Uncertainty about Uncertainty in Communication

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Abstract

I study the impact of higher order uncertainty on communication outcomes when the sender’s (he) preference is unknown to the receiver (she) and players have no conflict of interest with positive probability. When there is no higher order uncertainty, there exists an equilibrium in which the congruent sender can fully reveal his information conditional on not pooling with non-congruent ones. This is no longer true when the sender faces second order uncertainty. I show that in every equilibrium, the probability with which the sender fully reveals the state is zero. My proof uses a novel contagion argument, which exploits the interactions between higher order uncertainty, the sender’s indifference conditions and the receiver’s sequential rationality constraints.

Keywords: higher order uncertainty, cheap talk, communication

JEL Codes: D82, D83

1 Introduction

In many economic applications, strategic information transmission is accompanied by uncertainty and higher order uncertainty. As an example, consider an informed analyst (he) giving financial advice to an uninformed investor (she), he might be driven by hidden incentives which are in conflict with the investor’s. The investor might be unsure about the analyst’s trustworthiness, the analyst might be unsure about the investor’s credulity, the investor might also be unsure about the analyst’s belief about her credulity. This line continues. Similarly, a monarch may suspect his minister’s loyalty and latter can be uncertain about how much his prince trusts him; patients can be wary about their doctor’s hidden motives and the doctor is unsure about his patients’ perceptions about his honesty.

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In the stylized model of strategic communication, Crawford and Sobel (1982) assume that players’ preferences and the receiver’s prior belief are common knowledge. They show that the sender can fully reveal his information when players’ interests are aligned. Morgan and Stocken (2003) generalize this insight by allowing for unknown sender preferences: the sender’s payoff is either the same as the receiver’s or is different. They show there exists an equilibrium in which the congruent type can fully reveal his information conditional on not sending the same message as the misaligned types. However, assuming that the sender knowing the receiver’s prior precisely is unrealistic in many applications, so it is important to assess the significance of higher order uncertainty in shaping the communication outcomes.

This paper uses a simple example to demonstrate the dramatic effects of higher order uncertainty on the precision of information being transmitted. In my model, the receiver wishes to match her action with the state. The sender has two possible preferences. Either he is good, in which case his interest is aligned with the receiver’s. Or he is bad and aims to maximize the receiver’s action. The receiver could entertain various beliefs about the sender’s preferences and the sender does not know the receiver’s belief. I show that in the canonical uniform-quadratic model in which the sender faces higher order uncertainty, even the good sender can never fully reveal his information in any equilibrium. This insight extends to settings with richer type spaces for both players and more general forms of higher order uncertainty.

Intuitively, the good sender pools with the bad one iff the state is above an endogenously determined cutoff. Upon receiving a message that is sent with positive probability by both types of senders, the receiver’s best response depends on his belief about the sender’s congruence. When a good sender faces second order uncertainty and the state equals to the cutoff, his payoff from pooling with the bad type is strictly below his first best payoff. Since the good sender’s equilibrium payoff is continuous with respect to the state, his indifference condition at the cutoff state implies that there cannot exist any message which induces an action slightly below the cutoff. This implies that the good sender cannot fully reveal his private information in an interval of states below the cutoff.

Through the receiver’s sequential rationality constraints and the sender’s indifference conditions, non-full revelation in states slightly below the cutoff will propagate and prevent the good sender from fully revealing his information in a larger set of states. I show that when the prior belief about the state is uniform, or more generally, the density function of the state is non-decreasing, the impact of 2nd order uncertainty at the cutoff state is propagated to the entire state space, implying that the good sender can never fully reveal his information no matter how low the state is. In particular, for every equilibrium,
there exists a finite interval partition of the state space that represents the receiver’s information structure after communicating with the good sender. As in the Crawford and Sobel (1982) model in which the sender has a conflict of interest with the receiver, the number of partition elements is uniformly bounded from above given the values of the primitives.

**Related Literature:** There has been a growing literature that studies strategic communication when the sender’s preference is unknown. A few prominent examples include Morris (2001), Morgan and Stocken (2003), Li and Madarász (2008), etc. In these papers, the distribution of the sender’s bias is commonly known and he faces no higher order uncertainty. This paper shows that higher order uncertainty can change the qualitative features of players’ communication outcomes. Moreover, it develops a contagion argument to establish the impossibility of full revelation in every equilibrium. The argument is novel comparing with the existing ones in Rubinstein (1989), Kajii and Morris (1997), etc.

Complementary to this paper, Blume and Board (2014) study a cheap talk model in which higher order uncertainty is about the sender’s language competence, instead of his bias. Also related is Miura and Yamashita (2014) who examine the possibility of making robust predictions in cheap talk games *from an outside observer’s point of view*. They show a surprising result, that no robust predictions can be made despite the sender’s bias being arbitrarily low and it is common \((1 - \varepsilon)\)-believed by both players. Comparing with their paper, I characterize the set of equilibrium outcomes in a given type space, instead of considering the robustness across all type spaces. From a technical perspective, every type of receiver *mis-interprets* the sender’s message in Miura and Yamashita’s construction, despite her action being fully responsive to the state of nature. Unaware of the existence of certain types of opponents is another feature of their proof. In contrast, my argument relies on a full support condition, i.e. every type is aware of every other type’s existence. Moreover, non-full revelation in my model takes the form of ‘*pooling*’ between close-by states, instead of the receiver misinterpreting the sender’s messages.

## 2 The Model

In this section, I examine the canonical uniform-quadratic model in which the sender’s preference is unknown to the receiver (1st order uncertainty) and the receiver’s prior belief about the sender’s preference
is unknown to the sender (2nd order uncertainty). I show that the presence of 2nd order uncertainty changes the qualitative features of the equilibria. I characterize of the set of equilibria and discuss their properties. My results will be generalized in Appendix B by allowing for more general type spaces.

2.1 Setup

**Primitives:** A queen (she, receiver) needs her minister’s (he, sender) advice about the expenditure on a construction project. Let $\theta$ be the first best expenditure level, which is uniformly distributed on $[0, 1]$ and is the minister’s private information. Let $a \in A \equiv [0, 1]$ be the queen’s decision. The minister can be either good or bad, which is also his private information. Let $\omega \in \{g, b\}$, with $g$ representing the good type and $b$ representing the bad type. The timing of the game is as follows:

- The minister observes $\theta$ and $\omega$.
- The minister sends message $m \in M$ to the queen.
- The queen observes $m$ and chooses $a$.

The good minister’s preference is fully aligned with the queen’s, both want to match $a$ with $\theta$:

$$u^g(a, \theta) = u^r(a, \theta) = -(a - \theta)^2.$$ 

The bad minister’s payoff is given by $u^b(a, \theta) = a$, i.e. he always prefers the queen to spend more regardless of the state, so that he can divert more money into his own pocket. Players are risk neutral. Their payoffs from mixed actions, $\alpha \in \Delta(A)$, are defined naturally.

The queen does not know the minister’s preference. She believes that he is corrupt with probability $\eta \in (0, 1)$. I compare the equilibrium outcomes in two scenarios:

1. **No higher order uncertainty:** The minister knows $\eta$, i.e., there is only ‘1st order uncertainty’.

   This is the case studied in Morris (2001), Morgan and Stocken (2003), etc.

2. **Second order uncertainty:** The minister is unsure about the queen’s perception of his unrighteousness. Perhaps he is uncertain about whether the queen believes in the rumors his political

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1 Due to its convenience and tractability, the uniform-quadratic model has been the main focus of the communication literature. It has also been broadly applied in the political economy and organizational design literature.
enemies have spread in the marketplace, or he does not know whether the queen is suspicious or credulous. In this case, the minister faces ‘2nd order uncertainty’.

Suppose η can take two possible values, η₁ or η₂, and the minister believes that η = η₁ with probability p₁ and η = η₂ with probability p₂ = 1 − p₁. I assume that η₁ ≠ η₂ and p₁ ∈ (0, 1).

**Solution Concept:** As in Crawford and Sobel (1982), the solution concept is Bayes Nash Equilibrium (or ‘equilibrium’ for short), which consists a family of signalling rules for the minister, denoted by q(·|θ, ω) ∈ Δ(M) and an action rule for the queen, denoted by a(m, η) ∈ A such that:

1. For every θ ∈ [0, 1] and ω ∈ {g, b}, ∫m∈M q(m|θ, ω)dm = 1. If m’ ∈ supq(·|θ, ω), then:

   \[ m’ ∈ \arg\max_{m \in M} \left\{ p_1 u^\omega(a(m, \eta_1), \theta) + p_2 u^\omega(a(m, \eta_2), \theta) \right\}. \]

2. For every m and η ∈ {η₁, η₂},

   \[ a(m, \eta) ∈ \arg\max_{a ∈ A} \int_0^1 u^\prime(a, \theta) \pi(\theta|m, \eta) d\theta, \]

   where

   \[ \pi(\theta|m, \eta) = \frac{(1 - \eta)q(m|\theta, g) + \eta q(m|\theta, b)}{(1 - \eta)\int_0^1 q(m|\theta, g)d\theta + \eta \int_0^1 q(m|\theta, b)d\theta}. \]

The equilibrium outcome of the game is the joint distribution of (a, θ, ω, η). Two equilibria are outcome equivalent if they induce the same equilibrium outcome.

### 2.2 Benchmark: Equilibrium without Higher Order Uncertainty

I start from analyzing the no higher order uncertainty benchmark. As in most strategic communication games, there is a multiplicity of equilibria. The one that achieves the highest welfare is the one which features ‘low separating high pooling’.

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2Since the queen’s payoff is strictly concave in a, so it is without loss of generality to focus on pure actions.

3As in Crawford and Sobel (1982), we can consider the minister’s signalling rule induces a distribution on the Borel-measurable subsets of [0, 1] × M × {g, b}. In this setting, there exist regular conditional distributions q(M’|θ, ω) and π(Θ’|m, η) for all (θ, ω, η) ∈ [0, 1] × {g, b} × {η₁, η₂}, with M’ ⊂ M and Θ’ ⊂ [0, 1] being Borel subsets.
Lemma 2.1. In the no higher order uncertainty benchmark, the equilibrium that maximizes the queen’s expected payoff is the following ‘low separating high pooling equilibrium’. There exists $m^* \in M$ such that the bad minister always sends $m^*$, the good minister sends $m^*$ if and only if $\theta \geq \frac{1}{1 + \sqrt{\eta}}$. The queen chooses action $a = \frac{1}{1 + \sqrt{\eta}}$ after receiving $m^*$. The good minister fully reveals $\theta$ if and only if:

$$\theta < \frac{1}{1 + \sqrt{\eta}}.$$ \hspace{1cm} (2.1)

When information is fully revealed, the queen chooses $a = \theta$.

All omitting proofs, including this one can be found in Appendix A. The equilibrium is depicted on the left hand side of Figure 1. As we can see, since the good minister has no conflict of interest with the queen, he can fully reveal his information as long as he does not pool with the bad minister. First order uncertainty about the minister’s preference only prevents him from credibly conveying information when $\theta$ is large, but has no impact when $\theta$ is low. In equilibrium, the (ex ante) probability for him to fully reveal his private information is $\frac{1}{1 + \sqrt{\eta}}$, which is strictly positive and converges to 1 when $\eta \rightarrow 0$.

2.3 Equilibrium with Second Order Uncertainty:

In what follows, I examine the case where the minister faces 2nd order uncertainty. Proposition 1 shows that the above ‘low separating high pooling’ equilibrium no longer exists. Somewhat surprisingly, the probability with which the good minister can fully reveal his information is 0.

**Proposition 1.** The probability that the queen fully learns $\theta$ is 0 in every equilibrium.

The equilibrium communication strategy of the good minister is depicted on the right hand side of Figure 1. I present the proof since it is conducive for understanding the intuition behind the result.

**Proof of Proposition 1:** I start from introducing some extra notation. Recall that $M$ is the minister’s message set, with $m \in M$ a typical element. The type of queen whose prior belief is $\eta_i$ is referred to as ‘type $i’ (i \in \{1, 2\})$. Let $a_i(m)$ be type $i$’s equilibrium action after receiving message $m$.

For every message $m^b$ sent by the bad minister with strictly positive probability, the following condition is satisfied:

$$p_1a_1(m^b) + p_2a_2(m^b) \in \arg\max_{m \in M} \left\{ p_1a_1(m) + p_2a_2(m) \right\},$$ \hspace{1cm} (2.2)

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4Informative equilibrium do exists in my model, which I will characterize in Propositions 2 and 3.
where $M$ is the set of messages. Define $M^b \subset M$ as:

$$M^b \equiv \left\{ m \left| p_1a_1(m) + p_2a_2(m) = \max_{m' \in M} \{ p_1a_1(m') + p_2a_2(m') \} \right. \right\}.$$  \hspace{1cm} (2.3)

In equilibrium, the bad minister only sends messages in $M^b$. The next Lemma shows that it is without loss to focus on the case where $M \setminus M^b \neq \emptyset$:

**Lemma 2.2.** *Every equilibrium in which $M \setminus M^b = \emptyset$ is outcome equivalent to a babbling equilibrium, i.e. $a = \frac{1}{2}$ for every $\eta$, $\omega$ and $\theta$.*

When $M \setminus M^b \neq \emptyset$, the good minister prefers $m^b \in M^b$ to $m \in M \setminus M^b$ at $\theta$ only if:

$$\sum_i p_i(a_i(m^b) - \theta)^2 \leq \sum_i p_i(a_i(m) - \theta)^2,$$

which can be re-written as:

$$\frac{1}{2} \left[ \sum_i p_i a_i(m^b)^2 - \sum_i p_i a_i(m)^2 \right] \leq \theta \left[ \sum_i p_i a_i(m^b) - \sum_i p_i a_i(m) \right].$$  \hspace{1cm} (2.4)

By definition, $\sum_i p_i a_i(m^b) - \sum_i p_i a_i(m) > 0$, so there exists a unique $\theta_0$, such that the good minister sends messages in $M^b$ if and only if $\theta \geq \theta_0$.

**Lemma 2.3.** $\theta_0 < 1$. 

\begin{figure}[h]
\centering
\begin{subfigure}{0.49\textwidth}
  \centering
  \includegraphics[width=\textwidth]{fig1a.png}
  \caption{Good minister’s strategy with (right) and without (left) second order uncertainty.}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
  \centering
  \includegraphics[width=\textwidth]{fig1b.png}
\end{subfigure}
\end{figure}
The rest of the proof hinges on the following Lemma:

**Lemma 2.4.** There exists $\varepsilon > 0$ such that for every $m \in M^b$, 

$$|a_1(m) - \theta_0| + |a_2(m) - \theta_0| > \varepsilon. \quad (2.5)$$

**Proof of Lemma 2.4:** The proof is done by contradiction. Suppose such $\varepsilon$ does not exist, either one of the cases applies:

1. There exists $m^* \in M^b$ such that $a_i(m^*) = \theta_0$ for $i \in \{1, 2\}$.

2. There exists an infinite sequence of messages, $\{m_j\}_{j=1}^{\infty} \subset M^b$, such that 

$$\lim_{j \to \infty} \left\{|a_1(m_j) - \theta_0| + |a_2(m_j) - \theta_0|\right\} = 0. \quad (2.6)$$

**Case 1:** Suppose message $m^*$ exists. Then, the good minister strictly prefers $m^*$ to other messages in $M^b$, since every action distribution induced by every message in $M^b$ shares the same mean, but the one induced by $m^*$ has the lowest variance. Also, the good minister sends $m^*$ only if $\theta \geq \theta_0$, so the mean of $\theta$ when the bad minister sends $m^*$ must be smaller than $\theta_0$. The queen’s sequential rationality condition implies that her action must be strictly decreasing in $\eta$. Since $\eta_1 \neq \eta_2$, so $a_1(m^*) \neq a_2(m^*)$, which leads to a contradiction.

**Case 2:** Suppose there exists $\{m_j\}_{j=1}^{\infty} \subset M^b$, then $\lim_{j \to \infty} a_i(m_j) = \theta_0$ for $i = 1, 2$. Hence, $p_1a_1(m_j) + p_2a_2(m_j) = \theta_0$ for all $j$. Since the good minister’s preference is strictly concave in $a$, for every $m \in M^b$, $m \neq m^*$, there exists $n \in \mathbb{N}$, such that he strictly prefers sending $m_k$ to $m$ (for any $k \geq n$), so from the perspective of the good minister, every message in $M^b$ is being strictly dominated by a message in $\{m_j\}_{j=1}^{\infty}$. Therefore, no good minister will ever send message in $\{m_j\}_{j=1}^{\infty}$, but then, both types of queen’s actions upon receiving $m_j$ should be the same, which leads to a contradiction. \(\Box\)

When receiving $m \in M \setminus M^b$ on the support of the good minister’s strategy, both types of queens will choose the same action. Let $a^1$ be the largest equilibrium action induced by messages in $M \setminus M^b$. One can

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This follows from the fact that every $m_j$ is the bad minister’s best response, which implies the mean of the action taken must be the same. Since $\lim_{j \to \infty} a_i(m_j) = \theta_0$ for $i = 1, 2$, so the common mean must equal to $\theta_0$. 
show that $\theta_0 - a^1 > 0$. This is because otherwise, there exists $\varepsilon > 0$, such that the good minister strictly prefers to deviate and inducing $a^1$ when $\theta = \theta_0 + \varepsilon$.

The queen’s sequential rationality constraint requires that the good minister with $\theta \in [\theta_1, \theta_0]$ induces action $a^1$, where $\theta_1 \equiv 2a^1 - \theta_0$. When $\theta = \theta_1$, the good minister has to be indifferent between $a^1$ and the next equilibrium action, which we denote by $a^2$. Hence, $a^2 - \theta_1 = \theta_1 - a^1$. Iterating this process, the equilibrium is characterized by a partition of equi-length intervals between $[0, \theta_0]$. The good minister sends the same message within each interval and he can never fully reveal his information.

The comparison between the good minister’s equilibrium strategies with and without 2nd order uncertainty is shown in Figure 1. The red line is the range where the good minister pools with bad one, the green line is the fully separating range, and the blue line is the range where the good minister cannot fully reveal his information although he has convinced the queen that he is good.

In contrast to the case with only 1st order uncertainty, the possibility that the minister is biased together with the uncertainty about the queen’s belief adds friction to the information transmission process even when $\theta$ is very small. As shown in Proposition 1 even when the good minister can separate himself from the bad one by recommending a low enough action, the information he conveyed is still coarse.

Intuitively, higher order uncertainty about the sender’s preference induces different types of receivers taking different actions when the message does not fully reveal the sender’s preference. This implies that the good minister’s expected payoff is bounded away from 0 when the state is close to $\theta_0$. This creates an incentive for the ‘threshold type’ good sender ($\theta = \theta_0$) to pool with types with slightly lower $\theta_0$, since doing so can reduce the variance of the receiver’s action. The continuity and concavity of the good sender’s utility function as well as the connectedness of the state space provides him an opportunity to improve his payoff through such a deviation.

When $\theta \in [0, \theta_0]$, the good sender never pools with the bad sender. Upon receiving his message, there is common knowledge between the good sender and the receiver that $\omega = g$. Proposition 1 shows that an arbitrarily small amount of higher order uncertainty can permeate into this common knowledge region $[0, \theta_0]$ and can completely rule out the possibility of full information revelation. On the other hand, suppose that the state space is disconnected at $\theta_0$, for example, the prior of $\theta$ is such that $\Pr(\theta \in [\theta_0 - \varepsilon, \theta_0]) = 0$. The disconnectedness at $\theta_0$ acts as a ‘fire-wall’: higher order uncertainty can penetrate through this ‘wall’ and upset the common knowledge region (i.e. the interval $[0, \theta_0 - \varepsilon]$) only when it is sufficiently strong.
Equilibrium Characterization: I characterize the set of informative equilibria of this game, i.e. equilibria where the receiver’s action is not constant. I use the same notation as before. First, recall that the bad minister always sends messages in $M^b$, which is defined in (2.3). According to Lemma 2.2, I only need to consider the case in which $M \setminus M^b \neq \emptyset$. The following Lemma characterizes the equilibrium properties of the messages in $M^b$

**Lemma 2.5.** For every $m_1^b, m_2^b \in M^b$,

$$p_1 a_1(m_1^b) + p_2 a_2(m_1^b) = p_1 a_1(m_2^b) + p_2 a_2(m_2^b),$$  \hspace{1cm} (2.7)

$$p_1 a_1(m_1^b)^2 + p_2 a_2(m_1^b)^2 = p_1 a_1(m_2^b)^2 + p_2 a_2(m_2^b)^2.$$  \hspace{1cm} (2.8)

**Proof of Lemma 2.5:** Equation (2.7) is true by the definition of $M^b$. I establish equation (2.8) by contradiction. Suppose without loss of generality,

$$p_1 a_1(m_1^b)^2 + p_2 a_2(m_1^b)^2 < p_1 a_1(m_2^b)^2 + p_2 a_2(m_2^b)^2,$$

then the good minister always strictly prefers $m_1^b$ to $m_2^b$, implying that $m_2^b$ is only sent by the bad minister. Since the two types of queens differ only in their beliefs about the minister’s preference, which implies that $a_1(m_2^b) = a_2(m_2^b)$. This together with (2.7) imply that:

$$p_1 a_1(m_1^b) + p_2 a_2(m_1^b) = p_1 a_1(m_2^b) + p_2 a_2(m_2^b) = a_1(m_2^b) = a_2(m_2^b).$$

Using the Cauchy’s Inequality:

$$p_1 a_1(m_1^b)^2 + p_2 a_2(m_1^b)^2 \geq (p_1 a_1(m_1^b) + p_2 a_2(m_1^b))^2 = a_1(m_1^b)^2 = p_1 a_1(m_2^b)^2 + p_2 a_2(m_2^b)^2$$

which is a contradiction. \hfill \square

Therefore, we can define $x$ and $y$ as:

$$x \equiv p_1 a_1(m) + p_2 a_2(m)$$  \hspace{1cm} (2.9)

$$y \equiv p_1 a_1(m)^2 + p_2 a_2(m)^2$$  \hspace{1cm} (2.10)
where \( m \) is an arbitrary element in \( M^b \). Plugging (2.9) into (2.10), we have the following second order polynomials for \( a_1(m) \) and \( a_2(m) \):

\[
p_2a_1^2(m) - 2xp_1a_1(m) - yp_2 = 0,
\]

\[
p_1a_2^2(m) - 2xp_2a_2(m) - yp_1 = 0.
\]

Both polynomials admit a unique positive solution. Hence parameters \((x,y)\) uniquely pin down \((a_1(m), a_2(m))\), which implies the following Lemma:

**Lemma 2.6.** For every \( m_1^b, m_2^b \in M^b \), \( a_i(m_1^b) = a_i(m_2^b) \) for every \( i \in \{1, 2\} \).

Lemma 2.6 says that conditional on the queen’s type, her action upon receiving any message in \( M^b \) must be the same. Hence, it is without loss of generality to assume that \( M^b = \{m^b\} \) in the following discussions.

Recall that \( \theta_0 \) is the threshold such that the good minister sends messages in \( M^b \) if and only if \( \theta \geq \theta_0 \). Since the two types of queens only differ in their beliefs about the minister’s preference, for every \( m \in M \setminus M^b \), the queen will be fully convinced that the minister is good after receiving \( m \). This implies that the optimal action for both types of queens must be the same upon receiving such a message, i.e. \( a_1(m) = a_2(m) \). Let \( a^* \) be the largest action induced by good minister with \( \theta < \theta_0 \), and let \( m' \) to be the message inducing \( a^* \). The good minister is indifferent between \( m^b \) and \( m' \) at \( \theta = \theta_0 \) if and only if:

\[
(\theta_0 - a^*)^2 = (\theta_0)^2 - 2\theta_0x + y.
\]  

(2.11)

Moreover, as established in the proof of Proposition 1, \([0, \theta_0]\) is partitioned into intervals with equal length, and in equilibrium, the good minister tells the queen which interval \( \theta \) belongs to. Let \( n \) be the number of intervals. We have the following relationship between \( \theta_0 \) and \( a^* \):

\[
a^* = \frac{2n - 1}{2n} \theta_0.
\]  

(2.12)

According to Lemma 2.6 we can also derive \( a_1(m^b) \) and \( a_2(m^b) \) from \( \theta_0 \) using the queen’s incentive
compatibility condition after receiving message $m^b$:

$$a_i(m^b) = \frac{\eta_i \frac{1}{2} + (1 - \eta_i)(1 - \theta_0) \frac{1}{2} + \theta_0}{\eta_i + (1 - \eta_i)(1 - \theta_0)} = \frac{1 - \theta_0^2 + \eta_i \theta_0^2}{2(1 - \theta_0 + \eta_i \theta_0)}, \text{ for } i \in \{1, 2\}. $$

This establishes a relationship between $\theta_0$ and $x, y$:

$$\frac{1}{2} \sum_{i=1}^{2} p_i \frac{1 - \theta_0^2 + \eta_i \theta_0^2}{1 - \theta_0 + \eta_i \theta_0} = x, \quad (2.13)$$

$$\frac{1}{4} \sum_{i=1}^{2} p_i \left( \frac{1 - \theta_0^2 + \eta_i \theta_0^2}{1 - \theta_0 + \eta_i \theta_0} \right)^2 = y. \quad (2.14)$$

Hence, we have the following Proposition:

**Proposition 2.** Every informative equilibrium is characterized by $(x, y, \theta_0, a^*, n)$, such that (2.11), (2.12), (2.13) and (2.14) hold. Conversely, every $(x, y, \theta_0, a^*, n)$ characterizes an informative equilibrium if they satisfy (2.11), (2.12), (2.13) and (2.14). The equilibrium strategies are:

- The bad minister always sends $m^b$, the good minister sends $m^b$ when $\theta > \theta_0$.
- The good minister sends $m^j$ when $\theta \in (\frac{j-1}{n} \theta_0, \frac{j}{n} \theta_0]$, where $j \in \{1, 2, \ldots, n\}$.
- The queen takes action $a = \frac{2j-1}{2n} \theta_0$ when she receives message $m^j$, and takes action $a_i(m^b)$ when she receives message $m^b$ and her prior belief is $\eta_i$, with $a_i(m^b)$ given by:

$$a_i(m^b) = \frac{1 - \theta_0^2 + \eta_i \theta_0^2}{2(1 - \theta_0 + \eta_i \theta_0)}. \quad (2.15)$$

The proof of Proposition 2 is contained in the illustration above. Similar to Crawford and Sobel (1982), I show the existence of an upper bound $n$ on the number of partition elements in $[0, \theta_0]$. Moreover, for every positive integer $n$ below $\bar{n}$, there exists an equilibrium where $[0, \theta_0]$ is partitioned into $n$ intervals with equal length.

**Proposition 3.** For every $p_1 \in (0, 1)$, there exists $\bar{n} \in \mathbb{N}$ such that $n \leq \bar{n}$ for every equilibrium. Moreover, for every $1 \leq n' \leq \bar{n}$, there exists an equilibrium where $n = n'$. 

3 Conclusion

This paper studies strategic information transmission problems in which players face second order uncertainty about the sender’s preference. In the canonical uniform-quadratic framework, I show that higher order uncertainty is ‘contagious’ and has a significant effect on the game’s equilibrium outcomes. In particular, the sender can only convey coarse information despite having no conflict of interest with the receiver and his message can fully convince the receiver about his congruence. The insight of my analysis generalizes to richer type spaces.

A Appendix: Omitting Proofs

A.1 Proof of Lemma 2.1

First, I check that it is an equilibrium. The receiver’s incentive constraint is satisfied since:

1. When she receives a message indicating that $\theta = \theta'$ where $\theta' < \frac{1}{1 + \sqrt{\eta}} \equiv \theta_0$, then according to the sender’s messaging rule, her posterior belief assigns probability 1 for $\theta = \theta'$, thus taking $a = \theta'$ is incentive compatible.

2. When she receives message $m^*$, then her posterior expectation of $\theta$ is given by:

$$E[\theta|m, \eta] = \frac{\frac{1}{2}\eta + \frac{1}{2}(1 - \eta)(1 + \theta_0)(1 - \theta_0)}{\eta + (1 - \eta)(1 - \theta_0)} = \frac{1}{1 + \sqrt{\eta}} = \theta_0,$$

implying that it is incentive compatible for her to choose $a = \theta_0$.

The good sender’s incentive constraint is satisfied since $a = \theta$ is her favorite action when $\theta < \theta_0$, and when $\theta \geq \theta_0$, $a = \theta_0$ is her favorite action among which can be induced. The bad sender’s incentive constraint is satisfied since $\theta_0$ is the largest action he can induce.

Next, I show that this is the welfare highest equilibrium for the receiver. Let $a^b$ be the action induced by the bad sender and $a(\theta)$ be the action induced by the good sender when the state is $\theta$. Then $a(\theta) \leq a^b$ for all $\theta$. Moreover, let $m^*$ be the message sent by the bad sender,

$$a^b = E[\theta|m^*, \eta] \leq \max_{\theta \in [0, 1]} \left\{ \frac{\frac{1}{2}\eta + \frac{1}{2}(1 - \eta)(1 + \theta)(1 - \theta)}{\eta + (1 - \eta)(1 - \theta)} \right\} = \theta_0.$$

Hence, for every $\theta \geq \theta_0$,

$$u^g(a(\theta), \theta) \leq u^g(a^b, \theta) \leq u^g(\theta_0, \theta)$$

for all $a(\theta) \leq a^b$. 
The receiver’s expected payoff can be written as:

\[
V'(a(\theta), a^b) \equiv (1 - \eta) \int_0^1 u'(a(\theta), \theta)d\theta + \eta \int_0^1 u'(a^b, \theta)d\theta \\
\leq (1 - \eta) \int_0^{\theta_0} u'(a(\theta), \theta)d\theta + (1 - \eta) \int_{\theta_0}^1 u'(a(\theta), \theta)d\theta + \eta \int_0^1 u'(a^b, \theta)d\theta \\
\leq (1 - \eta) \int_0^{\theta_0} u'(\theta, \theta)d\theta + (1 - \eta) \int_{\theta_0}^1 u'(a^b, \theta)d\theta + \eta \int_0^1 u'(a^b, \theta)d\theta \\
= (1 - \eta) \int_0^{\theta_0} u'(\theta, \theta)d\theta + (1 - \eta) \int_{\theta_0}^1 u'(\theta_0, \theta)d\theta + \eta \int_0^1 u'(\theta_0, \theta)d\theta,
\]

and the last expression is the receiver’s expected payoff in the low separating high pooling equilibrium.

### A.2 Proof of Lemma 2.2

Since \(M \setminus M^b = \emptyset\), every message in \(M\) maximizes the bad minister’s expected payoff, implying that the expected action induced by these messages must be the same. Hence for every \(m_1, m_2 \in M\),

\[
p_1a_1(m_1) + p_2a_2(m_1) = p_1a_1(m_2) + p_2a_2(m_2).
\]

Since \(E[\theta] = \frac{1}{2}\), and the queen’s action equals to her expectation of \(\theta\), the martingale property of belief implies that:

\[
p_1a_1(m_1) + p_2a_2(m_1) = p_1a_1(m_2) + p_2a_2(m_2) = \frac{1}{2}.
\]  \hspace{1cm} (A.1)

I consider two separate cases.

First, suppose there exists \(m \in M\) such that only the bad minister sends message \(m\), then \(a_1(m) = a_2(m) = \frac{1}{2}\), and among all messages in \(M\), \(m\) is weakly preferred by the good minister at all \(\theta\) since all actions result in the same mean but this one has the minimal variance. Hence, for any other message \(m' \in M\), it must be that \(a_1(m') = a_2(m') = \frac{1}{2}\). This is outcome equivalent to a babbling equilibrium.

Second, if every \(m \in M\) is induced by the good minister with positive probability, then the following equation must also be true for every \(m_1, m_2 \in M\):

\[
p_1a_1(m_1)^2 + p_2a_2(m_1)^2 = p_1a_1(m_2)^2 + p_2a_2(m_2)^2.
\]  \hspace{1cm} (A.2)

Let

\[
x \equiv p_1a_1(m) + p_2a_2(m), \quad y \equiv p_1a_1(m)^2 + p_2a_2(m)^2,
\]

This gives:

\[
p_2a_1(m)^2 - 2xp_1a_1(m) - yp_2 = 0,
\]

\[
p_1a_2(m)^2 - 2xp_2a_2(m) - yp_1 = 0.
\]

Both polynomials admit a unique positive solution, implying that they uniquely pin down \(a_1(m)\) and \(a_2(m)\). Therefore, \(a_i(m_1) = a_i(m_2)\) for all \(m_1, m_2 \in M\) and \(i \in \{1, 2\}\).

But on the other hand, the two types of queen’s expected action (over all messages) must be the same, i.e. \(E[a_1(m)] = E[a_2(m)]\). Since for each type, her action is the same across all messages, so \(a_1(m) = a_2(m)\) for all \(m \in M\). This implies that \(a_1(m) = a_2(m) = \frac{1}{2}\), i.e. it is outcome equivalent to a babbling equilibrium.
A.3 Proof of Lemma 2.3

Suppose \( \theta_0 = 1 \), i.e. the good minister never sends messages in \( M^b \), then for every \( m^b \in M^b \), from the queen’s sequential rationality constraint, as well as the fact that the bad minister always prefers larger actions, \( a_i(m^b) = \frac{1}{2} \) for every \( i \in \{1, 2\} \). Since the different types of queens only differ in their beliefs about the minister’s preference, conditional on receiving \( m \in M \setminus M^b \), \( a_1(m) = a_2(m) \). By definition of \( M^b \), \( a_i(m) < \frac{1}{2} \) for every \( m \in M \setminus M^b \). The good minister strictly prefers \( m^b \) than any message in \( M \setminus M^b \) when \( \theta > \frac{1}{2} \), which leads to a contradiction.

A.4 Proof of Proposition 3

First, I show that there exists an upper bound for \( n \). For every \( \theta \in [0, 1] \), define two functions of \( \theta \):

\[
x(\theta) \equiv \sum_{i=1}^{2} p_i \frac{1 - \theta^2 + \eta_i \theta^2}{2(1 - \theta + \eta_i \theta)},
\]

and

\[
y(\theta) \equiv \sum_{i=1}^{2} p_i \left( \frac{1 - \theta^2 + \eta_i \theta^2}{2(1 - \theta + \eta_i \theta)} \right)^2.
\]

Since \( \theta_0 = a^* = \frac{\theta_0}{2n} \), the good minister’s indifference condition at \( \theta = \theta_0 \), i.e. equation (8), implies that:

\[
\left( \frac{\theta_0}{2n} \right)^2 = (\theta_0 - x)^2 + (y - x^2).
\]

Let

\[
h(\theta) \equiv (\theta - x(\theta))^2 + (y(\theta) - x(\theta))^2.
\]

Cauchy’s Inequality implies that \( y(\theta) \geq x(\theta)^2 \), with equality holds if and only if \( \theta \in \{0, 1\} \). When \( \theta \in \{0, 1\} \), \( x(\theta) = \frac{1}{2} \), and \( (\theta - x(\theta))^2 > 0 \). This implies that \( h(\theta) > 0 \) for all \( \theta \in [0, 1] \).

Moreover, since \( h(\theta) \) is continuous when \( \theta \in [0, 1] \), this implies that \( \min_{\theta \in [0, 1]} h(\theta) \) exists and is strictly positive. Thus

\[
\frac{1}{4n^2} \geq \left( \frac{\theta_0}{2n} \right)^2 = (\theta_0 - x)^2 + (y - x^2) = h(\theta_0) \geq \min_{\theta \in [0, 1]} h(\theta) > 0,
\]

which implies the existence of an upper bound for \( n \), which we denote by \( \pi \).

Next, I show that for every \( n \geq 2 \) such that there exists an equilibrium where \( [0, \theta_0] \) is partitioned into \( n \) intervals, there exists an equilibrium where \( [0, \theta_0] \) is partitioned into \( n - 1 \) intervals. Since there exists \( \theta_0 \in (0, 1) \) such that:

\[
\left( \frac{\theta_0}{2n} \right)^2 = h(\theta_0),
\]

then

\[
\left( \frac{\theta_0}{2(n-1)} \right)^2 > h(\theta_0).
\]
The left hand side is always strictly smaller than \( \frac{1}{4} \), while \( h(0) = \frac{1}{4} \). Since both the left hand side and the right hand side are continuous with respect to \( \theta_0 \), the Intermediate Value Theorem implies there exists \( \theta_1 \in (0, \theta_0) \) such that:

\[
\left( \frac{\theta_1}{2(n-1)} \right)^2 = h(\theta_1),
\]

which characterizes an equilibrium where \([0, \theta_1]\) is partitioned into \( n - 1 \) equi-length intervals.

### B Appendix: Generalizations

In this Appendix, I generalize the findings in section 2 by allowing for general utility functions and type spaces. Let \( \theta \in [0, 1] \) be the state of the world and \( a \in A = \mathbb{R} \) be the receiver’s (she) action. The sender (he) knows \( \theta \) and can be either ‘good’ or ‘bad’. The good sender and the receiver share the same preference, which is represented by utility function \( u(a, \theta) \). I assume that \( u \in C^2 \) and

\[
\frac{\partial^2 u}{\partial a \partial \theta} > 0, \quad \frac{\partial^2 u}{\partial a^2} < 0.
\]

Without loss of generality, I adopt the normalization that \( u(\theta, \theta) = \max_{a \in A} u(a, \theta) = 0 \).

The bad sender has utility function \( u^b(a, \theta) = a \). The receiver’s prior on \( \theta \) is \( F(\theta) \), which is absolutely continuous, has full support over \( \Theta \) and has strictly positive density function \( f(\theta) \).

#### B.1 Types and Beliefs

A player’s ‘type’ is characterized by his preference as well as his belief over the joint distribution of the fundamentals and the other players’ type (Mertens and Zamir 1985). In particular, the sender and the receiver’s ex ante type spaces are denoted by \((\Psi, \hat{u}, \hat{\pi})\) and \((\Phi, \hat{\tau})\) respectively. A sender type, \( \psi \in \Psi \), is characterized by his preference \( \hat{u}(\psi) \in \{g, b\} \) as well as his belief over the receiver’s type \( \hat{\pi}(\psi) \in \Delta(\Phi) \). A receiver type, \( \phi \in \Phi \), only contains her belief over the sender’s type: \( \hat{\tau}(\phi) \in \Delta(\Psi) \). For convenience, I assume both \( \Psi \) and \( \Phi \) are finite. When there is no higher order uncertainty, \( \Phi \) is a singleton and \( \Psi \) has two elements. I introduce several conditions on players’ type spaces. I start by defining ‘companion types’ on \( \Psi \):

**Definition 1.** \( \psi \) and \( \psi' \) are ‘companions’ if and only if \( \hat{u}(\psi) \neq \hat{u}(\psi') \) and \( \hat{\pi}(\psi) = \hat{\pi}(\psi') \).

Intuitively, two sender-types are companions if their preferences differ, but they share the same belief over the receiver’s type.

**Definition 2.** \( \Psi \) is rich if every type has a companion type,

Hence, when \( \Psi \) is rich, every possible belief of the good sender is also possible for the bad one and vice versa, which further implies that the ‘belief determine preference’ property fails. Next, I introduce a ‘full rank condition’ on the receiver’s type space:

**Definition 3.** Let \( N = \#\Psi \). \( \Phi \) has ‘full rank’ if there exists \( N \) elements \( \phi_1, \ldots, \phi_N \) in \( \Phi \), such that \( \{ \hat{\tau}(\phi_n) : \} \}_{n=1}^{N} \) are linearly independent vectors.

---

6This is without loss of generality since all elements above are independent of \( \theta \). Without further notice, ‘type’ refers to beliefs and higher order beliefs about the sender’s preference; and ‘state’ refers to the realization of \( \theta \).
The full rank condition is generically satisfied when $\Phi$ has more than $N$ elements, i.e. the number of receiver types is greater than the number of sender types. Next, I introduce the assumptions for my main result:

**Assumption 1** (Full Support). For any $\psi \in \Psi$ and $\phi \in \Phi$, $\hat{\pi}(\psi)[\phi] > 0$, $\hat{\tau}(\phi)[\psi] > 0$.

**Assumption 2** (Richness & Full Rank). $\Psi$ is rich and $\Phi$ has full rank.

Notice that Assumption 2 trivially fails when there is no higher order uncertainty. To separate belief types from payoff types, I use the following notation:

$$\Psi = \{g_1, \ldots, g_n, b_1, \ldots, b_n\}, \quad \Phi = \{\phi_1, \ldots, \phi_k\},$$

where $g_i$ and $b_i$ are companions for every $i \in \{1, 2, \ldots, n\}$. Some of my results require the following condition on $f(\theta)$:

**Assumption 3.** $f(\theta)$ is non-decreasing in $\theta$ and there exists $v$ such that $u(\theta, a) = v(|\theta - a|)$.

Notice that symmetric single-peaked preferences, which are standard in the political economy literature, satisfy Assumption 3.

**Remark:** Let us map back this formulation to the example I have presented in Section 2. Let $\Psi = \{g, b\}$, $\Phi = \{\phi_1, \phi_2\}$. Players’ beliefs are given by:

$$\hat{\pi}(g)[\phi_1] = \hat{\pi}(b)[\phi_1] = p_1, \quad \hat{\pi}(g)[\phi_2] = \hat{\pi}(b)[\phi_2] = p_2 = 1 - p_1,$$

$$\hat{\tau}(\phi_1)[g] = 1 - \eta_1, \quad \hat{\tau}(\phi_1)[b] = \eta_1,$$

$$\hat{\tau}(\phi_2)[g] = 1 - \eta_2, \quad \hat{\tau}(\phi_2)[b] = \eta_2.$$

We can check that assumptions 1 and 2 are satisfied in our example: $g$ and $b$ are companion types, who share the same belief over the receiver’s type. $\Psi$ has 2 elements, and the beliefs of types $\phi_1$ and $\phi_2$ are linearly independent.

### B.2 Equilibrium

Let $M$ be the message space. The sender’s communication strategy is $m : \Psi \times \Theta \rightarrow \Delta(M)$. After receiving $m$, type $\phi$ receiver updates her belief. Let $a_\phi(m)$ be her action, which satisfies:

$$a_\phi(m) \in \arg\max_{a \in A} \int_{\Theta} u(a, \theta)d\mathcal{F}(\theta|m, \phi),$$

where $\mathcal{F}(\cdot|m, \phi)$ is type $\phi$ receiver’s posterior belief on $\theta$ after receiving $m$. The concavity of $u$ guarantees that she always uses a pure strategy. The solution concept is Bayes Nash Equilibrium. I formally define ‘full revealing’:

**Definition 4.** In an equilibrium, type $g_i$ sender fully reveals his information at $\theta = \theta^*$ via message $m$ if every $\phi \in \Phi$ believes that $m$ is only sent when $\theta = \theta^*$.

Next, I introduce the concept of ‘conditional full revelation’ and ‘minimal full revelation’. 
**Definition 5.** An equilibrium is ‘conditional full revealing’ if

1. Full information revelation occurs with strictly positive probability;
2. The good sender fully reveals his information as long as he does not pool with any bad sender.

**Definition 6.** An equilibrium is ‘minimal full revealing’ if there exists a type of sender fully reveals $\theta$ with strictly positive probability.

The first requirement in the definition of conditional full revelation is necessary to rule out babbling equilibria. Intuitively, an equilibrium achieves ‘conditional full revelation’ if the good sender can fully reveal his information as long as his preference becomes common knowledge after communication and every type of receiver chooses her first best action upon receiving the message. By definition, minimal full revelation is a weaker requirement comparing with conditional full revelation. The following distribution condition about $\theta$ is introduced which will later be related to the possibility of minimal full revelation.

**Condition 1** (Contagion Condition). $f(\theta)$ is non-decreasing in $\theta$.

Apparently, the uniform distribution satisfies this condition.

I start from a benchmark result by showing that conditional full revelation equilibria always exist when there is no higher order uncertainty, and any welfare maximizing equilibrium takes this form. It has a ‘low separating high pooling’ feature, in which a good sender fully reveals his information when $\theta$ is small, and pools with the bad sender when $\theta$ is large (Morgan and Stocken 2003).

**Lemma B.1.** When $\Phi$ is a singleton, there exists a conditional full revelation equilibrium in which the good sender fully reveals his information if and only if $\theta < \theta_0$. The good sender pools with the bad sender when $\theta > \theta_0$, and both send a message which induces $a = \theta_0$.

Let $\theta_0$ be the ‘threshold point’, which characterizes a conditional full revelation equilibrium. Let $\theta^*_0$ be the maximum threshold point under $u$ and $F$, we have the following welfare property:

**Lemma B.2.** The conditional full revelation equilibrium characterized by threshold point $\theta^*_0$ is the ex ante welfare highest equilibrium for the receiver.

The proof of this Lemma is the same as the one for Lemma 2.1, which is omitted.

**B.3 Impossibility of Full Revelation**

In this subsection, I present and discuss my main result.

**Proposition 4.** Under Assumptions 1 and 2, there exists no conditional full revelation equilibrium. If the contagion condition is also satisfied, there exists no minimal full revelation equilibrium.

I sketch the proof by decomposing it into several Lemmas.

**Proof of Proposition 4:** The proof is done by contradiction. Pick any good type $g_1$. Suppose a conditional full revelation equilibrium exists, then:
Lemma B.3. There exists a unique threshold $\theta_0 \in (0, 1)$, such that every type of the good sender fully reveals his information if and only if $\theta < \theta_0$, and pools with the bad sender if and only if $\theta > \theta_0$.

**Proof of Lemma B.3:** I begin by showing the existence and uniqueness of the threshold. In any conditional full separation equilibria, there exists $\Theta_1, \Theta_2 \subset \Theta$ such that type $g_1$ good sender fully reveals his information if $\theta \in \Theta_1$ and pool with the bad sender if $\theta \in \Theta_2$. According to the definition of conditional full separation, $\Theta_1 \cup \Theta_2 = \Theta$. $\Theta_1 \cup \Theta_2 \neq \emptyset$, since $\Theta$ is connected under the Euclidean Topology on $\mathbb{R}$, so,

$$\Theta_1 \cap \Theta_2 \neq \emptyset.$$  

Next, I show that $\Theta_1 \cap \Theta_2$ must have a unique element. If not, let $\theta_0, \theta'_0 \in \Theta_1 \cap \Theta_2$ with $\theta_0 < \theta'_0$. There exists $\{\theta_{ij}\}_{i=1}^{\infty}$ and $\{\theta'_{ij}\}_{i=1}^{\infty}$ such that:

$$\lim_{i \to \infty} \theta_{ij} = \theta_0, \quad \lim_{i \to \infty} \theta'_{ij} = \theta'_0,$$

and

$$\{\theta_{ij}\}_{i=1}^{\infty} \subset \Theta_j, \quad \{\theta'_{ij}\}_{i=1}^{\infty} \subset \Theta_j.$$  

For any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|\theta_{i1} - \theta_{i2}| < \varepsilon$ when $i > n$. When $\theta = \theta_{i2}$, type $g_1$ good sender must prefer a message in $M^b$ to the one which induces $a = \theta_{i1}$. Then there exists $m \in M^b$ such that $|a_\phi(m) - \theta_0| \leq \psi(\varepsilon)$ where $\psi$ is strictly increasing with $\varepsilon$ and $\lim_{\varepsilon \to 0} \psi(\varepsilon) = 0$. The same is true for $\theta'_0$. Find $\varepsilon$ such that $2\psi(\varepsilon) < \theta'_0 - \theta_0$, then all types bad sender strictly prefer to send a message in $M^b$ which induces actions around $\theta'_0$ rather than $\theta_0$. So only good senders send the messages in $M^b$ which induces action around $\theta_0$. This leads to a contradiction.

Now, I show the second part of the Lemma. For any $\phi \in \Phi$, $\hat{\pi}(g_1)[\phi] > 0$. From the ‘no illusion’ assumption, $\hat{\pi}(g_1) > 0$. Also, for any $\psi \in \Psi$, there exists $\phi \in \Phi$, such that $\hat{\pi}(\phi)[\psi] > 0$. If the sender reports $\theta < \theta_0$, then all types of receivers believe that he is good. So every type of good sender can separate himself below $\theta_0$. Due to the companion type assumption, every type of bad sender can guarantee himself a deterministic payoff $\theta_0$.

To summarize, the proof of existence is based on the connectedness of $\Theta$ under the Euclidean Topology, which characterizes the threshold between full separation and pooling. The uniqueness is guaranteed by the bad sender’s preference. Full support of the sender’s belief guarantees that all types of receivers take the same action upon receiving a ‘fully separating message’, thus the separation thresholds must be the same for all types.

Let $U_{g_1}^*(\theta)$ be type $g_1$ sender’s expected equilibrium payoff when the state is $\theta$. The next Lemma is the crucial step to establish my result:

**Lemma B.4.** In every equilibrium,

$$\lim_{\theta \to \theta_0^+} U_{g_1}^*(\theta) < 0.$$  

(B.1)

**Proof of Lemma B.4:** I derive one contradiction for each case listed in the main text.
**Case 1:** \( g_1 \) sends \( m^* \) with positive probability. 

Since for all types, the good sender fully reveals his information when \( \theta < \theta_0 \), so he sends \( m^* \) only when \( \theta > \theta_0 \). Since \( a_\phi(m^*) = \theta_0 \), the receiver’s sequential rationality condition implies the existence of a type of bad sender, \( b_i \), who sends \( m^* \) with positive probability, and if the receiver believes that he is the only type sending that message, she will choose an action below \( \theta_0 \).

With an abuse of notation, let \( p(\psi, m) \) be the probability of type \( \psi \) sending message \( m \) and \( \psi(m) \) be the conditional probability distribution of \( \theta \) under \( \psi \) and \( m \).

\[
v(a, \psi|m) \equiv p(\psi, m) \int_\theta u'(a, \theta) d\psi(m).
\]

\( v \) is concave in \( a \). Since the receiver’s optimal action when \( g_1 \) and \( b_i \) sending message \( m^* \) differs,

\[
\left( \frac{\partial v(\theta_0, \psi|m^*)}{\partial a} \right)_{\psi \in \Psi} \neq 0.
\]

Also, if \( a = \theta_0 \) is the optimal action for all types of receivers, then:

\[
\hat{\tau}(\phi) \perp \left( \frac{\partial v(\theta_0, \psi|m^*)}{\partial a} \right)_{\psi \in \Psi}
\]

for all \( \phi \in \Phi \). This violates the full rank condition of \( \Phi \), which is a contradiction.

**Case 2:** \( g_1 \) never sends \( m^* \) (we allow for the non-existence of \( m^* \)).

If Lemma 3.4 is not true, then for any message \( m_j \in M^b \) (\( m_j \neq m^* \)) sent with positive probability by type \( g_1 \), there exists \( \varepsilon_1 > 0 \), such that \( g_1 \) strictly prefers \( a = \theta_0 - \varepsilon_1 \) (which is an equilibrium action in a conditional full separation equilibrium) to message \( m_j \). So there exists \( m_{j+1} \). Since \( g_1 \) assigns positive probability to all types of receivers and the number of receiver-types is finite, so for all \( \phi \):

\[
\lim_{j \to \infty} a_\phi(m_j) = \theta_0.
\]

According to definition, every message in \( \{m_j\} \) must be sent by the bad sender with positive probability, and there is a finite number of bad-sender types, so there exists type \( b_i \), who sends every element in an infinite subsequence, \( \{m_k\}_{j=1}^{\infty} \) of \( \{m_j\} \) with positive probability. This implies that:

\[
E_{b_i}a_\phi(m_k) = \theta_0 \geq E_{b_i}a_\phi(m), \tag{B.2}
\]

for all \( j \in \mathbb{N} \) and \( m \in M \).

Let us consider his companion type \( g_i \). For every \( m \in M^b \), \( m \neq m^* \), there exists \( n \in \mathbb{N} \), such that \( g_i \) strictly prefers \( m_{k_n} \) to \( m \) for any \( \theta \in [\theta_0, 1] \), which implies that \( g_i \) must send \( m^* \) with positive probability. Repeat the proof in Step 1, we get a contradiction. \( \square \)

According to Lemma B.4, if type \( g_1 \) good sender fully separate himself when \( \theta < \theta_0 \), then there exists \( \varepsilon > 0 \), such that

\[
U^*_g(\theta_0 + \varepsilon) < u(\theta_0 - \varepsilon, \theta_0 + \varepsilon), \tag{B.3}
\]

which implies that when \( \theta = \theta_0 + \varepsilon \), he can profitably deviate by mimicking type \( \theta_0 - \varepsilon \) and inducing action \( a = \theta_0 - \varepsilon \), which leads to a contradiction.

Furthermore, under Assumption 3, Lemma B.4 implies that the highest action induced by messages in \( M \setminus M^b \) is bounded away from \( \theta_0 \). Using the same argument as in the proof of Proposition 1, one can obtain a sequence of cutoffs \( \{\theta_0, \theta_1, \ldots\} \) based on the sender’s indifference condition. Moreover, since \( f(\theta) \) is non-decreasing, \(|\theta_i - \theta_{i-1}| \geq |\theta_{i-1} - \theta_{i-2}| \) for every \( i \geq 2 \), which finishes the proof. \( \square \)
References


