NOTES ON FINANCIAL ACCELERATOR

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1. A BASELINE FINANCIAL ACCELERATOR MODEL

- Stochastic model with non state contingent debt, collateral constraints and aggregate investment
- Full global solution in the spirit of Brunnermeier-Sannikov
- A rich problem with a two dimensional state space
- Entrepreneurs risk neutral, with discount factor $\beta$
- Lenders risk neutral, with discount factor $q$
- Entrepreneurs only agents that can hold capital
- Adjustment cost function
- When entrepreneurs selling capital goods, these are turned back into consumption goods (at some cost)
- Entrepreneur’s budget constraint

$$c_t + G(k_{t+1}, k_t) = A_t F(k_t, l_t) - w_t l_t - b_t + q b_{t+1}$$

- $G$ is a CRS investment cost function, which includes adjustment costs

$$G(k', k) \equiv k' - (1 - \delta)k + \zeta (k' - (1 - \delta)k)^2 / k$$

- Collateral constraint

$$b_{t+1} \leq \theta p_{t+1} k_{t+1}$$

for all realizations of $p_{t+1}$ that have positive probability at $t$

- More below on the price $p_t$ at which capital can be sold
- $A_t, w_t, p_t$ driven by Markov process $s_t$
- $G$ and $F$ are constant returns to scale
- Then the value function must satisfy

$$V(k_t, b_t, s_t) = v(b_t / k_t, s_t) k_t$$

for some function $v$

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• Bellman equation

\[ v(\tilde{b}_t, s_t) k_t = \max_{c_t, l_t, \tilde{b}_{t+1}, k_{t+1}} c_t + \beta E_t \left[ v(\tilde{b}_{t+1}, s_{t+1}) \right] k_{t+1} \]

subject to

\[ c_t + G(k_{t+1}, k_t) = A_t F(k_t, l_t) - w_t l_t - \tilde{b}_t k_t + q\tilde{b}_{t+1} k_{t+1} \]

and

\[ \tilde{b}_{t+1} \leq \theta p_{t+1} l_t \]

• Optimality for \( k_{t+1} \) yields

\[ \beta E_t \left[ v(\tilde{b}_{t+1}, s_{t+1}) \right] + q \lambda_t \tilde{b}_{t+1} = \lambda_t G_1(k_{t+1}, k_t) \]

• If it’s optimal to consume \( \lambda_t = 1 \), in this case

\[ G_1(k_{t+1}, k_t) = \beta E_t \left[ v(\tilde{b}_{t+1}, s_{t+1}) \right] + q\tilde{b}_{t+1} \]

• The LHS is marginal \( Q \) the RHS is average \( Q \) (Abel 1982 and Hayashi 1982)

• If the non-negativity of consumption is never binding this model yields standard \( Q \) theory predictions: asset price over capital stock is a sufficient statistic for the investment rate \( k_{t+1}/k_t \)

• In general we can have \( \lambda_t > 1 \) which implies marginal \( Q \) smaller than average \( Q \): firms have an incentive to issue more claims to finance investment, but entrepreneurs cannot buy these claims, since they are at \( c_t = 0 \)

• If \( \lambda_t > 1 \) it means that either the collateral constraint is binding today or it will be binding in the future

• Optimality condition with respect to \( \tilde{b}_{t+1} \) is

\[ \lambda_t q k_{t+1} + \beta E_t \left[ \frac{\partial v(\tilde{b}_{t+1}, s_{t+1})}{\partial \tilde{b}} \right] k_{t+1} - \mu_t = 0 \]

and using envelope condition

\[ q \lambda_t = \beta E_t [\lambda_{t+1}] + \mu_t/k_{t+1} \]

• Envelope condition for \( k_t \) is

\[ v(\tilde{b}_t, s_t) = \lambda_t \left[ A_t F(k_t, l_t) - G_2(k_{t+1}, k_t) - \tilde{b}_t \right] \]

• Combining with optimality for \( k_{t+1} \)

\[ (1) \quad \lambda_t = \frac{\beta E_t \left[ v(\tilde{b}_{t+1}, s_{t+1}) \right]}{G_1(k_{t+1}, k_t) - q\tilde{b}_{t+1}} = \frac{\beta E_t \left[ \lambda_{t+1} \left[ A_{t+1} F_{k,t+1} - G_{2,t+1} - \tilde{b}_{t+1} \right] \right]}{G_{1,t} - q\tilde{b}_{t+1}} \]
• Suppose now entrepreneurs can trade used capital from other entrepreneurs, before employing the adjustment cost technology.

• Then to reach capital $k_{t+1}$ they will choose to minimize total cost of achieving it

$$\min_{\hat{k}_t} G \left( k_{t+1}, \hat{k}_t \right) + p_t \left( \hat{k}_t - k_t \right)$$

• Representative entrepreneur, so no trade and $\hat{k}_t = k_t$ in equilibrium.

• First order condition

$$p_t = -G_2 \left( k_{t+1}, k_t \right)$$

gives us the price of capital that appears in the collateral constraint.

• Then the optimality condition can be rewritten as

$$E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} A_{t+1} F_{k,t+1} + p_{t+1} - \tilde{b}_{t+1} \right] = 1$$

This is an asset pricing equation where

$$\beta \frac{\lambda_{t+1}}{\lambda_t}$$

is the stochastic discount factor of the entrepreneurs and

$$\frac{A_{t+1} F_{k,t+1} + p_{t+1} - \tilde{b}_{t+1}}{G_{1,t} - q\tilde{b}_{t+1}}$$

is the levered return on entrepreneurial capital.

• We can also rewrite optimality for borrowing ratio as an asset pricing equation

$$1 = E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{q} \right] + \mu_t \frac{1}{q\lambda_t k_{t+1}}$$

which implies

$$E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{q} \right] \leq 1$$

here the expected return on bonds, discounted with the discount factor $\beta \lambda_{t+1}/\lambda_t$ can be $< 1$ if the collateral constraint is binding.

• Rewrite (1) as

$$E_t \left[ \beta \lambda_{t+1} \left( A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1} - \tilde{b}_{t+1} \right) \right] = \lambda_t \left( G_{1,t} - q\tilde{b}_{t+1} \right)$$

and then as

$$E_t \left[ \beta \lambda_{t+1} \left( A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1} \right) \right] = \lambda_t G_{1,t} - (\lambda_t q - E_t \left[ \beta \lambda_{t+1} \right]) \tilde{b}_{t+1}$$

$$= \lambda_t G_{1,t} - \mu_t \tilde{b}_{t+1}$$
so
\[ E_t \left[ \frac{\beta \lambda_{t+1} A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1}}{\lambda_t G_{1,t}} \right] \leq 1 \]

- Agents are willing to accept a lower return on capital, since holding capital helps to relax the collateral constraint
- Using the same condition and \( q \lambda_t \geq E_t [\beta \lambda_{t+1}] \) we also get that if \( \lambda_{t+1} \) and \( A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1} - \tilde{b}_{t+1} \)

are negatively correlated we have
\[ q \lambda_t E_t \left[ A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1} - \tilde{b}_{t+1} \right] \geq E_t \left[ \beta \lambda_{t+1} \left[ A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1} - \tilde{b}_{t+1} \right] \right] = \lambda_t \left( G_{1,t} - q \tilde{b}_{t+1} \right) \]

which imply
\[ E_t \left[ A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1} \right] \geq \frac{1}{q} \]

so the expected rate of return on capital is greater than the risk free interest rate
- New possibility: the collateral constraint can be slack even though the rate of return on entrepreneurial capital is greater than \( 1/q \)
- Rewrite condition as
\[ E_t \left[ \beta \lambda_{t+1} \left[ A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1} - \tilde{b}_{t+1} \right] \right] = \lambda_t \left( G_{1,t} - q \tilde{b}_{t+1} \right) \]

- If constraint is slack \( \mu_t = 0 \) and this becomes
\[ E_t [\beta \lambda_{t+1} [A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1}]] = \lambda_t G_{1,t} \]
or
\[ E_t \left[ \frac{\beta \lambda_{t+1} A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1}}{\lambda_t G_{1,t}} \right] = 1 \]

- If there are no shocks we have
\[ \beta \frac{\lambda_{t+1}}{\lambda_t} = q \]

and
\[ A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1} = \frac{1}{q} \]

so collateral constraint can be slack only if investment is efficient at date \( t \)
- With risk, rate of return on entrepreneurial capital is correlated with \( \lambda_{t+1} \)
- Temporary productivity shocks generate negative correlation: high return on entrepreneurial wealth, high net worth, economy closer to efficient investment, lower return on entrepreneurial capital
- Then
\[ 1 = E_t \left[ \beta \frac{\lambda_{t+1} A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1}}{\lambda_t G_{1,t}} \right] < E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} \right] E_t \left[ \frac{A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1}}{G_{1,t}} \right] \]
so
\[ E_t \left[ \frac{A_{t+1} F_k (k_{t+1}, l_{t+1}) + p_{t+1}}{G_{1,t}} \right] > \frac{1}{q} \]

- This is a form of precautionary behavior: entrepreneurs are avoiding excess leverage because they anticipate states of the world in which the rate of return on their wealth will be higher than today (high \( \lambda_{t+1}/\lambda_t \))
- Notice that entrepreneurs are risk neutral, so “precautionary behavior” is really driven by general equilibrium forces

2. Linear Technology

- Suppose \( F (k, l) = k \) and \( A_t = a_t \) that is an i.i.d. shock
- Model can be analyzed with single state variable
  \[ s_t = a_t - \bar{b}_t \]
- Recursive equilibrium is given by
  \[ \lambda (s), x (s), b (s) \]
  where
  \[ x_t = \frac{k_{t+1}}{k_t} \]
  the three functions must satisfy three sets of conditions for all \( s > s_0 \), where \( s_0 \) is a lower bound to be determined
- Recursive condition for \( \lambda \)
  \[ \lambda (s) = \beta E \left[ \lambda (a' - b (s)) \right] \frac{[a' - b (s) - G_2 (x (a' - b (s)), 1)]}{G_1 (x (s), 1) - qb (s)} \]
- Condition for \( x (s) \) that
  \[ s + qb (s) x (s) \geq G (x (s), 1) \]
  with strict equality if \( \lambda (s) > 1 \)
- Condition for the borrowing ratio \( b (s) \)
  \[ q\lambda (s) \geq \beta E [\lambda (a' - b (s))] \]
  and
  \[ b (s) \leq -\eta \min_{a'} G_2 (x (a' - b (s)), 1) \]
  with complementary slackness
- Equilibrium can be computed recursively
- As initial condition think of finite horizon problem, set \( \lambda = 1 \) in the final period and \( G_2 \) to some fixed value
- Code stoch_KM.m computes equilibrium using following algorithm
• Iteration, endogenous gridpoint method, find $\hat{b}$ that satisfies
  
  $$b = -\theta \min_{a'} G_2 (x (a' - b), 1),$$

• Choose candidate pairs $(b, \lambda)$ as follows
  
  • Set $b = \hat{b}$ and let
    $$\hat{\lambda} = \max \{ \frac{\beta}{q} E [\lambda (a' - b)], 1 \},$$
  
  then choose any $\lambda$ in $[\hat{\lambda}, \infty)$

  • Set $b < \hat{b}$ and if
    $$\frac{\beta}{q} E [\lambda (a' - b)] < 1$$
  
  discard, otherwise set
    $$\lambda = \frac{\beta}{q} E [\lambda (a' - b)]$$

  • For each pair $(b, \lambda)$ find $x$ that solves
    $$\lambda [G_1 (x, 1) - qb] = \beta E [\lambda (a' - b) [a' - b - G_2 (x (a' - b), 1)]] ,$$

  or

  $$\lambda [G_1 (0, 1) - qb] \geq \beta E [\lambda (a' - b) [a' - b - G_2 (x (a' - b), 1)]] ,$$

  if $\lambda = 1$ this is the optimal solution for all $s$ that satisfy

  $$s \geq G (x, 1) - qbx,$$

  if $\lambda > 1$ this is the optimal solution for

  $$s = G (x, 1) - qbx$$

• The lower bound for $s$ is

  $$s = \min_{x \geq 0} G (x, 1) - q\hat{b}x$$

  (which arises when $\lambda \to \infty$)

• Functional form used for $G$ is

  $$G (k', k) = k' - k + \frac{\xi (k' - k)^2}{2}$$

  or

  $$G (x, 1) = x - 1 + \frac{\xi}{2} (x - 1)^2$$

  so derivatives are

  $$G_1 = 1 + \xi (x - 1)$$

  and

  $$G_2 = -1 - \xi (x - 1) - \frac{\xi}{2} (x - 1)^2$$
• Then this equation
\[ \lambda [G_1(x, 1) - qb] = \beta E [\lambda (a' - b) \left[ a' - b - G_2(x(a' - b), 1) \right]], \]
becomes
\[ x = 1 + \frac{1}{\xi} \left\{ \frac{\beta E [\lambda (a' - b) \left[ a' - b - G_2(x(a' - b), 1) \right]]}{\lambda} + qb - 1 \right\} \]

• Frictionless benchmark
\[ G_1(x, 1) = qE[a - G_2(x, 1)] \]
investment constant with \( x \) solving
\[ 1 + \xi(x - 1) = q \left[ Ea + 1 + \xi(x - 1) + \frac{\xi}{2}(x - 1)^2 \right] \]
• Assume that
\[ r < Ea < r + \frac{\xi}{2}r^2 \]
where \( r = 1/q - 1 \) to ensure that a solution to the frictionless problem exists and is bounded

**Choose:** solution with \( x < 1 + r \) to satisfy transversality condition