1. A runnable entity

- A model inspired by He and Xiong
- Captures a simple entity with a balance sheet as follows:
  - a risky portfolio of uncertain value
  - short term liabilities, rolled over
- An entrepreneur needs to borrow 1 at date 0 to finance project with potential payoff \( y_0 \)
- Payoff of the project evolves according to the geometric random walk
  \[ y_t = y_{t-1} \varepsilon_t \]
- With \( E[\varepsilon_t] = 1 \) and \( \varepsilon_t \in [\underline{\varepsilon}, \overline{\varepsilon}] \)
- Each period the project is completed with probability \( \phi \), in which case it pays \( y_t \), or it continues
- A project only pays off when completed. The expected present value of the project is
  \[ v_t = \beta E_t \left( (1 - \phi) v_{t+1} + \phi y_{t+1} \right) \]
which yields
  \[ v_t = \frac{\phi}{1/\beta - 1 - \phi} y_t. \]
We assume
  \[ v_0 > 1, \]
so the project is profitable.
- Entrepreneur has no initial wealth
- E. finances the project selling debt contracts of random maturity (see below) to a large population of lenders
- Debt contracts have the following features
  - Each period \( t \geq 1 \), if the project is completed a lender gets
    \[ \min\left\{ 1, \frac{y_t}{d_{t-1}} \right\} \]
where \( d_{t-1} \) is the number of debt contracts outstanding from last period
NOTES ON PANICS

– If the project is not completed, a fraction \( \delta \) of the \( d_{t-1} \) contracts outstanding is drawn randomly and they are paid 1 each, at which point those debt contracts are fulfilled
– If a contract is not drawn at date \( t \), it remains outstanding
– The borrower finances the payment \( \delta d_{t-1} \) by issuing new debt contracts, so the budget constraint is

\[
 p_t \left[ d_t - (1 - \delta) d_{t-1} \right] \geq \delta d_{t-1}
\]

• Timing in case of rollover is as follows:
  – the borrower announces it will issue \( d_t \) debt contracts
  – the lenders observe \( d_t \) and the price \( p_t \) is determined (maybe on an auction)
  – if \( p_t \left[ d_t - (1 - \delta) d_{t-1} \right] \geq \delta d_{t-1} \) the borrower continues
  – if \( p_t \left[ d_t - (1 - \delta) d_{t-1} \right] < \delta d_{t-1} \) no new debt is issued, the borrower goes into default, the project is liquidated, and the existing debt holders receive

\[
 \min \left\{ 1, \frac{\lambda y_t}{d_{t-1}} \right\}
\]

• The parameter \( \lambda < 1 \) captures the costs of early liquidation
• Assumption: borrower always wants to continue and always wants to have minimum debt, therefore he will always issue the minimum \( d_t \) such that (1) is satisfied, if such a \( d_t \) exists
• We could microfound last assumption by assuming borrower is a risk neutral agent who receives the equity value after realization of \( y \) with no default
• Rational expectations require the price to satisfy

\[
p_t = \beta E_t \left[ (1 - \phi) \rho_{t+1} \left\{ \delta + (1 - \delta) p_{t+1} \right\} + (1 - \phi) (1 - \rho_{t+1}) \min \left\{ 1, \frac{\lambda y_{t+1}}{d_t} \right\} + \phi \min \left\{ 1, \frac{y_{t+1}}{d_t} \right\} \right]
\]

where \( \rho_{t+1} \) is an indicator equal to 1 in case of successful rollover and 0 otherwise
• Define the debt-to-capital ratio at the end of the period

\[
x_t = \frac{d_t}{y_t},
\]

• Markov equilibrium:
  – market price of debt is given by the decreasing continuous function \( \mathcal{P} \)

\[
p_t = \mathcal{P} \left( \frac{d_t}{y_t} \right)
\]
  – default occurs iff \( y_t < \xi d_{t-1} \) for some scalar \( \xi \)
• rewrite the budget constraint (1) as

$$pt \left( \frac{d_t}{y_t} - (1 - \delta) \frac{d_{t-1}}{y_t} \right) \geq \delta \frac{d_{t-1}}{y_t}$$

or

$$P \left( \frac{d_t}{y_t} \right) \left( \frac{d_t}{y_t} - (1 - \delta) \frac{d_{t-1}}{y_{t-1}} \right) \geq \delta \frac{d_{t-1}}{y_{t-1}}$$

$$P \left( x_t \right) \left( x_t - (1 - \delta) \frac{x_{t-1}}{\epsilon_t} \right) \geq \delta \frac{x_{t-1}}{\epsilon_t}$$

• Each period the borrower takes the function $P$ as given and chooses $x_t$ that satisfies last equation as an equality

• The following result is useful:

Claim 1. Given a non-negative, decreasing continuous function $P (.) \geq 0$, there is a $\xi$ (which could be $\infty$) such that if $x \leq \xi \epsilon$ there is a solution $x' \geq (1 - \delta) x / \epsilon$ to the equation

$$P \left( x' \right) x' = (\delta + (1 - \delta) P \left( x' \right)) \frac{x}{\epsilon},$$

and if $x > \xi \epsilon$ there is no solution

• Graphical argument focusing on first looking at the “debt Laffer curve”

$$P \left( x' \right) \left( x' - (1 - \delta) \frac{x}{\epsilon} \right)$$

and then translating it in terms of the solutions to

$$\frac{P \left( x' \right) x'}{\delta + (1 - \delta) P \left( x' \right)} = \frac{x}{\epsilon},$$

where function on LHS is independent of $x/\epsilon$

• Whenever $x \leq \xi \epsilon$ denote the smallest solution as $x' = f \left( \frac{x}{\epsilon} \right)$

• Notice that $f$ is defined only on domain $x/\epsilon \in [0, \xi]$

• Notice that the inverse of $f$ can be derived in closed form

$$f^{-1} \left( x' \right) = \frac{P \left( x' \right) x'}{\delta + (1 - \delta) P \left( x' \right)}$$

in some range $x' \in [(1 - \delta) x/\epsilon, f^{-1} (\xi)]$

• So if $\epsilon \geq \xi x$ the borrower offers $x' = f \left( \frac{\xi}{\epsilon} \right)$ new debt contracts and successfully rolls over

• The rational expectations condition (2) is

$$(3) \quad P \left( x \right) = \beta E[\phi \min \left\{ 1, \frac{\epsilon}{x} \right\} + (1 - \phi) \left( \ell \left( \frac{x}{\epsilon} \leq \xi \right) \left( \delta + (1 - \delta) P \left( f \left( \frac{x}{\epsilon} \right) \right) \right) + \ell \left( \frac{x}{\epsilon} > \xi \right) \min \left\{ 1, \frac{\lambda \epsilon}{x} \right\} ]$$
**Definition 2.** A Markov equilibrium is given by a scalar \( \xi > 0 \), a function \( f \) and a function \( \mathcal{P} \), such that \( \xi \) and \( f \) are constructed as in Claim 1 and \( \mathcal{P} \) satisfies (3).

- There can be an interval \([0, \bar{x}]\) where \( x \) is small enough and we are sure no liquidation or default will occur next period, then the price is given by
  \[
  \mathcal{P}(x) = \beta E[(1 - \phi) (\delta + (1 - \delta) \mathcal{P}(f(x/\varepsilon))) + \phi \min \{1, \varepsilon/x\}]
  \]
  However, eventually if \( \varepsilon < 1 \) is realized for many periods, we escape to the region where default possible. Therefore \( \mathcal{P}(x) < 1 \) for all \( x > 0 \)
- At date 0, we need to check that
  \[
  \mathcal{P}(x_0) x_0 \geq 1
  \]
  for some \( x_0 \), so the project can be financed

2. **Algorithm**

- We compute a finite horizon model where the project matures with probability 1 at date \( T \)
- At date \( T \) lenders get
  \[
  \min \left\{ 1, \frac{\varepsilon_T}{x_T} \right\}
  \]
- Calculate
  \[
  \mathcal{P}_{T-1}(x_T) = \beta E \left[ \min \left\{ 1, \frac{\varepsilon_T}{x_T} \right\} \right]
  \]
- Find \( \xi_{T-1} \)
  \[
  \xi_{T-1} = \max_x \frac{\mathcal{P}_{T-1}(x) x}{\delta + (1 - \delta) \mathcal{P}_{T-1}(x)} = \frac{\beta E[\varepsilon]}{\delta}
  \]
  and \( f_{T-1} \) from inverting
  \[
  \frac{\mathcal{P}_{T-1}(x) x}{\delta + (1 - \delta) \mathcal{P}_{T-1}(x)}
  \]
  (which is monotone everywhere in this first round)
- Iterate on \( \mathcal{P} \)
  \[
  \mathcal{P}_{T-2}(x) = \beta E[\phi \min \left\{ 1, \frac{\varepsilon}{x} \right\} + (1 - \phi) \left( \frac{x}{\varepsilon} \geq \xi_{T-1} \right) \left( \delta + (1 - \delta) \mathcal{P} \left( f_{T-1} \left( \frac{x}{\varepsilon} \right) \right) \right) + \left( \frac{x}{\varepsilon} > \xi_{T-1} \right) \min \left\{ 1, \lambda \frac{\varepsilon}{x} \right\}]
  \]