Notes on linear approximation for portfolio problems

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• These notes discuss how to approximate the solution to a portfolio problem

• They solve problem 2 in problem set 1, but they also introduce the general approach in Judd and Gu

• Problem, find \( \theta \) that solves

\[
E\left[u'(\theta A + (1 - \theta) A^*) (A - A^*)\right] = 0
\]

for the two random variables \( A, A^* \) defined in the problem set

• Solve

\[
E\left[\frac{a^\rho \epsilon - \epsilon^*}{\theta a^\rho \epsilon + (1 - \theta) \epsilon^*}\right] = 0
\]

or

\[
E\left[\frac{(a^\rho - 1) + a^\rho (\epsilon - 1) - (\epsilon^* - 1)}{\theta a^\rho + 1 - \theta + \theta a^\rho (\epsilon - 1) + (1 - \theta) (\epsilon^* - 1)}\right] = 0
\]

• Let \( \sigma^2 \) be the variance of the shocks \( \epsilon - 1 \)

• Problem, when \( \sigma^2 = 0 \) problem is not well defined

• We solve a sequence of problems that converge as \( \sigma \to 0 \)

• Let \( a^\rho = 1 + \pi \sigma^2 \) and \( a^\rho (\epsilon - 1) = (1 + \pi \sigma^2) \sigma \eta \) and \( \epsilon^* = \sigma \eta^* \)

• Consider the problem as \( \sigma \to 0 \)

\[
E\left[\frac{\pi \sigma^2 + (1 + \pi \sigma^2) \sigma \eta - \sigma \eta^*}{1 + \theta \pi \sigma^2 + \theta (1 + \pi \sigma^2) \sigma \eta + (1 - \theta) \sigma \eta^*}\right] = 0
\]

• Define

\[
H(\theta, \sigma) = E\left[\frac{\pi \sigma + (1 + \pi \sigma^2) \eta - \eta^*}{1 + \theta \pi \sigma^2 + \theta (1 + \pi \sigma^2) \sigma \eta + (1 - \theta) \sigma \eta^*}\right]
\]
• Problem

\[ H(\theta, 0) = 0 \]

for all \( \theta \! \)!

• Judd-Guu approach, us Bifurcation Theorem, which amounts to applying implicit function theorem to

\[
f(\theta, \sigma) = \begin{cases} 
H(\theta, \sigma) \sigma & \text{if } \sigma \neq 0 \\
H_\sigma(\theta, 0) & \text{if } \sigma = 0
\end{cases}
\]

• A simple example of the problem

• Apply implicit function theorem to

\[ H(\theta, \sigma) = \sigma(\theta - \sigma) \]

• If you divide by \( \sigma \) you find the arm that converges

• Now

\[
\lim_{\sigma \to 0} \frac{H(\theta, \sigma)}{\sigma} = H_\sigma(\theta, 0)
\]

\[ H_\theta(\sigma(\sigma), \sigma) \theta'(\sigma) + H_\sigma(\theta(\sigma), \sigma) = 0 \]

• At \( \sigma = 0 \) we have

\[ H_\theta(\theta, 0) = 0 \]

for all \( \theta \)

• So if we want to have a well defined \( \theta'(\sigma) \) at \( \sigma = 0 \) then we need

\[ H_\sigma(\theta_0, 0) = 0 \]

• This condition gives us the non-stochastic steady state!

• Better, it gives us the non-stochastic steady state that is a limit of the stochastic steady state

• Compute

\[
H_\sigma(\theta, \sigma) = E \left[ \frac{\pi \sigma + (1 + \pi \sigma^2) \eta - \eta^*}{1 + \theta \pi \sigma^2 + \theta (1 + \pi \sigma^2) \sigma \eta + (1 - \theta) \sigma \eta^*} \right]
\]

\[ \pi - (\eta - \eta^*) (\eta \eta + (1 - \theta) \eta^*) = 0 \]

\[ \theta(0) = \frac{1}{2} + \frac{\pi}{2} = \frac{1}{2} + \frac{a^2 - 1}{2\sigma^2} \]
• This is a “zero order” approximation we can then get a better approximation by computing 
  \[ \theta'(\sigma) \]

• How can we do it?

• Differentiate

  \[ H_\theta (\theta (\sigma), \sigma) \theta' (\sigma) + H_\sigma (\theta (\sigma), \sigma) = 0 \]

• Gives

  \[ H_\theta (\theta (\sigma), \sigma) \theta'' (\sigma) + H_{\theta \theta} (\theta (\sigma), \sigma) \theta' (\sigma) \theta' (\sigma) + 2H_{\sigma \theta} (\theta (\sigma), \sigma) \theta'(\sigma) + H_{\sigma \sigma} (\theta (\sigma), \sigma) = 0 \]

• Compute it \( \sigma = 0 \) the first two terms are 0 so we have

  \[ 2H_{\sigma \theta} (\theta (0), 0) \theta'(0) + H_{\sigma \sigma} (\theta (0), 0) = 0 \]