1. Consumption in the recession

**Figure 1.** Consumption

**Figure 2.** Durable goods

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2. PERMANENT INCOME HYPOTHESIS

- Basic idea: consumption smoothing
- Consumers’ objective
  \[ E \sum_{t=0}^{\infty} \beta^t u(c_t) \]
- Budget constraint
  \[ a_{t+1} = (1 + r) a_t + y_t - c_t \]
- Simple case
  - no uncertainty
  - \( \beta (1 + r) = 1 \)
- Optimality condition
  \[ u'(c_t) = \beta (1 + r) u'(c_{t+1}) = u'(c_{t+1}) \]
  so \( c_t \) constant
- Intertemporal budget constraint
  \[ \sum_{j=0}^{\infty} (1 + r)^{-j} (c_{t+j} - y_{t+j}) = (1 + r) a_t \]
- So we obtain
  \[ c_t = \frac{r}{1 + r} \sum_{j=0}^{\infty} (1 + r)^{-j} y_{t+j} + r a_t \]
- Main insights:
– consumption depends on expected future income (here with perfect foresight), not just on current income
– so income process matters
– marginal propensity to consume out wealth is small $(r)$

• Can we add uncertainty and get something like this?

$$c_t = \frac{r}{1 + r} E_t \sum_{j=0}^{\infty} (1 + r)^{-j} y_{t+j} + r a_t$$

• Yes, if we assume quadratic utility, so

$$u'(c_t) = E_t u'(c_{t+1})$$

becomes

$$c_t = E_t c_{t+1}$$

• Random walk property of consumption (rejected in data)

3. Income fluctuation problem

• Suppose i.i.d. income process $y_t$
• Utility function $u(.)$ strictly concave, with $\lim_{c \to 0} u'(c) = \infty$
• Borrowing constraint

$$a_t \geq -\phi$$

• Natural borrowing limit

$$\phi = \frac{y_{\min}}{r}$$

• Define cash-on-hand

$$z_t = a_t + y_t$$

• Bellman equation

$$V(z) = \max_{a'} u(z - a') + \beta E[V((1 + r) a' + y')]$$

• Euler equation

$$u'(c_t) \geq \beta (1 + r) E_t [u'(c_{t+1})]$$

• What happens if $\beta (1 + r) = 1$?
• We have

$$u'(c_t) \geq E_t [u'(c_{t+1})]$$

so $u'(c_t)$ is a supermartingale and has a limit distribution, but then $c_t$ has a limit distribution and if $c_t < \infty$ we obtain a violation of budget constraints
• Result (Bewley): when $\beta (1 + r)$ the optimal solution has $a_t \to \infty$ and $c_t \to \infty$
• Intuition: wealth provides self-insurance, as long as we are away from lower limit, with \( \beta (1 + r) = 1 \) no trade-off between self-insurance and impatience so agents accumulate unbounded wealth
• In general equilibrium supply of assets is “bounded”, so to have bounded asset demand the interesting case is \( \beta (1 + r) < 1 \)
• We’ll see this later in computations
• From now on we assume
  \[ \beta (1 + r) < 1 \]
• Properties of the value function
• \( V (z) \) is increasing, concave, differentiable (review)
• Properties of consumption and asset accumulation policies
• \( c (z) \) is increasing \( a' (z) \) is non-decreasing
• Proof: define \( \Psi (a') \equiv \beta E [V ((1 + r) a' + y')] \), then Bellman is just a 2 goods problem with separable utility
• Borrowing constraint is binding iff \( z \leq z^* \)
• Proof: If
  \[ u' (z + \phi) > \Psi' (-\phi) \]
  the inequality still holds for any \( z' \leq z \)
• An important property: with \( \beta (1 + r) < 1 \) the asset distribution is bounded above
• This property holds if the utility function satisfies
  \[ \lim_{c \to \infty} - \frac{u'' (c)}{u' (c)} = 0 \]
  that is if risk aversion not important at high levels of wealth, so trade-off now dominated by impatience and consumers stop accumulating wealth
• Proof (sketch)
• Define consumption tomorrow if highest realization of income is realized
  \[ \bar{c} (z) = c ((1 + r) a' (z) + y_{\text{max}}) \]
• Let \( z > z^* \) so Euler holds as equality
• Write Euler as
  \[ u' (c (z)) = \beta R \frac{E [u' (c (z'))]}{u' (\bar{c} (z))} u' (\bar{c} (z)) \]
• Suppose that
  \[ \lim_{z \to \infty} \frac{E [u' (c (z'))]}{u' (\bar{c} (z))} = 1 \]
  (we’ll prove it later)
• Then there is a $\bar{z}$ large enough such that if $z > \bar{z}$

$$u'(c(z)) < u'(\bar{c}(z))$$

and from envelope condition

$$V'(z) < V'((1 + r) a'(z) + y_{\text{max}})$$

• Concavity of $V$ then implies

$$(1 + r) a'(z) + y_{\text{max}} < z$$

so the map

$$(1 + r) a'(z) + y_{\text{max}}$$

crosses the 45 degree line at some $z$, that’s the upper bound for the distribution of $z$ in the long run

• It remains to prove (2), here we need

$$\frac{u'(c - A)}{u'(c)} \to 1$$

for $c \to \infty$

$$1 \leq \frac{u'(c - A)}{u'(c)} = 1 + \int_{c}^{c-A} \frac{u''(\tilde{c})}{u'(c)} d\tilde{c} \leq 1 + \int_{c-A}^{\tilde{c}} \frac{u''(\tilde{c})}{u'(c)} d\tilde{c}$$

and under condition (1)

$$\int_{c-A}^{\tilde{c}} \frac{u''(\tilde{c})}{u'(c)} d\tilde{c} \to 0$$