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Note

When does restricting your opponent's freedom hurt you?

Harry Di Pei [1]

Department of Economics, Massachusetts Institute of Technology, United States

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ABSTRACT

I examine the payoff consequences for a player when she removes a subset of her opponent's actions before playing a two-player complete information normal form game. When she faces a constraint on the maximal number of actions she can remove, she can be strictly better off by *not* removing any actions. I present such an example. I also establish sufficient conditions under which removing opponent's actions cannot hurt. As a corollary, I also characterize a necessary condition for a player's optimal Nash Equilibrium in games with generic payoffs when her opponent has strictly more actions than she does.

1. Introduction

Consider a complete information normal form game played by two players. Suppose before the game starts, one of the players, say player 1 (she), can choose to remove some of player 2's (he) actions from his action set. After that, players proceed to play the '*restricted game*'. Suppose player 1 only has the capability of removing k or fewer actions, then when does she prefer to restrict player 2's freedom in a non-trivial way, i.e. removing a non-empty subset of his actions, before the game starts?

This game theoretic question captures a number of economically interesting applications in political economy and organizational economics. For example, consider the two players being two political groups. One of the groups is in power and can restrict the other group's freedom by forbidding certain actions. However, such power is usually limited either due to the cost of policing and enforcement, or due to other exogenous restrictions (for example, the constitution). In my model, this '*limited amount of power*' is captured by the constraint that player 1 can remove at most k of player 2's actions. A larger k means that player 1 is more powerful. Similar problems can also arise within firms and organizations, in which superiors can impose rules on their subordinates, but the cost of monitoring as well as other concerns require her to leave at least some discretion to the latter.

Contrary to the conventional wisdom that restricting the freedom of an opponent should always be beneficial, I start with a counterexample, in which player 1 is *strictly* worse off by deleting any one of her opponent's actions, even when she can choose it optimally and can choose which equilibrium to coordinate on afterwards. The message from this example is clear: when player 1's ability to restrict her opponent's freedom is limited (i.e. she cannot reduce her opponent's action set to a singleton), then there exist circumstances in which she finds it *strictly* optimal to leave full discretion to her opponent.

E-mail address: harrydp@mit.edu.

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Next, I characterize sufficient conditions under which player 1 is weakly better off by removing at least one action from her opponent's action set. Aside from the trivial case in which her opponent has only $k+1$ actions, I show that player 1 is weakly better off when her opponent has strictly more actions than she does. The proof uses the Carathéodory Theorem to construct an equilibrium in the restricted game, which gives player 1 a payoff that is weakly greater than her highest equilibrium payoff in the original game.

As a byproduct of the proof, I also show that in 2-player games with generic payoffs, if player 2 has strictly more actions than player 1, then in every equilibrium that is optimal for player 1, player 2 must be playing a (possibly mixed) strategy that has at most n actions on its support, where n is the number of actions player 1 has.

Related literature: The question I asked is related to the literature on commitment games, *à la*, Renou (2009), Bade et al. (2009), in which there is an *ex ante stage*, during which every player commits to remove a subset of his actions.[2] Instead of examining the payoff consequences of commitment, i.e. a player restricting her own freedom, I examine the payoff consequences of restricting her opponent's freedom.

The underlying message of my paper is related to that in Bernheim and Whinston (1998), who show that if contracts must be incomplete due to non-verifiability, then it is often optimal for players to write contracts that are even more incomplete, i.e. leaving other verifiable aspects unspecified. However, the ways in which we model limiting freedom are very different. In their model, there is a partition for each player's action set and the court cannot distinguish actions within the same partition element. Therefore, under every contract, every player's allowable action set must be measurable with respect to that partition. In my model, all actions are verifiable but there is an upper bound on the number of actions that can be removed. This difference in modeling also leads to different results. In their model, signing an incomplete contract and leaving extra discretion to players is never optimal in static simultaneous move games, which is not true in my model.

2. Model setup

The original game: Consider a complete information normal form game: $\mathcal{G} = (I, A, U)$, where:

- The set of players is $I = \{1, 2\}$.
- Player $i \in I$ has a finite action set A_i, with typical element a_i. An action profile is denoted by $a \equiv (a_1, a_2) \in A \equiv A_1 \times A_2$.
- $U = (U_1, U_2)$, where $U_i : A \to \mathbb{R}$ maps action profiles to player i's payoff.

Let $\Delta(\cdot)$ be the set of probability measures on a finite set. The mixed extension of this game is defined naturally, with $\alpha_i \in \Delta(A_i)$ being player i's mixed action. For $i \in I$, let

$$U_i(\alpha_1, \alpha_2) \equiv \sum_{a_1 \in A_1} \sum_{a_2 \in A_2} \alpha_1(a_1)\alpha_2(a_2)U_i(a_1, a_2)$$

be player i's expected payoff from mixed strategy profile (α_1, α_2). Let the cardinalities of A_1 and A_2 be n and m, with:

$$A_1 \equiv \{a_1^1, a_1^2, ..., a_1^n\}, \quad A_2 \equiv \{a_2^1, a_2^2, ..., a_2^m\}.$$

I focus on the case in which $m \geq 2$.

A (possibly mixed) Nash Equilibrium (NE), (α_1^*, α_2^*), is defined as in Fudenberg and Tirole (1991, hereafter, FT), with $NE(\mathcal{G})$ the set of NEs in game $\mathcal{G}$. FT shows that the set of NEs is closed,[3] and players' payoffs are continuous in (α_1, α_2). As a result, player 1's highest NE payoff exists, which is defined as:

$$U_{1,max}(\mathcal{G}) \equiv \max_{(\alpha_1,\alpha_2) \in NE(\mathcal{G})} U_1(\alpha_1, \alpha_2).$$

Restricting freedom: Suppose before the game starts, player 1 (she) can forbid player 2 (he) from playing at most $k \in \mathbb{N}$ of his actions, where $1 \leq k \leq m-1$. I view k as a parameter that measures the ability of player 1 to restrict her opponent's freedom.

Let $\tilde{A}_2 \subset A_2$ be the set of remaining actions, with $m - k \leq \#\tilde{A}_2 \leq m - 1$, where '#' denotes the cardinality of a set. After $\tilde{A}_2$ is chosen, players play a '*restricted game*':

$$\mathcal{G}^{\tilde{A}_2} \equiv \left(I, (A_1, \tilde{A}_2), U^{\tilde{A}_2}\right),$$

[2] Romano and Yildirim (2005) studies dynamic games, in which players can only increase their actions over time. In this spirit, playing a higher action in the early stages commits a player to player higher actions in the future.

[3] This is implied by FT's claim in page 30 that players' best reply correspondences have closed graph property, which is a Lemma towards proving the existence of NE.

instead of $\mathcal{G}$ (the '*original game*'). Let $U^{\tilde{A}_2}: A_1 \times \tilde{A}_2 \to \mathbb{R}^2$ be players' payoffs in the restricted game, with $U^{\tilde{A}_2}(a') = U(a')$ for every $a' = (a'_1, a'_2) \in A_1 \times \tilde{A}_2$. Let $NE(\mathcal{G}^{\tilde{A}_2})$ be the set of NEs in $\mathcal{G}^{\tilde{A}_2}$, with

$$U_{1,max}(\mathcal{G}^{\tilde{A}_2}) \equiv \max_{(\alpha_1,\alpha_2)\in NE(\mathcal{G}^{\tilde{A}_2})} U_1(\alpha_1, \alpha_2)$$

being player 1's highest NE payoff in the restricted game.

3. Results

I investigate when will player 1 be weakly better off after deleting a non-empty subset of her opponent's actions, which are chosen optimally, under the constraint that she can delete at most k of them. The payoff consequences are evaluated according to the *best* NE for player 1, both in the original game and in the restricted game. Formally, for every $l \leq m-1$, let

$$\mathcal{A}(l) \equiv \left\{\tilde{A}_2 \subset A_2 \,\middle|\, \#\tilde{A}_2 = m - l\right\}.$$

I am interested in when does the following inequality hold:

$$\max_{\tilde{A}_2 \in \cup_{l=1}^{k} \mathcal{A}(l)} U_{1,max}(\mathcal{G}^{\tilde{A}_2}) \geq U_{1,max}(\mathcal{G}). \tag{1}$$

3.1. A counterexample

Somewhat surprisingly, (1) is not always true. This is because for a given equilibrium (α_1^*, α_2^*), removing player 2's action can change player 1's incentive to play α_1^*, which further affects player 2's incentives. This indirect effect can reduce player 1's equilibrium payoff. This message is conveyed via the following counterexample, in which player 1 *strictly* prefers not to exercise any control over her opponent when her capability of doing so is limited.

In this example, let $k = 1$. Consider the following 3×3 game, in which player 1 chooses the row, player 2 chooses the column and their payoffs are specified in the matrix below:

–	L	M	R
u	15, −5	45, −6	0, −10
m	0, −10	45, −6	15, −5
b	0, 0	50, −10	0, 0

This game admits three NEs. Two pure NEs, (u, L) and (m, R), both yield player 1 payoff 15, as well as a completely mixed NE:

$$\left(\frac{10}{23}u + \frac{10}{23}m + \frac{3}{23}b, \frac{1}{5}L + \frac{3}{5}M + \frac{1}{5}R\right),$$

in which player 1's expected payoff is 30.

Next, let us examine player 1's highest NE payoff after eliminating one of player 2's actions, i.e. whether there exists an action and a NE of the resulting restricted game, such that player 1's payoff is no less than 30, which is her highest NE payoff in the original game. I show that the answer to this question is no.

- First, notice that player 1's payoff can never exceed 15 after eliminating M, since her highest payoff when player 2 plays L or R is 15.
- Second, after player 1 eliminates R, then m is a weakly dominated by u, implying that it is played with positive probability only when player 2 always plays M. But if this is true, player 1 will play b which makes playing M not incentive compatible for player 2. Hence, m is never played by player 1. But then, after eliminating m, L strictly dominates M. So the unique NE is (u, L), which yields player 1 a payoff 15, which is strictly lower than 30.
- Similarly, after player 1 eliminates L, the unique NE is (m, R), and player 1's NE payoff is still 15, which is strictly lower than 30.

Intuition behind the counterexample: Player 1 would like player 2 to play the '*reward action*' M. But in order to motivate 2 to play M, she needs to be able to commit to play both u and m with positive probability. This is because player 2 will have a strict incentive to play L or R if either u or m is not played.

In the above counterexample, if R is removed, then it undermines player 1's incentive to play m, and this further undermines player 2's incentive to play M. Similar logic applies when player 1 removes L. Intuitively, even if a '*non-rewarding action*' is removed, it can still hurt player 1 if it alters her incentive to play her original equilibrium action.

3.2. A characterization theorem

In this subsection, I examine for any given k (interpreted as the amount of power player 1 has), when will optimally removing some of player 2's actions never hurt player 1. The following result provides sufficient conditions for (1):

Theorem 1. *If $m = k + 1$ or if $m \geq n + 1$, then:*

$$\max_{\tilde{A}_2 \in \cup_{l=1}^{k} \mathcal{A}(l)} U_{1,max}(\mathcal{G}^{\tilde{A}_2}) \geq U_{1,max}(\mathcal{G}).$$

Theorem 1 implies that even when player 1 does not have enough power to restrict player 2's action set to a singleton (i.e. $m \geq k + 2$), she still weakly prefers to remove some of player 2's actions as long as her opponent has strictly more actions than she does. In fact, when $m - n \geq l$ where $l \in \mathbb{N}$, then player 1 (weakly) prefers to remove l actions than to remove only $l - 1$ actions.

In the following proof, I will show that whenever $m \geq n + 1$, then in player 1's optimal equilibrium in the original game, call it (α_1^*, α_2^*), we can always find a '*redundant action*' of player 2's, such that after deleting this action, we can still find a mixed strategy of player 2 that weakly increases player 1's payoff and *does not* change player 1's incentive to play α_1^*. To do this, we need player 2 to have enough actions, so that we can have sufficiently many degrees of freedom to construct such a mixed strategy after an action is removed.

Proof of Theorem 1. The claim is trivially true if $m = k + 1$ (player 1 can achieve her best payoff in the matrix by forbidding player 2 from taking all other actions), or if in the optimal equilibrium, player 2's action is not completely mixed (by eliminating an action which is never taken by player 2, the equilibrium in the original game remains to be an equilibrium). So I will focus on equilibria in which player 2 uses a *completely mixed strategy*.

Let (α_1^*, α_2^*) be the best NE for player 1 in $\mathcal{G}$. The key idea of the proof is: can we find a_2 such that after eliminating it, there exists $\alpha_2' \in \Delta(A_2 \backslash \{a_2\})$, which can still give player 1 the incentive to play α_1^*, and can guarantee her a weakly higher payoff? If this is true, then player 2's incentive constraints are automatically satisfied, since any action on the support of α_2^* is a best reply to α_1^*.

To formalize this idea, I introduce some additional notation. For every $a_2 \in A_2$, let

$$v(a_2) \equiv \Big(U_1(a_1^1, a_2), ..., U_1(a_1^n, a_2)\Big) \in \mathbb{R}^n$$

be player 1's payoff vector (obtained via changing her own actions) conditional on player 2 playing a_2. Let $V \equiv \{v(a_2^1), ..., v(a_2^m)\} \subset \mathbb{R}^n$. Let $co(V)$ be the convex hull of V. Let

$$v^* \equiv \Big(U_1(a_1^1, \alpha_2^*), ..., U_1(a_1^n, \alpha_2^*)\Big).$$

To guarantee player 1's incentive to play α_1^*, I will construct α_2' in the next step that satisfies:

- For any $a_1, a_1' \in A_1$, the signs of $U_1(a_1, \alpha_2^*) - U_1(a_1', \alpha_2^*)$ and $U_1(a_1, \alpha_2') - U_1(a_1', \alpha_2')$ coincide.

In words, player 1's ordinal preferences over her own actions are exactly the same under α_2^* and α_2'. The proof uses the following result in convex geometry, which can be found in Eckhoff (1993):

Theorem 2 *(Carathéodory's Theorem). If $x \in \mathbb{R}^d$ lies in the convex hull of a set P, there is a subset P' of P consisting of $d + 1$ or fewer elements such that x lies in the convex hull of P'.*

Constructing α_2': The claim in Theorem 1 is established for two separate cases:

1. When $m \geq n + 2$, let $\lambda \equiv (\lambda_1, ..., \lambda_m)$, with λ_i the probability α_2^* attaches to a_2^i. We have:

$$\sum_{i=1}^{m} \lambda_i v(a_2^i) = v^*, \ \lambda_i \in (0, 1) \text{ and } \sum_{i=1}^{m} \lambda_i = 1.$$

Hence, $v^* \in co(V)$. By definition, both v^* and every $v(a_2)$ live in an n-dimensional Euclidean Space. According to Carathéodory's Theorem, there exists a subset V' of V with at most $n+1$ elements, such that $v^* \in co(V')$. Fig. 1 depicts v^*, V, V', $co(V)$ and $co(V')$ when $n = 2$. Since $m \geq n + 2$, we know that $V' \neq V$. Hence, there exists λ_i' such that

$$\sum_{i=1}^{m} \lambda_i' v(a_2^i) = v^*, \ \lambda_i' \in [0, 1], \sum_{i=1}^{m} \lambda_i' = 1 \text{ and } \lambda_i' = 0 \text{ if } a_2^i \notin V'.$$

Fig. 1. In this example, $n = 2$. In the left panel, the round dots are elements of V and the square is v^*. The dashed polytope depicts $co(V)$. In the right panel, the triangles are elements of V', the round dots are elements of $V \backslash V'$ and the square is v^*. The dashed polytope depicts $co(V')$. Since $n = 2$, V' can be chosen such that $\#V' \leq 3$.

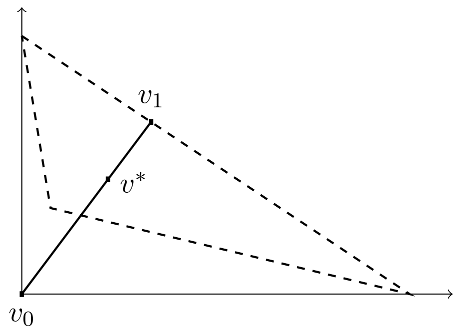

Fig. 2. Relationship between v_0, v^* and v_1, with the dashed line depicting $co(V)$.

Let α_2' be the mixed strategy such that a_2^i is played with probability λ_i' for $i = 1, 2, ..., n$. By construction, the support of α_2' is V'. Moreover, $U_1(a_1, \alpha_2^*) = U_1(a_1, \alpha_2')$ for all $a_1 \in A_1$, implying that player 1's preferences over her own actions are exactly the same under α_2^* and α_2', thus she has an incentive to play α_1^* when player 2 is playing α_2'. So her payoff cannot decrease after optimally deleting at least one of player 2's actions.

2. When $m = n + 1$, if $co(V)$ has dimensionality strictly less than n, then v^* lives in a Euclidean Space with dimension no more than $n - 1$, so we are back to the previous case in which the proof directly carries over. So, I will be focusing on the case where $co(V)$ is n-dimensional.
Since α_2^* is completely mixed, so v^* is in the interior of $co(V)$. Let

$$\underline{U}_1 \equiv \min_{a \in A} U_1(a),$$

and $v_0 \equiv (\underline{U}_1, ..., \underline{U}_1) \in \mathbb{R}^n$. By definition, every entry of v^* is weakly greater than $\underline{U}_1$.
Depict the line starting from v_0 and passing through v^*. Since v^* is in the interior of $co(V)$, this line will intersect one of the surfaces of $co(V)$ at a point v_1, such that v^* is strictly between v_0 and v_1. I show the three points, v_0, v^* and v_1 together in Fig. 2.
By construction, v_1 *strictly* dominates v^* at every entry.[4] Let $\alpha_2' \in \Delta(A_2)$ be the mixed strategy of player 2 such that player 1's payoff vector under α_2' is v_1.
I show that α_1^* is still incentive compatible for player 1 under α_2'. For any vector $v \in \mathbb{R}^n$, let $v[j]$ be its jth entry. As discussed before, I only need to show that the signs of $v_1[j_1] - v_1[j_2]$ and $v^*[j_1] - v^*[j_2]$ coincide for every $1 \leq j_1, j_2 \leq n$.
Since v_1, v^* and v_0 are along the same line, there exists $\mu > 0$ such that:

$$v_1 - v^* = \mu(v^* - v_0) \tag{2}$$

which implies that $v_1[j] - v^*[j] = \mu(v^*[j] - v_0[j])$ for every j. Hence, for every j_1, j_2, if $v^*[j_1] = v^*[j_2]$, then using (2) and the fact that $v_0[j] = \underline{U}_1$ for all j, we have:

$$v_1[j_1] = v^*[j_1] + \mu(v^*[j_1] - v_0[j_1]) = v^*[j_2] + \mu(v^*[j_2] - v_0[j_2]) = v_1[j_2].$$

If $v^*[j_1] > v^*[j_2]$, then using (2) and the fact that $v_0[j] = \underline{U}_1$ for all j as well as $\mu > 0$, we have:

$$v_1[j_1] = v^*[j_1] + \mu(v^*[j_1] - v_0[j_1]) > v^*[j_2] + \mu(v^*[j_2] - v_0[j_2]) = v_1[j_2].$$

[4] This is because v^* is in the interior of co(V), so v^* strictly dominates v_0 at every dimension. Since v^* is in the interior of the line between v_0 and v_1, v_1 also strictly dominates v^* at every dimension.

Similar argument can show that if $v^*[j_1] < v^*[j_2]$, then $v_1[j_1] < v_1[j_2]$.

But then, (α_1^*, α_2') is also a NE in the original game. Since v_1 strictly dominates v^* at every dimension, player 1's payoff in this new equilibrium is $\sum_{j=1}^n \alpha_1^*(a_1^j) v_1[j]$, which is strictly greater than his payoff under (α_1^*, α_2^*), which is $\sum_{j=1}^n \alpha_1^*(a_1^j) v^*[j]$. This contradicts the assumption that (α_1^*, α_2^*) is player 1's optimal equilibrium in the original game. $\square$

3.3. A further implication

An interesting implication arises as a byproduct of the proof, which is related to the structure of player 1's optimal NE in a normal form game. Recall that $V = \{v(a_2^1), ..., v(a_2^m)\}$ with

$$v(a_2^j) \equiv \Big(U_1(a_1^1, a_2^j), ..., U_1(a_1^n, a_2^j)\Big) \in \mathbb{R}^n.$$

I summarize the result in the following Proposition:

Proposition 1. *In a normal form game with $m \geq n+1$,*

1. *If co(V) is n-dimensional, then player 2 cannot play a completely mixed strategy in every optimal NE for player 1.*
2. *If co(V') is n-dimensional for any $V' \subset V$ with $\#V' \geq n+1$, then in every optimal NE for player 1, player 2 must be playing a (possibly mixed) strategy with at most n actions on its support.*

Notice that the assumptions in both parts of Proposition 1 are satisfied in games with generic payoffs. Therefore, Proposition 1 has uncovered a necessary condition for a player's optimal NE in generic games where her opponent has more actions than she does. Moreover, for every NE in which player 2 is playing a mixed strategy with strictly more than n actions on its support, the proof of Proposition 1 provides an algorithm for finding another NE which strictly dominates the original one in terms of player 1's payoff. The idea of the proof is similar to that of Theorem 1 when $m = n+1$. I provide a sketch below, with the full proof available upon request.

Proof of Proposition 1. I show the second part of the proposition as it directly implies the first part. Suppose towards a contradiction that (α_1^*, α_2^*) is an optimal NE for player 1 with #supp$(\alpha_2^*) \equiv l \geq n+1$. Without loss of generality, suppose supp$(\alpha_2^*) \equiv \{a_2^1, ..., a_2^l\}$.

Let $V' \equiv \{v(a_2^1), ..., v(a_2^l)\}$. By assumption, co($V'$) is n-dimensional. Let λ_j be the probability α_2^* attaches to the jth action on its support, with $\lambda_j \in (0,1)$ and $\sum_{j=1}^l \lambda_j = 1$. Let

$$v^* \equiv \sum_{j=1}^{l} \lambda_j v(a_2^j)$$

which is in the interior of co(V'). Define v_0 and v_1 as in the proof of Theorem 1. Let $\alpha_2' \in \Delta(\{a_2^1, ..., a_2^l\})$ be the mixed action that induces payoff vector v_1. We can check that (α_1^*, α_2') is also a NE, which delivers strictly higher payoff for player 1. This leads to a contradiction. $\square$

References

Bade, S., Haeringer, G., Renou, L., 2009. Bilateral commitment. J. Econ. Theory 144, 1817–1831.
Bernheim, D., Whinston, M., 1998. Incomplete contracts and strategic ambiguity. Amer. Econ. Rev. 88 (4), 902–932.
Eckhoff, J., 1993. Helly, Radon, and Carathéodory type theorems. In: Handbook of Convex Geometry. North-Holland, Amsterdam.
Fudenberg, D., Tirole, J., 1991. Game Theory. MIT Press, Cambridge, MA.
Renou, L., 2009. Commitment games. Games Econ. Behav. 66, 488–505.
Romano, R., Yildirim, H., 2005. On the endogeneity of Cournot–Nash and Stackelberg equilibria: games of accumulation. J. Econ. Theory 120, 73–107.