Monotone Equilibria in Signaling Games

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Abstract

We examine the monotonicity of sender's equilibrium strategy with respect to her type in signaling games. We show that when the sender's return from the receiver's action depends on her type, the Spence-Mirrlees condition cannot rule out equilibria in which a higher-type sender chooses a strictly lower action than a lower-type. We provide sufficient conditions under which all equilibria are monotone, which require the sender's payoff decreases in her action, increases in the receiver's action, and has strictly increasing differences between her type and the action profile. We apply our sufficient conditions to education signaling, advertising, and warranty provision.

Keywords: signaling game, monotone equilibrium, Spence-Mirrlees condition, monotone-supermodularity, increasing absolute differences over distributions.

JEL Classification: C72, D82

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1 Introduction

Starting from the seminal contribution of Spence (1973), signaling games have become powerful tools to study strategic interactions under asymmetric information. In a standard signaling game, an informed sender, who has private information about the payoff environment (or her *type*), takes an action that influences the behavior of an uninformed receiver. Typically, the sender's payoff function satisfies a *Spence-Mirrlees condition*, which says that her actions and her types can be ranked, such that a higher type enjoys a comparative advantage in taking higher actions compared to a lower type. For example, more talented workers face lower cost to receive education (Spence 1973), and more efficient firms benefit more from cutting prices (Milgrom and Roberts 1982). A natural prediction under this condition is that the sender's equilibrium strategy is a *nondecreasing* function of her type.

This paper studies signaling games with general payoff functions. Our results address the robustness and limitations of the above monotonicity prediction by examining whether it applies to *all equilibria*.[1] In our model, the set of types and players' actions are complete lattices, and the sender's payoff function has strictly increasing differences between her type and her action. This generalizes the Spence-Mirrlees condition by allowing types and actions to be multi-dimensional. In addition, we allow the sender's return from the receiver's action to depend on her type. We say that an equilibrium is monotonically nondecreasing, or *monotone* for short, if every action played by a higher-type sender is not strictly lower than any action played by a lower-type.

Our results are interesting for two reasons. First, many seminal works on signaling games, such as Milgrom and Roberts (1986) and Mailath (1987), establish the *existence* of monotone equilibria and focus their analysis on those equilibria. This is because for Bayesian games, monotone equilibria are straightforward to interpret, tractable to analyze, and easy to compute.[2] However, it remains unclear whether this monotonicity property applies to *all* equilibria, especially when the sender's returns from the receiver's action depends on her type. Second, from a more practical perspective, as pointed out by Ho and Rosen (2017) and other works on partial identification, identifying properties that apply to all equilibria is an important step towards empirical estimation of game theoretic models.

[1]To address the concern that there is a plethora of equilibria in signaling games, we adopt the following "double standard": In our counterexamples, we focus on equilibria that can survive refinements in Kreps and Wilson (1982), Kohlberg and Mertens (1986), Cho and Kreps (1987), Banks and Sobel (1987), and so on. For positive results, we use weak solution concepts such as Bayes Nash equilibrium, which makes our results stronger and implies that our monotonicity prediction is robust against equilibrium selection.

[2]For instance, McLennan (2018) argues that for many Bayesian games "the existence of equilibria with strategies that are monotonic functions of the agents' types is important because nonmonotonic equilibria are intractable."

We start by formally showing that the Spence-Mirrlees condition *cannot* guarantee the monotonicity of all equilibria. To illustrate, consider an example between a student and an employer. The student, who privately knows whether she enjoys coding or managing, needs to choose whether to do an MBA or an MS in statistics. Observing the student's education choice, the employer then decides whether to hire her as a data analyst or as an account manager. The players' payoffs are as follows:

Coder Type	Analyst	Manager
MS	$4,4$	$2,2$
MBA	$3,3$	$1,1$

Manager Type	Analyst	Manager
MS	$1,1$	$3,3$
MBA	$2,2$	$4,4$

Types and actions are ordered according to *Coder Type* $\succ$ *Manager Type*, *MS* $\succ$ *MBA*, and *Analyst* $\succ$ *Manager*. The Spence-Mirrlees condition is satisfied because, regardless of the job assigned, the Coder Type has a higher marginal benefit from pursuing the MS degree compared to the Manager Type. In fact, the above game even satisfies a more demanding condition: both players' payoff functions are *strictly supermodular* with respect to the sender's type and the action profile.

However, there exists a *non-monotone equilibrium* in which the Coder Type chooses to do an MBA and the Manager Type chooses to do an MS in statistics; the employer assigns the data-analyzing job to an MBA graduate, and assigns the managerial job to an MS graduate.[3] This equilibrium survives standard refinements since both players have strict incentives to choose their equilibrium strategies and the sender plays each of her actions with strictly positive probability. Intuitively, non-monotone equilibria can arise because the sender's return from the receiver's action depends on her type, which may countervail the effect of supermodularity. This observation is important because in many economic applications, the sender's return from the receiver's action would naturally be type-dependent. For example, in addition to the application on education and job assignment, it is also relevant when sellers signal their product quality through prices and introductory advertising (as the markup of a product can depend directly on its quality; see Milgrom and Roberts 1986).

Next, we provide sufficient conditions under which all equilibria are monotone. At the heart of our analysis is a *monotone-supermodularity* condition, which requires that the sender's payoff (1) is strictly decreasing in her action and is strict increasing in the receiver's action, and (2) has

[3]Spence (2002) also provides an example of education signaling game where a low-type worker chooses a strictly higher level of education than a high-type one. However, in his example the worker's payoff function is submodular with respect to her own type and action (because, e.g., the opportunity costs associated with spending time on education are larger for very talented people).

strictly increasing differences between her type and the action profile.[4] In words, our monotone-supermodularity implies that all types of the sender pay higher signaling costs by sending higher signals and enjoy higher "signaling benefits" when the receiver takes higher actions. Moreover, the marginal increase of such costs (benefits) are higher (lower) for a high type than a low type. This fits into the application on education and job assignment when workers of all types face strictly positive cost to receive more education, and strictly benefit from better jobs, though for the more talented workers *both* the cost is lower (Spence 1973) *and* the benefit is higher (Waldman 1984; Gibbons and Waldman 1999).[5]

Our Theorem 1 shows that all equilibria are monotone when the sender's payoff function is monotone-supermodular and the receiver's action choice is binary. Intuitively, this is because distributions over the receiver's actions can be ranked according to first-order stochastic dominance (FOSD). Since playing a higher action a is strictly more costly than a lower action a' for the sender, she would have an incentive to do so only when the distribution over the receiver's action induced by a first-order stochastically dominated the one induced by a'. This implies that in every equilibrium, the ranking over the sender's equilibrium actions coincides with the ranking over the receiver's mixed actions that they induce. Since a higher type sender has a stronger preference towards higher *action profiles*, she would never play a strictly lower action than a low type in any equilibrium.

When the receiver has three or more actions, we show by example that non-monotone equilibria exist even when the sender's payoff function is monotone-supermodular. This is because not every pair of the receiver's mixed actions can be ranked according to FOSD. Therefore, a higher action of the sender does not necessarily induce a more favorable distribution over the receiver's actions.

We introduce two sets of sufficient conditions to address this issue. First, we show in Theorem 2 that every equilibrium is monotone if the sender's payoff is monotone-supermodular, the ranking over the receiver's actions is complete, and the receiver's payoff satisfies a *quasiconcavity-preserving property* (QPP). QPP requires the receiver's payoff to be strictly quasi-concave in his own action under *every* belief about the sender's type, which implies that the receiver has at most two pure best replies against each action of the sender and under each of his posterior belief about the sender's

[4] Although we cannot establish the necessity of our monotone-supermodularity condition, we use counterexamples to show that none of its components is superfluous.

[5] Different types of workers receive different returns from a "higher level" job can be due to, for instance, that the talent of a worker affects her prospects of promotion, or the expected compensation she receives under piece-rate incentive schemes, etc.

type.[6] Furthermore, these best replies are adjacent elements in the receiver's action set, which implies that every pair of the receiver's *mixed best replies* can be ranked according to FOSD. One can then apply the argument in Theorem 1 to establish the monotonicity of all equilibria.

A sufficient condition for QPP is that the receiver's payoff being strictly concave in his own action. For example, with the sender being a worker choosing education and the receiver being an employer assigning jobs, QPP will be satisfied if the employer faces a quadratic loss when there is a mismatch between the levels of the assigned job and the "ideal" job assignment (i.e., the job that maximizes the employer's payoff when he can observe the worker's talent).

Second, we identify a novel condition on the sender's payoff function under which every pair of the receiver's mixed actions can be ranked endogenously. We call this property *increasing absolute differences over distributions* (IADD). Theorem 3 shows that all equilibria are monotone when the sender's payoff function is monotone-supermodular and satisfies IADD.[7] We also establish a representation result that characterizes IADD (Proposition 1), which facilitates the application of our Theorem 3 to economic modeling. For example, the characterization shows that in the education signaling context, a worker's payoff function satisfies IADD if it can be written as:

$$u_1(\theta,a_1,a_2) = \underbrace{f(\theta,a_1)g(a_2)}_{\text{worker's return from job assignment}} - \underbrace{c(\theta,a_1)}_{\text{cost of education}}, \tag{1.1}$$

where $\theta \in \mathbb{R}$ is her talent, $a_1 \in \mathbb{R}$ is the amount of education she receives, $a_2 \in \mathbb{R}$ is the level of job she is assigned to. This payoff function captures new economic forces that are relevant in practice but are absent in Spence (1973). For example, suppose jobs that have higher a_2 are those that offer more promotion opportunities (such as tenure-track positions), then a worker's benefit from those jobs relative to other jobs depends on his talent θ and his education background a_1. This is because both θ and a_1 affect his chances of being promoted given the opportunities. The above interaction, which is shown to be empirically relevant by Gibbons and Waldman (1999), is omitted in Spence (1973) but can be captured by the term $f(\theta,a_1)g(a_2)$ in our model.

[6] In Appendix B.3, we relate QPP to a strict version of the signed-ratio monotonicity condition in Quah and Strulovici (2012) and provide a full characterization of this property.

[7] Since Theorems 1 and 3 require no assumption on the receiver's payoff function, their conclusions extend to richer environments such as the sender signals to a population of heterogeneous receivers, the receiver has private information about his payoff, and so on.

Related Literature: Starting from Spence (1973), monotone equilibria have been a primary focus in many applications of signaling games. Our counterexamples point out that this is *not* without loss of generality when the sender's return from the receiver's action depends on her type. This finding contrasts to alternative channels through which non-monotone equilibria can arise, such as the receiver observing exogenous signals that are informative about the sender's type (Feltovich, Harbaugh and To 2002, Araujo, Gottlieb and Moreira 2007), or the sender's actions being imperfectly observed (Gal-Or 1989, Balachander 2001).

Our paper also contributes to the study of supermodular incomplete information games. Most papers in this literature focus on simultaneous-move games and provide sufficient conditions for the *existence* of monotone equilibrium (e.g. Athey 2001, McAdams 2003, Van Zandt and Vives 2007, Reny 2011).[8] A separate strand of literature establishes the monotonicity of all equilibria in simultaneous-move games, such as Morris and Shin (1998) on coordination games, and McAdams (2006) on multi-unit auctions. In contrast, we provide sufficient conditions for the monotonicity of *all* equilibria in *sequential-move* supermodular games, with one-shot signaling games a natural starting point.[9] In particular, our sufficient conditions, together with the counterexamples, highlight the key economic force guaranteeing equilibrium monotonicity - the monotone-superdularity of the sender's payoff, which has been implicit in signaling game models that assume separable payoffs.

Lastly, our sufficient conditions for equilibrium monotonicity are related to two existing results. The first one is in Cho and Sobel (1990). They restrict attention to equilibria in which the receiver uses a pure strategy, and obtain a sufficient condition under which the sender's equilibrium strategy is monotone (Lemma 4.1, p. 393). Different from theirs, our results examine the common properties of *all* equilibria. The second one is in a recent work by Kartik, Lee and Rappoport (2019). Similar to our Theorem 3, they also identify a class of utility functions under which the sender's strategy is monotone in all equilibria. The difference between our condition and theirs is explained in the education signalling context in section 4.

[8] Mensch (2018) provides sufficient conditions on the existence of monotone equilibrium in dynamic games of incomplete information.

[9] Complementarities in dynamic games are studied by Echenique (2004a, 2004b), who show that intertemporal incentives weaken the implications of supermodularity. Although we establish the monotonicity of all equilibria, a signaling game with supermodular payoff functions is not necessarily supermodular in its normal form. Therefore, the other attractive properties of simultaneous move supermodular games, such as the existence of extremal equilibria and monotone comparative statics, are not applicable to our setting.

2 The Model

Consider the following two-player signaling game. Player 1 (or sender, she) privately observes her type $\theta \in \Theta$, and then chooses $a_1 \in A_1$. Player 2's (or receiver, he) prior belief is $\pi \in \Delta(\Theta)$. He chooses $a_2 \in A_2$ after observing a_1. For $i \in \{1,2\}$, player i's payoff is $u_i(\theta, a_1, a_2)$. Each player maximizes his/her expected payoff. For every $\alpha_2 \in \Delta(A_2)$, let $u_1(\theta, a_1, \alpha_2) \equiv \int_{a_2} u_1(\theta, a_1, a_2) d\alpha_2$ be type θ sender's expected payoff from (a_1, α_2).

Throughout the paper, we assume that Θ, A_1 and A_2 are finite lattices and π has full support.[10] We use $\succ$ and $\succsim$ to denote strict and weak orders on lattice sets. Following Topkis (1998), for two lattices X and Y, a mapping $f : X \times Y \to \mathbb{R}$ exhibits *increasing differences* if for every $x, x' \in X$ and $y, y' \in Y$ with $x \succ x'$ and $y \succ y'$:

$$f(x,y) - f(x',y) \geq f(x,y') - f(x',y'). \tag{2.1}$$

The mapping exhibits *strictly increasing differences* if the above inequality is strict. We introduce a condition on the sender's payoff, which generalizes the Spence-Mirrlees condition to discrete lattice sets:

Definition 1 (Generalized Spence-Mirrlees Condition)**.** *u_1 satisfies the generalized Spence-Mirrlees condition if it exhibits strictly increasing differences in (θ, a_1).*[11]

Intuitively, the above condition requires a higher type sender to have a comparative advantage in playing higher actions relative to a lower type. This fits into most applications of signaling games, which include education games where receiving education is less costly for a more talented worker (Spence 1973), the beer-quiche game where drinking beer is more pleasant for the strong-type sender (Cho and Kreps 1987), warranty provision games where providing lengthier warranty is less costly for a high-quality firm (Gal-Or 1989), and so on. Our condition is also satisfied in many multi-dimensional signaling models (e.g., Quinzii and Rochet 1985, Araujo, Gottlieb, and Moreira 2007).

Strategies & Equilibrium: The sender's strategy is $\sigma_1 : \Theta \to \Delta(A_1)$ and the receiver's strategy is $\sigma_2 : A_1 \to \Delta(A_2)$. Let $\sigma_1^\theta \in \Delta(A_1)$ be the (possibly mixed) action played by type θ. We also write $\sigma_1 = (\sigma_1^\theta)_{\theta \in \Theta}$.

[10] A set X is a lattice if there exists a partial order $\succeq$ such that for every $x, x' \in X$, $x \vee x', x \wedge x' \in X$, where $x \vee x'$ is the smallest element above both x and x', $x \wedge x'$ is the largest element below both x and x'.

[11] For alternative versions of the Spence-Mirrlees condition, see Engers (1987) and Cho and Sobel (1990).

The solution concept is Bayes Nash equilibrium (or *equilibrium*), which is a strategy profile $\sigma \equiv (\sigma_1, \sigma_2)$ such that for every $i \in \{1,2\}$, σ_i best replies against σ_{-i}. Since the game is finite, an equilibrium exists. We introduce our notions of *monotone strategy* and *monotone equilibrium*:

Definition 2 (Monotone Strategy & Monotone Equilibrium). σ_1 *is a monotone strategy if for every* $\theta \succ \theta'$, *there exist no* $a_1 \in supp(\sigma_1^{\theta})$ *and* $a_1' \in supp(\sigma_1^{\theta'})$ *such that* $a_1 \prec a_1'$. *An equilibrium* (σ_1, σ_2) *is monotone if* σ_1 *is a monotone strategy.*

According to Definition 2, a strategy is monotone if a lower type sender never plays a strictly higher action than a higher type. When the order on A_1 is complete (or equivalently A_1 is one-dimensional), the monotonicity of σ_1 is equivalent to:

$$\min_{a_1} \left\{ \text{supp}(\sigma_1^{\theta}) \right\} \succsim \max_{a_1} \left\{ \text{supp}(\sigma_1^{\theta'}) \right\} \text{ for every } \theta \succ \theta'. \tag{2.2}$$

That is to say, if type θ' plays a_1 with positive probability, then every type higher than θ' plays actions that are higher or equal to a_1 with probability 1.[12]

We examine whether *all* equilibria are monotone in signaling games that satisfy the generalized Spence-Mirrlees condition. Our choice of solution concept, namely Bayes Nash equilibrium, implies that our positive results (Theorems 1, 2, and 3) are robust against equilibrium selection. To address the concern that there is a plethora of equilibria in signaling games and some of them are driven by unreasonable off-path beliefs, we adopt more stringent solution concepts in our counterexamples, such as sequential equilibrium (Kreps and Wilson 1982). We also require those equilibria to survive refinements in Kohlberg and Mertens (1986), Cho and Kreps (1987), and Banks and Sobel (1987).

3 Existence of Non-Monotone Equilibria: Examples

In this section, we present two counterexamples which show that *neither* the generalized Spence-Mirrlees condition *nor* the more demanding requirement that both players' payoff functions being strictly supermodular is sufficient to guarantee the monotonicity of the sender's equilibrium strategy.

[12] When the order on A_1 is incomplete, i.e., A_1 is multi-dimensional, our notion of monotonicity is no longer equivalent to, but is only implied by condition (2.2). One may wonder whether our robust prediction results can be strengthened by replacing Definition 2 with (2.2). In Appendix B.1, we provide an example that illustrates the difficulty of such a generalization when A_1 is multi-dimensional.

Example 1. Let us revisit the example in the introduction, in which a student privately knows his preference between coding and managing, chooses between an MBA degree and an MS degree, and an employer decides whether to offer a data analyst job or a account manager job. The employer's prior belief remains unspecified since it is irrelevant. One can check that according to the orders *Coder Type* $\succ$ *Manager Type*, *MS* $\succ$ *MBA*, and *Analyst* $\succ$ *Manager*, this game satisfies the generalized Spence-Mirrlees condition. In fact, players' payoffs in this game satisfy a more demanding notion of complementarity, that is, both u_1 and u_2 are strictly supermodular functions of the triple (θ, a_1, a_2).[13] As already mentioned in the Introduction, this game admits the following non-monotone equilibrium, that the coder type pursues the MBA degree, and the manager type pursues the MS degree; the employer offers the data analyst position upon observing the MBA degree and offers the account manager position upon observing the MS degree.

A notable feature of this example is that the student's ordinal preference over job offers depend on her type. In equilibrium, she chooses to receive the training she dislikes in order to get the job offer she desires. The next example shows that such non-monotone equilibria can arise even when the sender's ordinal preference over the receiver's actions is the same for all types, but different types have different cardinal preferences.

Example 2. Consider the following $2 \times 2 \times 2$ game:

$\theta = \theta_1$	h	l
H	$1,2$	$1,0$
L	$2,1$	$4,0$

$\theta = \theta_0$	h	l
H	$0,0$	$3,1$
L	$2,0$	$7,2$

The sender observes θ and choose between H and L, and the receiver chooses between h and l after observing the sender's action choice. We again leave the receiver's prior belief unspecified since it is irrelevant. The sender's payoff satisfies the generalized Spence-Mirrlees condition, and u_1 and u_2 are both strictly supermodular functions of (θ, a_1, a_2) once we rank the sender's types and players' actions according to $\theta_1 \succ \theta_0$, $H \succ L$, and $h \succ l$.

Different from Example 1, both types of sender's payoffs are strictly increasing in a_1 and is strictly increasing in a_2. In the education signaling context of Example 1, this implies that both the coder-

[13] Let X be a lattice. A function $f : X \to \mathbb{R}$ is *strictly supermodular* if $f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$ for every $x, x' \in X$, and the inequality is strict if $\{x, x'\} \neq \{x \vee x', x \wedge x'\}$.

type and the manager-type students would prefer to do an MBA rather than do an MS in statistics (perhaps because the latter is more difficult to graduate), and prefer the account manager job to the data analyst job (perhaps because the salary of the former is higher). Despite the type-independent ordinal preferences, this game also admits a non-monotone equilibrium in which type θ_1 plays L, type θ_0 plays H, and the receiver plays h after observing L and plays l after observing H. Hence, in this equilibrium a manager-type student has to do the more challenging MS in statistics in order to receive the more well-paid managerial job, while the coder-type student gets the MBA degree without studying very hard but receives the less well-paid coding job in the end.

Note that in both non-monotone equilibria discussed above, both players have strict incentives to play their equilibrium strategies, and all actions of the sender are played with strictly positive probability. As a result, those equilibria are robust against perturbations of players' payoff functions and can survive the refinements proposed by Kreps and Wilson (1982), Kohlberg and Mertens (1986), Cho and Kreps (1987), and Banks and Sobel (1987).

We argue that the existence of non-monotone equilibria are driven by the following three features shared by the examples: (1) players move sequentially, (2) the sender's return from the receiver's action depends on her type, and (3) values are interdependent. Since players move sequentially, every a_1 induces a distribution over a_2. As a result, the sender is effectively choosing a distribution over *action profiles* instead of only her own action. Since u_1 is *non-separable* with respect to θ and a_2, the sender's action choice depends not only on her comparative advantage in a_1, but also on her preference over a_2. Since the receiver's best reply against a_1 depends on his belief about the sender's type (i.e. values are *interdependent*), his strategy of choosing h after observing L and choosing l after observing H is rationalizable despite the complementarities between players' actions. This strategy of the receiver's provides the sender an incentive to take the low (high) action when her type is high (low), making the receiver's belief self-fulfilling.

While sequential-move and interdependent values are standard in signaling games, non-separability of the sender's payoff distinguishes our model from the education game in Spence (1973) and the beer-quiche game in Cho and Kreps (1987). In these classic examples, the sender's returns from wages and fighting do not depend on her type. Nevertheless, the sender's return from the receiver's action does depend on her type in many scenarios. In the education signaling context, a worker's benefit from a job depends both on her preferences and on her talents, the relevance of which has been pointed out in the labor economics literature, both in the case where jobs are horizontally differentiated (Roy 1951), and in the case where jobs are vertically differentiated (Gibbons and Waldman 1999). As we will discuss in detail in section 5, such non-separable payoffs also naturally arise in many applications in industrial organization.

4 Sufficient Conditions for Monotone Equilibria

In this section, we introduce sufficient conditions that guarantee the monotonicity of all equilibria. At the heart of our analysis is the following *monotone-supermodular* condition on the sender's payoff:

Definition 3 (Monotone-Supermodular Condition)**.** *The sender's payoff is monotone-supermodular if (1) u_1 is strictly decreasing in a_1 and is strictly increasing in a_2, and (2) u_1 exhibits strictly increasing differences in (θ, a_1) and increasing differences in (θ, a_2).*

Compared to the requirement that both u_1 and u_2 are strictly supermodular functions of (θ, a_1, a_2), our monotone-supermodularity condition *does not* require any complementarities between players' actions, nor does it impose any restrictions on the receiver's payoff function. Nevertheless, it introduces an additional requirement that the sender's payoff is *strictly decreasing* in her own action and is *strictly increasing* in the receiver's action. This requirement rules out Examples 1 and 2, and fits into applications to education and job assignment when jobs are vertically ranked such that all types of workers prefer better jobs (e.g., jobs that offer more promotion opportunities and/or provide more retirement benefits), but face more opportunity costs to receive more education. It also fits into warranty provision games (see section 5) if all types of firms find it costly to provide lengthier warranties and higher refund, but beneficial when consumers increase their purchasing quantities.

4.1 Binary Action Games

In this subsection, we focus on games in which the receiver's action choice is binary. This is the case in the education and job assignment example when the employer have two jobs to offer. Theorem 1 shows that monotone-supermodularity is sufficient to guarantee the monotonicity of all equilibria.

Theorem 1. *If $|A_2| = 2$ and u_1 is monotone-supermodular, then all equilibria are monotone.*

PROOF. Let $A_2 \equiv \{\overline{a}_2, \underline{a}_2\}$ with $\overline{a}_2 \succ \underline{a}_2$. Suppose towards a contradiction that in some equilibrium σ, there exist $\theta \succ \theta'$ and $a_1 \succ a_1'$ such that $\sigma_1^{\theta}(a_1') > 0$ and $\sigma_1^{\theta'}(a_1) > 0$. Let $\alpha_2 \equiv \sigma_2(a_1)$ and $\alpha_2' \equiv \sigma_2(a_1')$ be the (potentially mixed) actions played by the receiver after observing a_1 and a_1', respectively. Since type θ prefers (a_1', α_2') to (a_1, α_2) and type θ' prefers (a_1, α_2) to (a_1', α_2'), we have:

$$u_1(\theta, a_1', \alpha_2') \geq u_1(\theta, a_1, \alpha_2) \tag{4.1}$$

and

$$u_1(\theta', a_1, \alpha_2) \geq u_1(\theta', a_1', \alpha_2'). \tag{4.2}$$

These together imply that:

$$u_1(\theta, a_1', \alpha_2') - u_1(\theta, a_1, \alpha_2) \geq 0 \geq u_1(\theta', a_1', \alpha_2') - u_1(\theta', a_1, \alpha_2). \tag{4.3}$$

Since u_1 is strictly decreasing in a_1, it also follows from (4.2) that $u_1(\theta', a_1', \alpha_2) > u_1(\theta', a_1, \alpha_2) \geq u_1(\theta', a_1', \alpha_2')$. This further implies that α_2 attaches a higher probability to $\overline{a}_2$ compared to α_2', as the sender's payoff is strictly increasing in a_2. Therefore, we have $\theta \succ \theta'$, $a_1 \succ a_1'$, and α_2 dominates α_2' in the sense of first-order stochastic dominance (FOSD). Since u_1 has strictly increasing differences in (θ, a_1) and increasing differences in (θ, a_2), we have:

$$u_1(\theta, a_1', \alpha_2') - u_1(\theta, a_1, \alpha_2) < u_1(\theta', a_1', \alpha_2') - u_1(\theta', a_1, \alpha_2). \tag{4.4}$$

This contradicts (4.3) and establishes Theorem 1. $\square$

Intuitively, when $|A_2| = 2$, every pair of distributions over the receiver's action can be ranked according to FOSD. Since playing a higher action is more costly for the sender, she only has an incentive to do so when it induces a more favorable response from the receiver. This implies that the ranking over the sender's equilibrium actions coincides with the ranking over the receiver's mixed actions that they induce. Since a higher type sender has a stronger preference towards higher action profiles, she never plays a strictly lower action than a lower type.

Given that our proof makes no reference to the receiver's incentives, our monotonicity property also applies to every ex ante rationalizable strategy (Bernheim 1984; Pearce 1984). In fact, only monotone strategies can survive the first round of elimination (of non-rationalizable strategies). The irrelevance of the receiver's payoffs also implies that our result extends to cases in which the receiver has private information about his preferences, the sender signals to a population of receivers with heterogeneous preferences, and so on.

Our result also generalizes to signaling games with infinite A_1 and Θ with two caveats. First, Bayes Nash equilibrium needs to be defined at the interim stage after the sender observes her type. This is to ensure that all types of the sender play a best reply. Second, when A_1 is infinite, some actions in the support of a sender's strategy can be suboptimal. Nevertheless, we show in Appendix B.2 that the sender's equilibrium strategy is *almost surely monotone*: for every $\theta \succ \theta'$ and $a_1 \in \text{supp}(\sigma_1^\theta)$, the probability that type θ' plays an action strictly higher than a_1 equals zero.

Theorem 1 is also applicable to the study of *repeated* signaling games in which the sender's stage game payoff is monotone-supermodular, the receiver's action choice is binary, and the sender's type is perfectly persistent. In particular, for every pair of types $\theta \succ \theta'$ and every equilibrium (σ_1, σ_2) of the repeated signaling game, if playing the highest action in every period is type θ' sender's best reply against σ_2, then according to every best reply of type θ sender, she plays the highest action with probability 1 at every on-path history. As shown in Pei (2019), this is an important step towards establishing the commitment payoff theorem and the uniqueness of the sender's on-path equilibrium behavior in monotone-supermodular reputation games.

Finally, note that Theorem 1 only shows that the monotonicity property applies to all equilibria, but it does not imply the uniqueness of equilibrium or the uniqueness of equilibrium outcome. For example, consider the following $2 \times 2 \times 2$ signaling game:

$\theta = \theta_1$	h	l
H	$1, 1$	$-1, 0$
L	$2, -1$	$0, 0$

$\theta = \theta_0$	h	l
H	$\frac{1}{2}, -1$	$-\frac{3}{2}, 0$
L	$2, -2$	$0, 0$

The sender's payoff is monotone-supermodular according to the orders $\theta_1 \succ \theta_0$, $H \succ L$ and $h \succ l$. Since the action set of the sender is binary, Theorem 1 implies that all equilibria in this game are monotone. Now suppose the receiver's prior belief attaches probability more than $1/2$ to type θ_0. Then, there exists a pooling equilibrium in which both types of senders play L and the receiver chooses l no matter which action he observes. However, neither this equilibrium nor its outcome is unique. There is also a partially separating equilibrium in which type θ_1 chooses H, while type θ_0 mixes between H and L with probabilities such that the receiver's posterior belief attaches probability $1/2$ to type θ_1 after observing H. The receiver chooses l after observing L, and chooses h with probability $3/4$ after observing H, which makes type θ_0 sender indifferent between H and L.

4.2 Games with $|A_2| \geq 3$

We generalize our findings to games in which the receiver has three or more actions. We start from a counterexample showing that monotone-supermodularity of the sender's payoff function is no longer sufficient to guarantee the monotonicity of all equilibria.

Example 3. Consider the following game in which the sender chooses row and the receiver chooses column:

$\theta = \theta_1$	h	m	l
H	$2-\varepsilon, 1$	$1-2\varepsilon, 0$	$-3\varepsilon, -2$
L	$2, 0$	$1, 1$	$0, 0$

$\theta = \theta_0$	h	m	l
H	$2-3\varepsilon, 0$	$1-3\varepsilon, 0$	$4\varepsilon, 0$
L	$2-\varepsilon, 0$	$1, 2$	$8\varepsilon, 3$

Suppose $\varepsilon \in (0, 1/8)$. The sender's types and players' actions are ranked according to $\theta_1 \succ \theta_0$, $H \succ L$, and $h \succ m \succ l$.[14] One can verify that the sender's payoff is monotone-supermodular. However, the game admits the following non-monotone equilibrium. The sender plays L when $\theta = \theta_1$, and plays H when $\theta = \theta_0$. The receiver plays m after observing L, and plays h and l with equal probabilities after observing H. One can check that first, the sender's strategy is non-monotone, and second, the above strategy profile and its induced belief constitute a sequential equilibrium.[15]

Why does monotone-supermodularity alone cannot guarantee equilibrium monotonicity? Reviewing Theorem 1's proof suggests that the main issue is that the distributions over the receiver's actions *cannot* be completely ranked via FOSD when $|A_2| \geq 3$. In what follow, we proceed along

[14] Mapping Example 3 into an education signaling context similar to Example 1, we can again interpret the sender's type as whether she is more talented in coding (θ_0) or managing (θ_1), and her action as whether to do an MS in statistics (L) or an MBA (H). The positions which the receiver (an employer) is recruiting are a manager (h), a coder (m), and a customer service staff (l). Given the payoff matrix of Example 3, one can check that, other things equal, the sender would always prefer to do an MS in statistics rather than an MBA degree (perhaps due to the difference in tuition fees). In addition, both types of the sender prefer the manager position to the coder position, and the coder position to the customer service position (perhaps due to the difference in salaries).

[15] This counterexample is not driven by the receiver's non-generic payoff function. Even when the receiver has strict preferences over A_2 conditional on $(\theta, a_1) = (\theta_0, H)$, there still exists a non-monotone partial pooling equilibrium in which type θ_1 mixes between H and L, and type θ_0 always plays H.

two directions to address this challenge. First, we introduce a property on the receiver's payoff under which every pair of his *mixed best replies* can be ranked via FOSD. This together with the monotone-supermodularity condition on the sender's payoff implies the monotonicity of all equilibria (Theorem 2). Second, we identify a condition on the sender's payoff under which one can endogenously construct a complete order on $\Delta(A_2)$. When the sender's payoff satisfies this condition and monotone-supermodularity, every equilibrium is monotone regardless of the receiver's payoff (Theorem 3).

4.2.1 Quasiconcavity-Preserving Property

In this subsection, we assume that $A_2 \equiv \{a_2^1, ..., a_2^n\}$ is completely ranked with $a_2^1 \prec a_2^2 \prec ... \prec a_2^n$.[16] For every $(\theta, a_1) \in \Theta \times A_1$ and $i \in I \equiv \{1, 2, ..., n-1\}$, let

$$\gamma_\theta^{a_1}(i) \equiv u_2(\theta, a_1, a_2^i) - u_2(\theta, a_1, a_2^{i+1}), \tag{4.5}$$

be the receiver's payoff gain by locally decreasing his action. Given a belief about the sender's type $\tilde{\pi} \in \Delta(\Theta)$, let

$$\Gamma_{\tilde{\pi}}^{a_1}(i) \equiv \int \gamma_\theta^{a_1}(i) d\tilde{\pi} \tag{4.6}$$

be the receiver's expected payoff gain under $\tilde{\pi}$. Recall the *strict single-crossing property* in Milgrom and Shannon (1994):

Definition 4. *Function $\gamma : I \to \mathbb{R}$ satisfies the strict single-crossing property (SSCP) if for every $i \in I$, $\gamma(i) \geq 0$ implies that $\gamma(j) > 0$ for every $j \in I$ with $j > i$.*

If $\gamma_\theta^{a_1}(\cdot)$ satisfies SSCP for every $(\theta, a_1) \in \Theta \times A_1$, then $u_2(\theta, a_1, \cdot)$ is strictly quasi-concave in a_2. In that case, the receiver has at most two pure best replies to every (θ, a_1), which are adjacent elements in A_2. This further implies that every pair of his mixed best replies against a degenerate distribution on $\Theta \times A_1$ can be ranked according to FOSD. However, some actions of the sender may induce a *non-degenerate* posterior belief, which gives rise to an issue of aggregating the single-crossing property. This motivates us to introduce a *quasiconcavity-preserving property* (QPP) on the receiver's payoff.

[16]Our monotonicity result in this subsection (Theorem 2) can be extended to settings where A_2 is a multi-dimensional convex set.

Definition 5 (Quasiconcavity-Preserving Property). *u_2 satisfies QPP if $\Gamma_{\tilde{\pi}}^{a_1}(\cdot)$ satisfies SSCP for every $(\tilde{\pi}, a_1) \in \Delta(\Theta) \times A_1$.*[17]

Under QPP, the receiver's mixed best replies against any element in $\Delta(\Theta) \times A_1$ can be ranked according to FOSD. This leads to our second result:

Theorem 2. *If the order on A_2 is complete, u_1 is monotone-supermodular, and u_2 satisfies QPP, then all equilibria are monotone.*

The proof is similar to that of Theorem 1, which is omitted to avoid repetition. Theorem 2 applies to games in which u_2 is *strictly concave* in a_2, which is equivalent to $\gamma_\theta^{a_1}(\cdot)$ being strictly increasing for every (θ, a_1).[18] In the education-signaling example with vertically differentiated jobs, this is the case if the employer's suffers a convex loss when there is a mismatch between the worker and the job position that maximizes the employer's payoff under complete information, where the latter can depend both on the worker's talent and her level of education.

One thing to note is that Theorem 2 only requires the receiver to play a best reply against *some* posterior belief $\tilde{\pi} \in \Delta(\Theta)$. Therefore, the monotonicity property does not depend on the details of the receiver's posterior beliefs and applies to all outcomes under weaker solution concepts such as $S^\infty W$ in Dekel and Fudenberg (1990) and iterative conditional dominance in Shimoji and Watson (1998).[19] Moreover, when applying the elimination procedure for $S^\infty W$, all non-monotone strategies are deleted after one round of elimination of weakly dominated strategy followed by another round of elimination of strictly dominated strategy. When applying iterative conditional dominance, all surviving strategies are monotone after two rounds of elimination.

[17] A more general version of the QPP property when A_2 is any subset of $\mathbb{R}$ is introduced and characterized by Choi and Smith (2017). In the case where A_2 is finite, our condition is equivalent to a strict version of theirs.

[18] Strict concavity is sufficient but not necessary for QPP. In Appendix B.3, we provide a full characterization of QPP by relating it to a strict version of the signed-ratio monotonicity condition introduced in Quah and Strulovici (2012).

[19] These solution concepts are variants of rationalizability that rule out the receiver's suboptimal plays at off-path information sets. In particular, $S^\infty W$ is the solution concept when applying one round elimination of weakly dominated strategies followed by iterative elimination of strictly dominated strategies. Dekel and Fudenberg (1990) show that it characterizes the set of rationalizable strategies when players entertain small amount of uncertainty about their opponents' payoffs. Shimoji and Watson (1998) show that iterative conditional dominance generalizes rationalizability in normal-form games to extensive-form games.

4.2.2 Increasing Absolute Differences over Distributions

In this subsection, we take an alternative approach by introducing a condition on the sender's payoff that can guarantee the monotonicity of all equilibria irrespective of the receiver's payoff. For this result, we allow the order on A_2 to be incomplete. Instead, we introduce the following *increasing absolute differences over distributions* (IADD) condition on the sender's payoff under which one can construct a *complete order* on $\Delta(A_2)$ endogenously.

Definition 6 (Increasing Absolute Differences over Distributions)**.** *The sender's payoff satisfies IADD if for every $a_1 \in A_1$ and every $\alpha_2, \alpha_2' \in \Delta(A_2)$, we have $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ being either increasing in θ and non-negative for all $\theta \in \Theta$, or decreasing in θ and non-positive for all $\theta \in \Theta$.*

To make sense of the terminology of IADD, note that it implies that the absolute value of the difference $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_2, \alpha_2')$ is increasing in θ.[20] Intuitively, if u_1 satisfies IADD, then for every $a_1 \in A_1$, there exists a complete ordinal preference on $\Delta(A_2)$ (denoted by $\succsim_{a_1}$) that is shared among all types of senders. In addition, this ordinal ranking coincides with the one based on the intensity of the sender's preferences. In other words, if $\alpha_2 \succsim_{a_1} \alpha_2'$, then the difference in the sender's payoffs between (a_1, α_2) and (a_1, α_2') is increasing in θ. This leads to our third theorem:

Theorem 3. *If u_1 is monotone-supermodular and satisfies IADD, then all equilibria are monotone.*

PROOF. Suppose towards a contradiction that in some equilibrium (σ_1, σ_2), there exist $\theta \succ \theta'$ and $a_1 \succ a_1'$ such that $\sigma_1^{\theta}(a_1') > 0$ and $\sigma_1^{\theta'}(a_1) > 0$. Let $\alpha_2 \equiv \sigma_2(a_1)$, $\alpha_2' \equiv \sigma_2(a_1')$ with $\alpha_2, \alpha_2' \in \Delta(A_2)$. Since type θ prefers (a_1', α_2') to (a_1, α_2), and type θ' prefers (a_1, α_2) to (a_1', α_2'), we have:

$$u_1(\theta, a_1', \alpha_2') \geq u_1(\theta, a_1, \alpha_2) \tag{4.7}$$

and

$$u_1(\theta', a_1, \alpha_2) \geq u_1(\theta', a_1', \alpha_2'). \tag{4.8}$$

Since u_1 is strictly decreasing in a_1, we have $u_1(\theta', a_1', \alpha_2) > u_1(\theta', a_1, \alpha_2)$. Inequality (4.8) then

[20] IADD is also necessary for $\left|u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_2, \alpha_2')\right|$ to be increasing in θ when Θ is a continuum and u_1 is a continuous function of θ.

implies that $u_1(\theta', a_1', \alpha_2) > u_1(\theta', a_1', \alpha_2')$. Applying (4.7) and (4.8) we have:

$$u_1(\theta, a_1', \alpha_2') - u_1(\theta, a_1, \alpha_2) \geq u_1(\theta', a_1', \alpha_2') - u_1(\theta', a_1, \alpha_2). \tag{4.9}$$

Meanwhile, note that

$$u_1(\cdot, a_1', \alpha_2') - u_1(\cdot, a_1, \alpha_2) = u_1(\cdot, a_1', \alpha_2') - u_1(\cdot, a_1', \alpha_2) + u_1(\cdot, a_1', \alpha_2) - u_1(\cdot, a_1, \alpha_2).$$

Since u_1 exhibits strictly increasing differences between θ and a_1, we have:

$$u_1(\theta, a_1', \alpha_2) - u_1(\theta, a_1, \alpha_2) < u_1(\theta', a_1', \alpha_2) - u_1(\theta', a_1, \alpha_2). \tag{4.10}$$

In addition, IADD and $u_1(\theta', a_1', \alpha_2) - u_1(\theta', a_1', \alpha_2') > 0$ imply that:

$$u_1(\theta, a_1', \alpha_2') - u_1(\theta, a_1', \alpha_2) \leq u_1(\theta', a_1', \alpha_2') - u_1(\theta', a_1', \alpha_2). \tag{4.11}$$

Inequalities (4.10) and (4.11) together lead to a contradiction against (4.9), which implies that all equilibria are monotone. □

We make two remarks. First, since the order on $\Delta(A_2)$ can be constructed endogenously under IADD, our result does not rely on the pre-specified order on A_2, nor does it rely on the monotone-supermodularity condition on u_1 with respect to a_2. In fact, it is clear from the above proof that once u_1 satisfies IADD, all equilibria are monotone if u_1 is strictly decreasing in a_1 and exhibits strictly increasing differences in (θ, a_1).

Second, since the proof makes no reference to the receiver's incentives, Theorem 3 shares the same robust properties with Theorem 1. That is, all ex ante rationalizable strategies of the sender are also monotone. This monotonicity result also applies when the receiver has private information about his payoff, when the sender is signaling to a population of receivers with heterogeneous preferences, and so on. In addition, Theorem 3 immediately extends to cases where A_2 is infinite as the cardinality of A_2 plays no role in its proof. Finally, extensions of Theorem 3 to cases where Θ and A_1 are infinite are subject to the same cautions mentioned in subsection 4.1 and Appendix B.2. This is because the support of the sender's strategy may include a measure zero of suboptimal actions.

The key step to apply Theorem 3 is the verify the IADD condition, which we fully characterize in the following proposition:

Proposition 1. *u_1 satisfies IADD if and only if there exist functions $f : A_1 \times A_2 \to \mathbb{R}$, $v : \Theta \times A_1 \to \mathbb{R}_+$ and $c : \Theta \times A_1 \to \mathbb{R}$ with $v(\theta, a_1)$ increasing in θ, such that:*

$$u_1(\theta, a_1, a_2) = f(a_1, a_2)v(\theta, a_1) + c(\theta, a_1). \tag{4.12}$$

PROOF. See Appendix A. □

According to Proposition 1, Theorem 3 is applicable to education-signaling games in which the worker's payoff function takes the form of (4.12). For an economic interpretation, $f(a_1, a_2)v(\theta, a_1)$ is the worker's return from her assigned job, and $c(\theta, a_1)$ captures her cost of receiving education. That is to say, our result applies to situations in which a worker's benefit from a job depends both on her talent θ and her education background a_1. This is relevant in practice as both talent and education background can affect a worker's chances of receiving bonuses, being promoted, and so on.

Remark on the IADD Condition: The significance of IADD is that it endogenously provides a complete order on $\Delta(A_2)$. Similar conditions are provided in the monotone comparative statics literature, such as *single-crossing expectational differences* (SCED) and *monotone expectational differences* (MED) in Kartik et al. (2019), and *increasing differences over distributions* (IDD) in Kushnir and Liu (2019).[21]

When applied to the same probability space, IADD is more demanding than MED and SCED. This is because, for example, IADD on $\Delta(A_2)$ requires that first, $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_2, \alpha_2')$ does not change sign when we vary θ, and second, its absolute value is increasing in θ. These together imply that the expected difference $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_2, \alpha_2')$ is monotone in θ.[22]

[21] Formally, a utility function u defined on a partially ordered set Θ and an arbitrary set A has SCED on the probability space $\Delta(A)$ if $\forall \alpha, \alpha' \in \Delta(A)$, the expected utility difference $u(\theta, \alpha) - u(\theta, \alpha')$ is single-crossing in $\theta \in \Theta$ (i.e, its signum function is monotone in θ). The function has *MED* if $u(\theta, \alpha) - u(\theta, \alpha')$ is monotone in θ. Finally, the function has IDD if $u(\theta, \alpha) - u(\theta, \alpha')$ is strictly monotone in θ either strictly increasing, strictly decreasing, or constant in θ.

[22] In our context, IDD on $\Delta(A_2)$ would require that for every $a_1 \in A_1$ and every $\alpha_2, \alpha_2' \in \Delta(A_2)$, the expected payoff differences $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ is either strictly increasing, strictly decreasing, or constant in θ. In contrast, IADD only implies that these differences are either increasing or decreasing. Thus, in general Kushnir and Liu (2019)'s IDD is only implied by the strict version of our IADD (i.e. for every $a_1 \in A_1$ and $\alpha_2, \alpha_2' \in \Delta(A_2)$, $|u_2(\theta, a_1, \alpha_2) - u_2(\theta, a_1, \alpha_2')|$ is either constant or strictly increasing in θ).

However, neither MED, SCED nor IDD on $\Delta(A_2)$ is sufficient for our purpose. It is also important to note that IADD on $\Delta(A_2)$ neither implies nor is implied by MED or SCED on $\Delta(A_1 \times A_2)$. To illustrate, we note that in the context of education signaling, Kartik et al. (2019) show that if the worker's payoff function has SCED on $\Delta(A_1 \times A_2)$, then every equilibrium is monotone. According to their characterization result, u_1 has SCED on $\Delta(A_1 \times A_2)$ if and only if it takes the following functional form:

$$u_1(\theta, a_1, a_2) = g_1(a_1, a_2) f_1(\theta) + g_2(a_1, a_2) f_2(\theta) + h(\theta), \tag{4.13}$$

where both f_1 and f_2 are single-crossing functions that satisfy a ratio-ordered condition.[23] Hence, their monotonicity result is not applicable to education signaling games where the worker's payoff is given by (4.12) but the cost of education $c(\theta, a_1)$ cannot be written as a product of two functions $c_1(\theta)$ and $c_2(a_1)$.[24] In contrast, our Theorem 3 allows for these cases.

5 Application to Industrial Organization

We apply our results to study firms' decisions in advertising and warranty provision. While the seminal work by Spence (1977) suggests that a longer warranty is a signal of higher product quality, empirical studies have shown that the opposite may often be the case (Priest 1981, Wiener 1985, Cooper and Ross 1985). To resolve this puzzle, the literature has offered explanations that are related to, e.g., consumer moral hazard (Lutz 1989), or firm competition (Balachander 2001, Gal-Or 1989). Our counterexamples contribute to the literature by showing that once we account for that the firm's return from the consumer's purchase may depend on its product's quality (perhaps due to word-of-mouth effects), the above empirical evidence about warranty provision can also be explained by a very simple and standard signaling model.

Furthermore, our theorems delineate the theory of Spence (1977) by providing general sufficient conditions under which we can expect to discover a monotone relation between a firm's decision (such as warranty provision) and the quality of its product. To illustrate this, consider a firm (sender) selling

[23] According to Kartik et al. (2019), for two single-crossing functions $f_1, f_2 : \Theta \to \mathbb{R}$, f_1 ratio dominates f_2 if first, $\forall \theta \succeq \theta'$, $f_1(\theta') f_2(\theta) \leq f_1(\theta) f_2(\theta')$, and second, $\forall \theta \succeq \hat{\theta} \succeq \theta'$, $f_1(\theta') f_2(\theta) = f_1(\theta) f_2(\theta')$ if and only if $f_1(\theta') f_2(\hat{\theta}) = f_1(\hat{\theta}) f_2(\theta')$ and $f_1(\hat{\theta}) f_2(\theta) = f_1(\theta) f_2(\hat{\theta})$. Functions f_1 and f_2 are ratio-ordered if either f_1 ratio dominates f_2 or f_2 ratio dominates f_1.

[24] The cost function is not multiplicative separable in applications when $c(\theta, a_1) = k(\theta) a_1 + t(a_1)$, with $t(a_1)$ being a fixed cost, interpreted as the cost of tuition, and $k(\theta) a_1$ being a variable cost which depends on the worker's talent.

products to a consumer (receiver). Let $\theta \in \Theta \subset \mathbb{R}$ be the product's quality, which is the firm's private information. For simplicity, we assume that the per unit sales price is exogenous, which is normalized to 1. Every product sold has a positive probability of breakdown, which depends on its quality. The firm chooses a 3-dimensional action: $a_1 \equiv (a_1^{ad}, a_1^{len}, a_1^{re}) \in A_1 \subset \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1]$, where a_1^{ad} is the intensity of advertising, a_1^{len} is the length of warranty, and a_1^{re} is the (per unit) refund the firm commits to pay if the product breaks down during the length of the warranty. The consumer chooses how many units to purchase after observing a_1, which is denoted by $a_2 \in A_2 \subset \mathbb{N}$.

Our monotone-supermodular condition requires that first, u_1 is strictly decreasing in the triple $(a_1^{ad}, a_1^{len}, a_1^{re})$ and is strictly increasing in a_2, and second, u_1 has strictly increasing differences in (θ, a_1^{ad}), (θ, a_1^{len}) and (θ, a_1^{re}), and increasing differences in (θ, a_2). The first part of the monotonicity requirement is straightforward, since advertising, providing lengthier warranty and refund are all costly for firms. Monotonicity also requires that, keeping other factors fixed, the firm's profit is strictly higher when consumers purchase larger quantities. This rules out cases in which the low-quality seller loses money when offering the equilibrium warranty/refund policy of the high-quality seller. Nevertheless, it still fits into a number of cases of economic interest.

Next, we justify the supermodularity part of our assumption. First, there are complementarities between θ and a_1^{ad} when the cost of promoting a good product is lower than the cost of promoting a bad one. This can be driven by repeat purchase (Nelson 1974), reputation concerns (Klein and Leffler, 1981), umbrella branding (Wernerfelt 1988), and so on. Second, there are complementarities between θ and a_1^{re} when higher quality product has lower probability of breakdown, therefore committing to a higher per unit refund is less costly. Similarly, the firm's per unit profit (defined as sales price minus expected refund payment) is strictly increasing in the product's quality, and therefore, u_1 has strictly increasing differences in (θ, a_2). Finally, there are complementarities between θ and a_1^{len} when breakdown arrives according to a time homogeneous Poisson process with intensity strictly decreasing in the product's quality (Gal-Or 1989).

Given the multidimensionality of the firm's action set, in general it would be difficult to fully characterize the equilibrium outcomes of the game. Nevertheless, our theorems can still be applied to provide robust predictions about the sender's equilibrium strategies. In particular, provided that the firm's payoff is monotone-supermodular, it would use a monotone strategy in every equilibrium when the consumer has unit demand (Theorem 1), when the consumer faces decreasing marginal returns to quantities (Theorem 2), or when its payoff can be written as follows (Theorem 3):

$$u_1(\theta, a_1, a_2) = \Big(1 - \underbrace{g(\theta, a_1^{len})}_{\substack{\text{prob. of breakdown} \\ \text{within } a_1^{len} \text{ periods}}} \cdot \overbrace{a_1^{re}}^{\text{per unit refund}} + f(\theta)\Big) a_2 - \underbrace{c(\theta, a_1^{ad})}_{\text{cost of advertising}}, \tag{5.1}$$

in which

1. $g : \Theta \times \mathbb{R}_+ \to [0,1]$ is strictly decreasing in θ, strictly increasing in a_1^{len} and exhibits strictly decreasing differences in (θ, a_1^{len}),

2. $f : \Theta \to \mathbb{R}_+$ is strictly increasing, which captures the firm's benefit from initial sales beyond the sales price in a reduced form,[25] and

3. $c : \Theta \times \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing in a_1^{ad} and exhibits strictly decreasing differences in (θ, a_1^{ad}).

6 Conclusion

Our paper makes two contributions. First, we show that equilibrium monotonicity does not follow from the Spence-Mirrlees condition nor is it implied by the complementarities in players' payoff functions. Our counterexamples are robust against equilibrium refinements and highlight the problems that can arise when the sender's returns from the receiver's action depend on her type. Second, we provide sufficient conditions under which all Bayes Nash equilibria are monotone. These conditions are easy to verify and fit into a number of applications, including advertising, warranty provision, education and job assignment, and so on.

[25] This is relevant when the product is a newly introduced experience good, such as the environments studied by Nelson (1974) and Milgrom and Roberts (1986). In addition, the absence of $f(\theta)$ does not affect the applicability of our monotonicity result.

Appendix A

We establish Proposition 1 by proving a series of equivalence statements.

Lemma A1. *u_1 has IADD if and only if for every $a_1 \in A_1$ and every $\alpha_2, \alpha_2' \in \Delta(A_2)$, we have*

$$u_1(\tilde{\theta}, a_1, \alpha_2) > u_1(\tilde{\theta}, a_1, \alpha_2') \text{ for some } \tilde{\theta} \in \Theta \implies u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2') \text{ is increasing in } \theta. \tag{A.1}$$

PROOF. The only-if part of the lemma is straightforward. Let us focus on the if part. To show that (A.1) implies IADD, it suffices to show that if $u_1(\tilde{\theta}, a_1, \alpha_2) > u_1(\tilde{\theta}, a_1, \alpha_2')$ for some $\tilde{\theta}$, then for all $\theta \in \Theta$, we have $u_1(\theta, a_1, \alpha_2) \geq u_1(\theta, a_1, \alpha_2')$.

Suppose towards a contradiction that there exist $a_1 \in A_1, \alpha_2, \alpha_2' \in \Delta(A_2)$, and $\tilde{\theta}, \hat{\theta} \in \Theta$, such that $u_1(\tilde{\theta}, a_1, \alpha_2) > u_1(\tilde{\theta}, a_1, \alpha_2')$ and $u_1(\hat{\theta}, a_1, \alpha_2) < u_1(\hat{\theta}, a_1, \alpha_2')$. Then, condition (A.1) implies that we have both $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ and $u_1(\theta, a_1, \alpha_2') - u_1(\theta, a_1, \alpha_2)$ being increasing in θ. Hence, $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ is constant for all θ, which leads to a contradiction. □

Next, notice that an immediate implication of u_1 satisfying IADD is that for every $a_1 \in A_1$ and $\alpha_2, \alpha_2' \in \Delta(A_2)$, the expected payoff difference $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ is monotone in θ. The following lemma fully characterizes this necessary condition of IADD.

Lemma A2. *$u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ is monotone in θ for every $(a_1, \alpha_2, \alpha_2') \in A_1 \times \Delta(A_2) \times \Delta(A_2)$ if and only if the sender's payoff has the following representation:*

$$u_1(\theta, a_1, a_2) = f(a_1, a_2) v(\theta, a_1) + c(\theta, a_1) + g(a_1, a_2), \tag{A.2}$$

where $v : \Theta \times A_1 \to \mathbb{R}$ is an increasing function of θ.

The proof of Lemma A2 is omitted as it immediately follows from the characterization results in Kartik et al. (2019) and Kushnir and Liu (2019). Therefore, it is without loss of generality to assume u_1 taking the functional form in (A.2). We now proceed to characterize condition (A.1). To do this, let us first introduce some useful notation. Let $A_2 \equiv \{a_2^1, .., a_2^n\}$ with $n \geq 2$. For every $a_1 \in A_1$, let

$$\underline{v}^{a_1} \equiv \min_{\theta \in \Theta} v(\theta, a_1) \in \mathbb{R},$$

and

$$f^{a_1} \equiv (f(a_1, a_2^1), ..., f(a_1, a_2^n)), \; g^{a_1} \equiv (g(a_1, a_2^1), ..., g(a_1, a_2^n)) \in \mathbb{R}^n.$$

Finally, let $\Gamma \equiv \{\gamma \in \mathbb{R}^n | \mathbf{1} \cdot \gamma = 0\}$, where $\mathbf{1} \equiv (1, 1, ..., 1) \in \mathbb{R}^n$ and '$\cdot$' denotes the inner product of two vectors. We establish the following result.

Lemma A3. *Suppose that u_1 has representation (A.2). Then, u_1 satisfies* (A.1) *if and only if*

$$\forall (a_1, \gamma) \in A_1 \times \Gamma, \; (\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma > 0 \Longrightarrow f^{a_1} \cdot \gamma \geq 0. \tag{A.3}$$

PROOF. (*If statement*) First, note that given the representation (A.2), condition (A.1) is equivalent to the requirement that for every $(a_1, \gamma) \in A_1 \times \Gamma$ and every $v \geq \underline{v}^{a_1}$, $(v f^{a_1} + g^{a_1}) \cdot \gamma > 0 \Rightarrow f^{a_1} \cdot \gamma \geq 0$. Suppose towards a contradiction that this does not hold for some $(a_1, \gamma) \in A_1 \times \Gamma$ and some $v \geq \underline{v}^{a_1}$. That is, we have $(v f^{a_1} + g^{a_1}) \cdot \gamma > 0$ but $f^{a_1} \cdot \gamma < 0$. Then, (A.3) implies that $(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma \leq 0$. Hence, we have

$$0 < (v f^{a_1} + g^{a_1}) \cdot \gamma = \underbrace{(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma}_{\leq 0} + \underbrace{(v - \underline{v}^{a_1}) f^{a_1} \cdot \gamma}_{\leq 0} \leq 0,$$

which leads to a contradiction.

(*Only-if statement*) Suppose that (A.3) is violated for some $(a_1, \gamma) \in A_1 \times \Gamma$, i.e. $(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma > 0$ but $f^{a_1} \cdot \gamma < 0$. Let $\xi > 0$ be small enough such that:

$$\max\{|\xi \gamma_1|, ..., |\xi \gamma_n|\} < 1/n.$$

Consider two probability distributions $\alpha_2, \alpha_2' \in \Delta(A_2)$, where $\alpha_2 \equiv \sum_{i=1}^n \frac{1}{n} \delta_{a_2^i}$, $\alpha_2' \equiv \sum_{i=1}^n (\frac{1}{n} - \xi \gamma_i) \delta_{a_2^i}$, and $\delta_{a_2^i}$ denotes the Dirac measure on $a_2^i \in A_2$. Let $\underline{\theta}$ be the smallest element in Θ, which exists since Θ is a complete lattice. By construction, when playing a_1, type $\underline{\theta}$ sender strictly prefers α_2 to α_2'. However, since $f^{a_1} \cdot \gamma < 0$, $u_1(\cdot, a_1, \alpha_2) - u_1(\cdot, a_1, \alpha_2')$ is strictly decreasing in θ. Hence, condition (A.1) is violated. $\square$

Next, consider the linear operator $\tau : \mathbb{R}^n \to \mathbb{R}^{n-1}$ with

$$\tau(w) \equiv (w_1 - w_n, ..., w_{n-1} - w_n), \quad \forall w \in \mathbb{R}^n.$$

By construction, $\tau(w) = 0$ if and only if w is a constant vector. In addition, for every $\gamma \in \Gamma$ and $w \in \mathbb{R}^n$, we have $w \cdot \gamma = \sum_{i=1}^{n-1}(w_i - w_n)\gamma_i = \tau(w) \cdot \gamma$. Our next lemma provides a further characterization of condition (A.1) via the linear mapping τ.

Lemma A4. *Suppose that u_1 has the representation (A.2). Then, u_1 satisfies condition* (A.3) *if and only if for every $a_1 \in A_1$, there exist $\lambda, \mu \in [0, +\infty)$ with $(\lambda, \mu) \neq (0,0)$ such that*

$$\lambda \tau(f^{a_1}) = \mu \tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}). \tag{A.4}$$

Proof. For every $w \in \mathbb{R}^n$, let us partition Γ into $\Gamma^+(w), \Gamma^-(w), \Gamma^0(w)$, such that $w \cdot \gamma > 0$ (resp., $w \cdot \gamma < 0$) for every $\gamma \in \Gamma^+(w)$ (resp., $\gamma \in \Gamma^-(w)$), and $\Gamma^0(w) = \Gamma \setminus (\Gamma^+(w) \cup \Gamma^-(w))$. Now we can equivalently state condition (A.3) as

$$\Gamma^+(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \subset \Gamma^0(f^{a_1}) \cup \Gamma^+(f^{a_1}), \quad \forall a_1 \in A_1. \tag{A.5}$$

(*If statement*) Pick any $a_1 \in A_1$ and suppose there exist λ and μ such that (A.4) holds. If either λ or μ is 0, then since $(\lambda, \mu) \neq (0,0)$, we have either $\tau(f^{a_1}) = 0$ or $\tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}) = 0$. In both cases, (A.5) is satisfied. If $\lambda\mu \neq 0$, then by (A.4) we have for every $\gamma \in \Gamma^+(\underline{v}^{a_1} f^{a_1} + g^{a_1})$,

$$f^{a_1} \cdot \gamma = \tau(f^{a_1}) \cdot \gamma = \frac{\mu}{\lambda} \tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma = \frac{\mu}{\lambda} (\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma > 0.$$

Hence, $\gamma \in \Gamma^+(f^{a_1})$.

(*Only-if statement*) Pick any $a_1 \in A_1$ and consider the two $n-1$ dimensional vectors $\tau(f^{a_1})$ and $\tau(\underline{v}^{a_1} f^{a_1} + g^{a_1})$. Suppose that the required λ and μ do not exist. Then, there exists no $\kappa \geq 0$ such that

$\kappa\tau(f^{a_1}) = \tau(\underline{v}^{a_1} f^{a_1} + g^{a_1})$. By Farkas' Lemma, there exists $\tilde{\gamma} \equiv (\tilde{\gamma}_1, ..., \tilde{\gamma}_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$\tau(f^{a_1}) \cdot \tilde{\gamma} < 0 \text{ but } \tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \tilde{\gamma} > 0.$$[26]

Let $\gamma \equiv (\tilde{\gamma}_1, ..., \tilde{\gamma}_{n-1}, \tilde{\gamma}_n)$, where $\tilde{\gamma}_n \equiv -\sum_{i=1}^{n-1} \tilde{\gamma}_i$. The construction of $\tilde{\gamma}$ implies that $\gamma \in \Gamma^+(\underline{v}^{a_1} f^{a_1} + g^{a_1})$ but $\gamma \in \Gamma^-(f^{a_1})$. This violates (A.5) and thus also violates (A.3). □

To conclude the proof of Proposition 1, we derive (4.12) from (A.4). According to the definition of τ, Lemma A4 implies that for every $(a_1, a_2) \in A_1 \times A_2$,

$$\lambda\left(f(a_1,a_2) - f(a_1,a_2^n)\right) = \mu\left[(\underline{v}^{a_1} f(a_1,a_2) + g(a_1,a_2)) - (\underline{v}^{a_1} f(a_1,a_2^n) + g(a_1,a_2^n))\right],$$

or, equivalently,

$$\mu g(a_1,a_2) = (\lambda - \mu\underline{v}^{a_1}) f(a_1,a_2) + h(a_1),$$

where

$$h(a_1) = \mu(\underline{v}^{a_1} f(a_1,a_2^n) + g(a_1,a_2^n) - \lambda f(a_1,a_2^n).$$

On the one hand, if $\mu \neq 0$, let

$$\hat{v}(\theta,a_1) \equiv v(\theta,a_1) + (\lambda - \mu\underline{v}^{a_1})/\mu \text{ and } \hat{c}(\theta,a_1) \equiv c(\theta,a_1) + h(a_1),$$

which obtains representation (4.12). Note that by construction, $\min_{\theta\in\Theta} \hat{v}(\theta,a_1) = \lambda/\mu \geq 0$.

On the other hand, if $\mu = 0$, then we have $\lambda \neq 0$ and $f(a_1,a_2) = h(a_1)/\lambda$. In this case, let

$$\hat{f}(a_1,a_2) \equiv g(a_1,a_2),\ \hat{v}(\theta,a_1) \equiv 1 \text{ and } \hat{c}(\theta,a_1) \equiv c(\theta,a_1) + h(a_1)v(\theta,a_1)/\lambda,$$

which obtains representation (4.12). □

[26] Farkas' Lemma implies the existence of $\hat{\gamma} \in \mathbb{R}^{n-1}$ such that $\tau(f^{a_1}) \cdot \hat{\gamma} \leq 0$ and $\tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \hat{\gamma} > 0$. But given that $\tau(f^{a_1}) \neq 0$, if $\tau(f^{a_1}) \cdot \hat{\gamma} = 0$, there exist $\tilde{\gamma} \in \mathbb{R}^{n-1}$ close to $\hat{\gamma}$ such that $\tau(f^{a_1}) \cdot \tilde{\gamma} < 0$ and $\tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \tilde{\gamma} > 0$.

Appendix B

B.1 Strongly Monotone Equilibria

Theorems 1,2, and 3 in the main text provide sufficient conditions under which the sender uses a monotone strategy in every Nash equilibrium. As discussed in section 2, our notion of monotonicity can be strengthened in environments when A_1 is multi-dimensional. This leads to the definitions of strongly monotone strategy and strongly monotone equilibrium:

Definition B1. *σ_1 is a strongly monotone strategy if* $\min_{a_1}\{supp(\sigma_1^{\theta})\} \succeq \max_{a_1}\{supp(\sigma_1^{\theta'})\}$ *for every $\theta \succ \theta'$. An equilibrium (σ_1, σ_2) is strongly monotone if σ_1 is strongly monotone.*

According to the above definition, a strategy is strongly monotone if a high type sender always plays a higher action than a low type. Plainly, strong monotonicity is strictly more demanding than monotonicity when A_1 is multi-dimensional. In what follows, we use an example to illustrate why it is fundamentally more challenging to establish a result stating that all equilibria are strongly monotone.

Let $\Theta = A_2 = \{0,1,2,3\}$, and $A_1 = \{(0,0),(1,0),(0,1),(1,1)\}$. All sets are endowed with the product order. Note that A_1 is two-dimensional, and the sender's actions are only partially ordered. For all $\theta \in \Theta$, $a_1 = (a_{11}, a_{12}) \in A_1$ and $a_2 \in A_2$, the sender's payoff is given by

$$u_1(\theta, a_1, a_2) = \sqrt{a_2} - c(\theta, a_1),$$

where

$$c(\theta, a_1) = \begin{cases} 3a_{11} + 3a_{22} & \text{if } \theta = 0, \\ 0.9a_{11} + 2a_{22} & \text{if } \theta = 1, \\ 0.8a_{11} + a_{22} & \text{if } \theta = 2, \\ 0.2a_{11} + 0.2a_{22} & \text{if } \theta = 3. \end{cases}$$

The receiver's payoff is given by $u_2(\theta, a_1, a_2) = 1$ if $a_2 = \theta$, and $u_2(\theta, a_1, a_2) = 0$ otherwise. We leave the receiver's prior belief π unspecified as it plays no role.

In this example, the sender's payoff is *separable* and monotone-supermodular, which is sufficient to guarantee that she uses a monotone strategy in every equilibrium (Theorem 3). However, even the sender's payoff takes such a simple form, one cannot assert that all equilibria are strongly monotone. In particular, consider the following strategy profile: Type $\theta = 0$ sender chooses $a_1 = (0,0)$, type 1 chooses $a_1 = (1,0)$, type 2 chooses $a_1 = (0,1)$, and type 3 chooses $a_1 = (1,1)$. The receiver plays $a_2 = 0$ if he observes $a_1 = (0,0)$, 1 if he observes $a_1 = (1,0)$, 2 if he observes $a_1 = (0,1)$, and 3 if $a_1 = (1,1)$ is observed. One can check that this strategy profile constitutes a sequential equilibrium, in which every action of the sender is played with positive probability, and the players' incentives are strict. The sender's strategy is monotone but not strongly monotone, because the action $(0,1)$ taken by type 2 is not higher than the action $(1,0)$ taken by type 1.

In sum, the above example suggests that strong assumptions on the players' payoffs would need to be made if we want to further strengthen the robust monotonicity prediction about the sender's equilibrium strategies. In particular, the difficulty due to the incompleteness of the order on A_1 cannot be easily bypassed even in the simplest settings where the sender's payoff is separable.

B.2 Generalized Results with Infinite A_2

We generalize our monotonicity results to cases where A_1 is infinite. For simplicity, we shall assume that $A_1 \subset \mathbb{R}^n$ with $n \geq 1$ and it is a complete lattice with to the product order on the Euclidean space. With infinite A_1, a technical difficulty is that some of the actions in the support of σ_1^θ can be suboptimal. Therefore, the notion of monotonicity in Definition 2 does not apply.

For every $a_1 \in A_1$ and $\alpha_1 \in \Delta(A_1)$, let $\Pr(\alpha_1 \succ a_1)$ be the probability that the realization of α_1 is strictly higher than a_1. We introduce the following weaker version of monotonicity:

Definition B2. *σ_1 is an almost surely monotone strategy if for every $\theta \succ \theta'$ and $a_1 \in supp(\sigma_1^\theta)$, we have $\Pr(\sigma_1^{\theta'} \succ a_1) = 0$. An equilibrium (σ_1, σ_2) is almost surely monotone if σ_1 is almost surely monotone.*

We establish the following result, which generalizes Theorem 1.

Theorem B1. *If $|A_2| = 2$ and the sender's payoff is monotone-supermodular, then every Nash equilibrium is almost surely monotone.*

PROOF. The proof of Theorem 1 implies the following lemma.

Lemma B1. *Given the receiver's strategy σ_2, for every $\theta \succ \theta'$ and $a_1 \succ a_1'$, if a_1' is a best response* *to σ_2 for type θ, then a_1 is not a best response to σ_2 for type θ'.*

For every $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ be the open ball around x with radius r. For every $\theta \succ \theta'$ and $a_1 \in \text{supp}(\sigma_1^\theta)$, we have $\sigma_1^\theta(B(a_1,r)) > 0$ for every $r > 0$. That is to say, there exists $a_1' \in B(a_1,r)$ such that a_1' is optimal for type θ. Let a_1^r be the smallest element that is above every element in $B(a_1,r)$. Lemma B1 implies that $\Pr(\sigma_1^{\theta'} \succ a_1^r) = 0$ for every $r > 0$. For every strictly positive decreasing sequence $\{r_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} r_i = 0$, we have:

$$\lim_{i\to\infty}\{a_1'|a_1' \succ a_1^{r_i}\} = \{a_1'|a_1' \succ a_1\} \text{ and } \{a_1'|a_1' \succ a_1^{r_i}\} \supset \{a_1'|a_1' \succ a_1^{r_j}\} \text{ for every } i > j.$$

The monotone convergence theorem implies that:

$$\Pr(\sigma_1^{\theta'} \succ a_1) = \Pr(\sigma_1^{\theta'} \succ \lim_{i\to\infty} a_1^{r_i}) = \lim_{i\to\infty} \Pr(\sigma_1^{\theta'} \succ a_1^{r_i}) = 0.$$

B.3 Characterizing the Quasiconcavity-Preserving Property

We characterize the quasiconcavity-preserving property using the primitives of our model. We first introduce a strict version of the signed-ratio monotonicity condition in Quah and Strulovici (2012).

Definition B3 (Strict Signed-Ratio Monotonicity). *A pair of functions $\gamma_\theta^{a_1}, \gamma_{\theta'}^{a_1} : I \to \mathbb{R}$ obeys strict signed-ratio monotonicity (or SSRM) if*

(1) for every i such that $\gamma_\theta^{a_1}(i) < 0$ and $\gamma_{\theta'}^{a_1}(i) > 0$, we have

$$\frac{\gamma_\theta^{a_1}(i)}{\gamma_{\theta'}^{a_1}(i)} < \frac{\gamma_\theta^{a_1}(j)}{\gamma_{\theta'}^{a_1}(j)} \quad \textit{for every } j > i, \textit{ and}$$

(2) for every i such that $\gamma_\theta^{a_1}(i) > 0$ and $\gamma_{\theta'}^{a_1}(i) < 0$, we have

$$\frac{\gamma_{\theta'}^{a_1}(i)}{\gamma_\theta^{a_1}(i)} < \frac{\gamma_{\theta'}^{a_1}(j)}{\gamma_\theta^{a_1}(j)} \quad \textit{for every } j > i.$$

The next result characterizes the quasiconcavity-preserving property in our setting, which is a straightforward extension of Theorem 1 in Quah and Strulovici (2012):

Proposition B1. *The receiver's payoff is quasiconcavity-preserving if and only if (i) $\gamma_\theta^{a_1}$ satisfies*

SSCP for every $(\theta, a_1) \in \Theta \times A_1$, and (ii) $\gamma_\theta^{a_1}$ and $\gamma_{\theta'}^{a_1}$ obey SSRM for every $a_1 \in A_1$ and every $\theta, \theta' \in \Theta$.

PROOF. (*Only-if statement*) Suppose that the receiver's payoff is quasiconcavity-preserving, i.e., $\Gamma_{\tilde{\pi}}^{\theta}$ has the strict single-crossing property for every $(a_1, \tilde{\pi}) \in A_1 \times \Delta(\Theta)$. Then, (i) immediately follows by taking the degenerate distributions over $\Delta(\Theta)$. For (ii), pick any pair of functions $\gamma_\theta^{a_1}$ and $\gamma_{\theta'}^{a_1}$. Suppose that $\gamma_\theta^{a_1}(i) < 0$ and $\gamma_{\theta'}^{a_1}(i) > 0$. Let

$$\beta = \frac{-\gamma_\theta^{a_1}(i)/\gamma_{\theta'}^{a_1}(i)}{1-\gamma_\theta^{a_1}(i)/\gamma_{\theta'}^{a_1}(i)},$$

so that $\beta \in (0,1)$ and $\beta\gamma_\theta^{a_1}(i) + (1-\beta)\gamma_{\theta'}^{a_1}(i) = 0$. Since $\beta\gamma_\theta^{a_1} + (1-\beta)\gamma_{\theta'}^{a_1}$ has the strict single-crossing property, we have $\beta\gamma_\theta^{a_1}(j) + (1-\beta)\gamma_{\theta'}^{a_1}(j) > 0$ for all $j > i$. Given that $\gamma_{\theta'}^{a_1}$ satisfy SSCP and thus $\gamma_{\theta'}^{a_1}(j) > 0$, we can further obtain

$$\frac{1-\beta}{\beta} = -\frac{\gamma_\theta^{a_1}(i)}{\gamma_{\theta'}^{a_1}(i)} > -\frac{\gamma_\theta^{a_1}(j)}{\gamma_{\theta'}^{a_1}(j)}.$$

Hence, $\gamma_\theta^{a_1}$ and $\gamma_{\theta'}^{a_1}$ obey SSRM for every $a_1 \in A_1$ and every $\theta, \theta' \in \Theta$.

(*If-statement*) Let $\Theta \equiv \{\theta_1, ..., \theta_K\}$. We need to show that $\forall \mu \equiv (\mu_1, ..., \mu_K) \in [0,1]^K$ such that $\sum_{k=1}^K \mu_i = 1$, the function $\Gamma_\mu^{a_1} : I \to \mathbb{R}$ with $\Gamma_\mu^{a_1}(i) \equiv \sum_{k=1}^K \mu_k \gamma_{\theta_k}^{a_1}(i)$ satisfies the strict single-crossing property. Since SSCP is preserved under positive scalar multiplication, and if $\gamma_\theta^{a_1}$ and $\gamma_{\theta'}^{a_1}$ obey SSRM then so do $\beta\gamma_\theta^{a_1}$ and $\gamma_{\theta'}^{a_1}$ for all $\beta \geq 0$, it suffices for us to show that $\Gamma^{a_1} \equiv \sum_{k=1}^K \gamma_{\theta_k}^{a_1}$ satisfies SSCP.

Suppose that $\Gamma^{a_1}(i) \geq 0$. We want to show that $\Gamma^{a_1}(j) > 0$ for every $j > i$. If $\gamma_{\theta_k}^{a_1}(i) \geq 0$ for all $k = 1, ..., K$, then we are done because each $\gamma_{\theta_k}^{a_1}$ satisfies SSCP. Now suppose that $\gamma_{\theta_k}^{a_1}(i) < 0$ for some $\theta_k \in \Theta$. In this case, let us partition Θ into three subsets, Θ^+, Θ^0 and Θ^-, such that $\theta_{k'} \in \Theta^+$ if $\gamma_{\theta_{k'}}^{a_1}(i) > 0$, $\theta_{k'} \in \Theta^0$ if $\gamma_{\theta_{k'}}^{a_1}(i) = 0$, and $\theta_{k'} \in \Theta^-$ if $\gamma_{\theta_{k'}}^{a_1}(i) < 0$. Hence, we have

$$\sum_{\theta_k \in \Theta^+ \cup \Theta^-} \gamma_{\theta_k}^{a_1} = \sum_{\ell=1}^{L} \gamma_\ell^{a_1},$$

where each function $\gamma_\ell : I \to \mathbb{R}$ is a positive linear combination of at most two functions $\gamma_{\theta_k}^{a_1}, \gamma_{\theta_{k'}}^{a_1}$ such that $\theta_k, \theta_{k'} \in \Theta^+ \cup \Theta^-$, and $\gamma_\ell(i) \geq 0$ for all $\ell = 1, ..., L$.

To complete the proof, it now suffices to show that for every $\ell = 1, ..., L$, if $\gamma_\ell^{a_1} = \alpha\gamma_{\theta_k}^{a_1} + \beta\gamma_{\theta_{k'}}^{a_1}$ for some $\alpha, \beta > 0$ and $\gamma_{\theta_k}^{a_1}, \gamma_{\theta_{k'}}^{a_1}$ such that $\gamma_{\theta_k}^{a_1}(i) < 0$ and $\gamma_{\theta_{k'}}^{a_1}(i) > 0$, we would then obtain $\gamma_{\ell}^{a_1}(j) > 0$ for every $j > i$. This is true because by SSRM, we have

$$\frac{\beta}{\alpha} \geq -\frac{\gamma_{\theta_k}^{a_1}(i)}{\gamma_{\theta_{k'}}^{a_1}(i)} > -\frac{\gamma_{\theta_k}^{a_1}(j)}{\gamma_{\theta_{k'}}^{a_1}(j)} \quad \text{for every } j > i,$$

and hence $\gamma_{\ell}^{a_1}(j) = \alpha \gamma_{\theta_k}^{a_1}(j) + \beta \gamma_{\theta_{k'}}^{a_1}(j) > 0$. □

B.4 Insufficiency of Single-Crossing Differences

We show that our strictly increasing difference condition on u_1 cannot be replaced with the *strict single-crossing difference property* in Milgrom and Shannon (1994).

Definition B4. *u_1 has strict single-crossing differences (SSCD) if for every $\theta \succ \theta'$ and every $(a_1, a_2) \succ (a_1', a_2')$, $u_1(\theta', a_1, a_2) - u_1(\theta', a_1', a_2') \geq 0$ implies that $u_1(\theta, a_1, a_2) - u_1(\theta, a_1', a_2') > 0$.*

By definition, SSCD is implied by strictly increasing differences. The following example shows that SSCD cannot guarantee the monotonicity of all equilibria in signaling games, even when u_1 satisfies the monotonicity part of our monotone-supermodular condition.

Example B1. Consider the following game in which the sender chooses row and the receiver chooses column:

$\theta = \theta_1$	h	l
H	$1, 2$	$-3, 0$
L	$3, 1$	$-1, 0$

$\theta = \theta_0$	h	l
H	$1, 0$	$-2, 0$
L	$2, -1$	$-1, 0$

The orders are $\theta_1 \succ \theta_0$, $H \succ L$, and $h \succ l$. One can verify that first, u_1 is strictly increasing in a_2 and is strictly decreasing in a_1. Second, u_1 has SSCD but it does not have increasing differences. To see this, let $\alpha_2 \equiv \frac{2}{3}h + \frac{1}{3}l$ and $\alpha_2' \equiv \frac{1}{3}h + \frac{2}{3}l$. Although α_2 FOSD α_2', we have

$$u_1(\theta_0, h, \alpha_2) - u_1(\theta_0, l, \alpha_2') = 0 > -\frac{2}{3} = u_1(\theta_1, h, \alpha_2) - u_1(\theta_1, l, \alpha_2').$$

When the receiver's prior belief attaches probability $1/3$ to state θ_1, the game admits the following non-monotone equilibrium. Type θ_1 sender plays L, type θ_0 sender plays H and L each with probability $1/2$, the receiver plays α_2 after observing H and α_2' after observing L.

In the above example, the receiver's best reply against the sender's action is mixed. SSCD only requires that $u_1(\theta,a_1,a_2)-u_1(\theta,a_1',a_2')$ has the strict single-crossing property for every pair of *pure action profiles* that can be ranked. This does not imply that $u_1(\theta,a_1,\alpha_2)-u_1(\theta,a_1',\alpha_2')$ satisfies strict single-crossing property for every $(a_1,\alpha_2),(a_1',\alpha_2')\in A_1\times\Delta(A_2)$ with $a_1\succ a_1'$ and α_2 FOSDs α_2'.[27] This opens up the possibility of non-monotone equilibria.

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[27] In fact, since $|A_1|=|A_2|=2$ in this example, the payoff function u_1 also has SCED on both $\Delta(A_1)$ and $\Delta(A_2)$ (Kartik et al. 2019). However, it does not have SCED on the larger space $\Delta(A_1\times A_2)$.

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