 - 4(h_{11}h_{22} - h_{21}h_{12}) < 0$$

- A saddle cannot be a spiral
5. Back to the (continuous time) model

- Notice that if multiple steady states are possible, then the loci have to cross multiple times and they cannot cross always from the same side.
- Result: if there are multiple ss, if one is a saddle, the one immediately next to it cannot be a saddle.
- Result: if a ss is a spiral, then multiple equilibria exist (sufficient condition, not necessary).
- In our example above, the high $u$ steady state was a sink/spiral.
- Compute $Dh$ for the model

\[
\begin{align*}
  h_1(u, S) &= \delta (1 - u) - f(S) u \\
  h_2(u, S) &= (r + \delta + \gamma f(S)) S - \rho(u) + b \\
  h_{11} &= -\delta - f(S) \\
  h_{12} &= -f'(S) u \\
  h_{21} &= -\rho'(u) \\
  h_{22} &= r + \delta + \gamma f(S) + \gamma f'(S) S
\end{align*}
\]

- Determinant

\[
 h_{11} h_{22} - h_{12} h_{21} = -(\delta + f(S)) (r + \delta + \gamma f(S) + \gamma f'(S) S) - f'(S) u \rho'(u)
\]

- Notice that if

\[
 \rho' = 0
\]

(as in baseline DMP) then determinant is always $< 0$ and we can only have a saddle (so ss must be unique).
- But if $\rho' < 0$ and large enough in absolute value, we can have other configurations.