Children’s Understanding of the Natural Numbers’ Structure

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Abstract

When young children attempt to locate numbers along a number line, they show logarithmic (or other compressive) placement. For example, the distance between “5” and “10” is larger than the distance between “75” and “80.” This has often been explained by assuming that children have a logarithmically scaled mental representation of number (e.g., Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Siegler & Opfer, 2003). However, several investigators have questioned this argument (e.g., Barth & Paladino, 2011; Cantlon, Cordes, Libertus, & Brannon, 2009; Cohen & Blanc-Goldhammer, 2011). We show here that children prefer linear number lines over logarithmic lines when they do not have to deal with the meanings of individual numerals (i.e., number symbols, such as “5” or “80”). In Experiments 1 and 2, when 5- and 6- year-olds choose between number lines in a forced-choice task, they prefer linear to logarithmic and exponential displays. However, this preference does not persist when Experiment 3 presents the same lines without reference to numbers, and children simply choose which line they like best. In Experiments 4 and 5, children position beads on a number line to indicate how the integers 1-100 are arranged. The bead placement of 4- and 5-year-olds is better fit by a linear than by a logarithmic model. We argue that previous results from the number line task may depend on strategies specific to the task.

Keywords: numerical cognition, number concepts, number development
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By five years of age, children understand enough about the positive integers to retrieve the right number of items in response to a question. They can use their counting routine to respond to requests to “Give me one balloon,” “Give me two balloons,” and so on, up to the largest numeral on their list of count terms—for example, “Give me twenty balloons” (Fuson, 1988; Gelman & Gallistel, 1978; Wynn, 1992). But how much do these children know about the overall structure of the integers? Of course, adults know that each positive integer has a unique immediate successor (2 is the immediate successor of 1, 3 is the immediate successor of 2, …), and they understand that, for any integer, the arithmetic difference between its immediate successor and that integer is the same (the difference between 1 and 2 is the same as the difference between 2 and 3). This fact induces a linear structure on the integers: Successive integers are equally spaced.

It is an open question, however, whether children have a similar grasp of this linear structure. According to one prominent theory of number representation, which we will discuss momentarily, children do not fully understand that this structure applies to the integers 0 to 100 until they are in second grade and do not know that the property applies to 0 to 1000 until about fourth grade. The theory in question implies that children of younger ages believe that the difference between successive integers shrinks, with smaller differences between larger integers. For such children, the difference between 13 and 14, for example, is smaller than that between 3 and 4.

The experiments we report here provide evidence that four- to six-year-olds prefer the integers between 0 and 100 to be linearly arrayed, even though they may not display this knowledge in some tests of their number abilities. If this is so, then children just beginning to learn school-based arithmetic are in a stronger position to appreciate facts about addition and subtraction than the earlier theory implies. For example, they should find it natural to think that $2 - 1 = 3 - 2 = \ldots$ and that $1 + 1 = 2$, $2 + 1 = 3$, $3 + 1 =$
4, and so on up. The evidence here comes from novel tasks in which children choose between alternative structures for the integers or in which they position tokens to represent the integers. Both types of tasks suggest that children favor a linear distribution of integers over rival distributions in external displays.

The Number-Line Placement Task

One popular method for exploring number representation in children is a number-line placement task (e.g., Siegler & Opfer, 2003). In this task, participants (typically kindergarten or grade-school children) receive a bounded number line with labeled endpoints of 0 and 100 (or 0 and 1000), but which is otherwise unmarked. Participants then receive a numeral within this range, like “78,” and they are asked to place a mark at the appropriate location on the line. This process is repeated with new numerals (and new blank number lines) over a series of trials. (Alternatively, participants may be asked to estimate the numbers corresponding to given marks on the line.) Investigators can then compare the children’s subjective estimates to the actual values to reveal how their placement of the numerals relates to the true distribution. Researchers have used the number-line placement task to examine number representation in children (e.g., Barth & Paladino, 2011; Berteletti et al., 2010; Booth & Siegler, 2006; Ebersbach, Luwel, Frick, Onghena, & Verschaffel, 2008; Moeller, Pixner, Kaufmann, & Nuerk, 2009; Siegler & Opfer, 2003), adults (e.g., Candia, Deprez, Wernery, & Nunez, 2015; Cohen & Blanc-Goldhammer, 2011; Landy, Silbert, & Goldin, 2013; and Rips, 2013), and cross-culturally in indigenous Amazonian and New Guinean people (Dehaene, Izard, Spelke, & Pica, 2008; Nuñez, Cooperrider, & Wassmann, 2012). In this article, we will refer to this standard procedure as the number-line placement task, and we will use number-line task as a more general label for procedures involving number lines, such as the ones we report here. We will also use “numeral” to stand for a symbol for a number, such as a number word (“five”) or an Arabic number (“5”), and we reserve “number” itself for the abstract entity that these symbols stand for. Thus, in the number-line placement task, children are positioning numerals on a number line.

In some experiments, a logarithmic function provides the best fit to young children’s number-line estimates (e.g., Siegler & Opfer, 2003; Booth & Siegler, 2006). Although children widely space the
numerals 1 to 10, they place numerals larger than 10 increasingly close together (see the log function in Figure 2 for an idealized version). The distribution of numerals shifts with development: By grade two, children’s responses are approximately linear for the number range 0 to 100, but still logarithmic for larger ranges such as 0 to 1000. Around the fourth grade, as children accumulate education and experience, their estimates become increasingly linear for the 0 to 1000 range.

A leading interpretation of these results is that they reflect the mental representation of number by the approximate number system. According to one such theory, the approximate number system depicts a number $n$ as a continuous quantity proportional to $\log(n)$. Just as the nervous system translates many physical continua (e.g., air pressure, mass, and luminosity) into psychological representations (e.g., perceived loudness, weight, and brightness) that vary as a log or power function of their physical counterparts, the approximate number system similarly translates the number of objects in the perceptual field into a continuous psychological representation that is proportional to the log of that number. Children’s understanding of the number words, at least in the context of the number-line placement task, initially maps onto the same representation. But with age and education, children acquire the linear representation we see in adults (Dehaene et al., 2008; Siegler & Opfer, 2003). Because performance with number lines predicts performance in other numerical tasks (e.g., Booth & Siegler, 2008; Geary, 2011; Laski & Siegler, 2007; Schneider, Grabner, & Paetsch, 2009; Siegler, Thompson, & Schneider, 2011; but see also LeFevre, Jimenez Lira, Sowinski, Cankaya, Kamawar, & Skwarchuk, 2013; Link, Nuerk, & Moeller, 2014), investigators have used the number-line placement task both as a measure of the effectiveness of mathematical interventions (Siegler & Ramani, 2008, 2009; Whyte & Bull, 2008) and as an intervention in itself. Improving linear representation improves mathematical abilities in other areas (e.g., Ramani, Siegler, & Hitti, 2012).

**Alternative Interpretations**

However, these conclusions have come under scrutiny in recent research. One issue is whether children’s performance on the number-line placement task provides convincing evidence of a logarithmic-to-linear shift in number representation. On the one hand, children’s estimates in other numerical tasks do
not always show a similar log-to-linear pattern with age (Barth, Starr, & Sullivan, 2009; Lipton & Spelke, 2005). On the other hand, children also show an apparent log-to-linear shift in mapping sequences that do not have any inherent linear magnitude, such as the letters of the alphabet (Hurst, Monahan, Heller, & Cordes, 2014). In addition, investigators have questioned whether children’s initial estimates are truly logarithmic rather than following some other decelerating form (e.g., Ebersbach et al., 2008; Moeller et al., 2009).

Some alternative explanations of the initial number-line placements center on the greater variability that appears in estimates of larger numerals. Mental representations of many quantitative dimensions exhibit scalar variability: greater variance in estimates of larger magnitudes (Gallistel & Gelman, 2000; Gibbon, Church, & Meck, 1984). When combined with other factors, this may lead to log-like data that do not necessarily reflect a logarithmically scaled internal number line (Cantlon, Cordes, Libertus, & Brannon, 2009). One such factor is that number lines in the estimation task are typically bounded with numerical anchors (e.g., at 0 and 100). These endpoints necessarily constrain participants’ responses, since they must put their marks on the line, within these boundaries. Bounded number lines truncate the variability of estimates for larger numerals, and as a result, the mean estimates can be artificially compressed as they approach the boundary, giving rise to log-like distributions (Cohen & Blanc-Goldhammer, 2011; Rips, 2013).

Boundary points may influence performance in other ways. Barth and colleagues (Barth & Paladino, 2011; Slusser, Santiago, & Barth, 2013; Sullivan, Juhasz, Slattery, & Barth, 2011) argue that the number-line placement task is best seen as a proportion task (see Hollands & Dyre, 2000; Spence, 1990): Participants judge the distance of a numeral along the line relative to the distance between the two endpoints (or other reference points). The data are well fit by a power function appropriate for a proportion task (Rouder & Geary, 2014). (Because of the uncertainty surrounding the exact function that best describes these data, we will refer to the initial distribution of numeral placements as log-like rather than logarithmic.) In accord with this evidence, when children know the interval on the number line that
corresponds to one unit (the distance between 0 and 1) and when the line itself is unbounded above, they do not show the typical log-to-linear shift with age (Cohen & Sarnecka, 2014).

**Issues with the Number-line Placement Task**

As we’ve already hinted, a central question about the number-line estimates is how faithfully they depict children’s mental representations of number. Investigators have often taken the number-line estimates to be direct and accurate reflections of these mental representations, but recent evidence suggests that the task may not be so straightforward. The effect of boundary points, just discussed, is one such complicating factor (e.g., Slusser et al., 2013). So is familiarity with the number word stimuli.

Even adults create log-like distributions for familiar number ranges, such as 0 to 1,000, when they receive the numeric stimuli in an unfamiliar format (e.g., when 64 is presented as .002 x 10⁴.⁵) (Chesney & Matthews, 2013). Similarly, college students, who know the linear structure of natural numbers, show log-like placement of unfamiliar numerals. These college students placed both conventional but large numerals (e.g., twenty-one duodecillion and thirty-two) and fictitious numerals (thirty katrillion and fifty-five) in a compressed log-like distribution (Rips, 2013). The surface structure of conventional number labels may also influence where people place numerals along the number line. About a third of college-student participants behave as if large numerals comprising the short scale (e.g., thousand, million, billion, trillion) were equally spaced, although in fact these magnitudes increase exponentially (Landy, Silbert, & Goldin, 2013).

However, magnitudes need not be fictitious, extremely large, or presented in unusual formats to drive log-like placement in adults. Adults respond logarithmically in a number-line placement task with atypical anchors (1639 to 2897), although they respond linearly to the same task with standard anchors of similar magnitude (2000 to 3000) or range (0 to 1258) (Hurst et al., 2014). Likewise, under conditions of uncertainty (e.g., attentional demands or unfamiliarity with numeric format), adults may rely on central tendency, leading to a log-like pattern of results (Anobile, Cicchini, & Burr, 2012; Cicchini, Anobile, & Burr, 2014). Together, these findings suggest caution in taking the number placement results as faithful evidence for a log-like mental representation of number.
Overview of the Present Experiments

The experiments reported here make the case that young children recognize the correct linear distribution of the numbers 1-100 when they see the distribution (or create it) on an external number line. Although these children are of the same age as those who exhibit log-like placement in many earlier studies, they prefer linearly to logarithmically distributed number lines. Correct placement of an individual numeral requires children to understand the meaning of the target number word or symbol, such as “eighty-three” or “83,” and to map this meaning to the appropriate position on the line. Because terms for larger numerals are less frequent in natural language than terms for smaller ones (Dehaene & Mehler, 1992), children may be correspondingly less familiar with them. Uncertainty about the meanings of these larger items can then produce log-like distributions for the reasons just discussed. This could be true even though the children know that successive integers are equally spaced. To overcome this problem here, we directly ask children about the correct distribution of the positive integers rather than asking them about the position of individual numerals. We predict a preference for a linear distribution over a logarithmic distribution (and over other alternatives) under these conditions.

One hint that younger children (first-graders) prefer linear over log spacing is that they find it easier to learn (through feedback) linear placement of the numerals on a number line than logarithmic placement of the same numerals (Huber, Moeller, & Nuerk, 2014). Similarly, first to third graders show a linear preference, given a choice of several “rulers” that might help them locate the position of specific numerals in a number-line placement task (Petitto, 1990). The “rulers” were strips marked off with divisions labeled “0,” “10,” “20,” “30,”…, “100,” but where the divisions and their labels appeared on different strips with: (a) equal spacing, (b) increasing spacing as the numbers increased, (c) decreasing spacing, (d) alternating spacing (big space between 0 and 10, small space between 10 and 20, big space between 20 and 30, etc.), and (e) irregular spacing. When asked “If you were going to use these rulers to help you solve number-line problems, which one would be best to use?” children either selected the ruler with equal spacing (37.5% to 76.2% of children across grade level) or claimed that no ruler was best. No child selected the decreasing (log-like) or increasing (exponential-like) rulers. Of course, children were
expressing their preference for the rulers rather than for the spacing of the numerals, and the children themselves were at or approaching the age at which earlier studies have found linear placement of numerals in the 0-100 range (the youngest participants were in late first grade). But we can take advantage of a similar technique to probe younger children’s knowledge of the correct distribution for the integers.

Experiments 1 and 2 ask children to choose which of two lines best represents the numbers 1 to 100. Children see pairs of number lines containing unlabeled points that (they are told) represent numbers. On one trial in Experiment 1, for example, they see the linear and logarithmic lines from the collection in Figure 1. For each pair, we ask them to choose which line shows the way the numbers should go on the line. Experiment 3 compares the results of this task to one in which children simply express their personal preference (Which one do you like best?) for the lines from Experiment 2 rather than their belief about numbers (How do the numbers go on the line?). Experiments 4 and 5 ask children to place beads on a line to indicate the correct distribution of the same numbers. We then fit linear and logarithmic functions to the distributions they create to see which best predicts their positioning.

These experiments provide some of the first direct evidence that five-year-olds think the natural numbers 1-100 are uniformly distributed. Although children of this age show robust log-like results in the standard placement task, the present findings suggest that this task does not exhaust their knowledge of the positive integers. Of course, these experiments cannot resolve the issue of where this additional knowledge comes from. Experiment 3 shows that children’s preference for linear arrangements is not due to the lines’ pure non-numeric attractiveness, but it could derive from a variety of other sources, including an innate understanding of the successor function (Leslie, Gallistel, & Gelman, 2008; Leslie, Gelman, & Gallistel, 2008), explicit instruction at home or at school, or experience with linear displays of numerals on school number lines, rulers, or analog thermometers. Our task here is simply to document the linear preference as a first step in understanding it.
Experiment 1: Number Line Comparison

Our aim in this experiment is to find out whether children favor linear number lines over those with alternative distributions. We use the five lines that appear in Figure 1: a linearly distributed number line, a logarithmically distributed line, an exponentially distributed line, and two random lines, each with 100 points. We present each child with all ten possible pairs of these lines, and for each pair they choose which line best shows the way the numbers 1-100 go on the line. Our main interest is the comparison between linear and logarithmic lines: Children should favor the linear over the logarithmic distribution. We choose the paired-comparison method since it provides a clear test of this difference. However, we also include the exponential and random lines as baselines for comparison. No current theory attributes to children an exponential or random representation for the positive integers. So if the children do prefer linear or logarithmic distributions, this should show up as a preference for the linear or log patterns over the exponential and random ones.

Method

To introduce the participants to the concept of a number line, we created a story about two boys, Johnny and Ernie, who liked to think about and draw pictures of numbers on their playground at school (with a wink to Benacerraf, 1965). In the story, the boys draw simple line-segment arrows and place stones on them to represent numbers. After this introduction, participants saw a pair of number lines on each trial, and the experimenter asked them to decide which of the two lines best showed how the numbers 1-100 go on the line. The participants saw all ten possible pairs of the number lines, and the percentage of choices for each line provides the dependent measure for this experiment.

Participants. Fourteen 5- and 6-year-old children (M = 6.35 years, Range = 5 years 7 months to 6 years 11 months) were recruited in the Danville, Kentucky community. Seven children were kindergarteners and seven were first graders. No additional demographic information is available for these participants or for those in later studies.

Procedure. After a warm-up period of crayon drawing with the experimenter, each child was introduced to the task by hearing a story about Johnny and Ernie, and the way they talk about and model
numbers. Children viewed images of number lines (an empty, unlabeled number line and a number line with blue circles in a haphazard arrangement) that were consistent with the story in order to acclimate them to the diagrams for the subsequent number-line task. The experimenter explained that because Ernie and Johnny did not always agree about the pictures that they made, she wanted to hear what the participant thought. To make sure the participants understood the information in the story, the experimenter showed the children a blank “arrow” and asked them to point to where the smaller numbers should go [correct answer: on the left] and where the larger numbers should go [correct answer: on the right]. The experimenter also asked what the stones (small circles on the line) were supposed to represent [correct answer: numbers]. If a child answered these questions correctly, the experimenter went on to the number-line choice task. If the child did not, the experimenter explained Johnny and Ernie’s game until the child was able to answer the questions correctly.

After verifying that the children understood the diagrams, the experimenter presented a series of forced-choice judgments for each possible pairing of the lines in Figure 1. For each judgment, the experimenter explained, “One of the boys thinks that numbers 1 to 100 should go on the arrow like this [gesture to 1st line]. The other boy says, ‘That’s silly!’ He thinks they should go on that way [gesture to 2nd line]. Which of the boys do you think is more likely to be right?”

After the completion of the line judgments, the experimenter asked each participant questions to establish number knowledge, including how high they could count, and estimation problems, such as the number of students in their school. The correct answers to the estimation questions vary from one participant to another, and we do not analyze them here. However, we discuss the highest number to which the children reported they could count after we take up the results from the main task.

**Materials.** Each of the five number lines consisted of 100 unlabeled 1 mm circles, distributed on a 30 cm right-oriented arrow in either a linear, logarithmic, exponential, or random arrangement. Figure 1 displays these number lines at a reduced size. In the actual stimuli, the distance between the points was distinct because of the enlarged size of the line. To produce the comparison pairs of lines for the forced-choice task, we created 10 11” x 17” cards, one for each possible pairing of the number lines.
On each card, we printed in color one of the 10 possible pairs. The lines were centered on the card, one above the other and separated by 11 cm. In order to control for the positioning of the number lines, we created a second set of stimuli in which the positions of the lines were inverted. Participants were randomly assigned to a stimulus condition. The presentation order of the line comparisons was randomized anew for each participant.

Results and Discussion

As predicted, participants preferred linear lines (chosen 82% of the time across the pairs in which they appeared) to both logarithmic (64%) and exponential (52%) lines, and they preferred log and exponential lines to the two random ones (32% for random line 1 and 20% for random line 2, as shown in Figure 1). For each participant, we ranked the number of times he or she chose the linear, log, exponential, and random line types over all pairs. We then applied Friedman’s analysis of variance by ranks to these data. The results indicated a significant difference between line types, $\chi^2(4) = 30.04$, $p < .001$. Planned comparisons based on the same rank test found a significant advantage for the linear over the exponential line ($\chi^2(1) = 4.57$, $p = .032$) and an advantage for the linear over the logarithmic line that was on the cusp of significance ($\chi^2(1) = 3.77$, $p = .052$).

We also examined the children’s choice of a linear over a log line as a function of the highest number to which they could count (as self-reported at the end of the test session). For this purpose, we used logistic regression, scoring a child’s response 1 if she chose the linear over the log line and 0 if she chose the log over the linear line. Participants’ self-reported highest count term ranged from 30 to 1,000,000 with a median of 105. However, this variable did not prove a significant predictor of the choice, $b = -1.10 \times 10^{-6}$, $SE = 1.56 \times 10^{-6}$, Wald $\chi^2(1) = 0.492$, $p = .48$. (A similar regression using the log of the self-reported highest number the child could count, rather than the number itself, produced the same null result.) Of course, children’s self-reported highest count may not be an accurate measure of their true counting abilities, and it remains possible that their actual counting would do a better job of predicting their number-line performance. However, we also found no correlation between the age of the children (in days) and the number of times they selected the linear over the log line, $b = 0.002$,
SE = 0.004, Wald $\chi^2(1) = 0.350, p = .55$. One might similarly expect children in first grade to be more likely than kindergarteners to select the linear line, but a comparison of the ranked choices for the two grades produced no interaction between grade and line choice, $F(4,44) = 1.08, MS_e = 1.163, p = .38$.

If the children’s preference in this number-line choice task followed their performance on the number-line placement task, we would expect them to favor a logarithmic distribution. This was not the case. The participants tended to prefer a linear distribution as a more appropriate representation of numbers 1 to 100 over the other possible arrangements in Figure 1. However, one possibility is that the logarithmic arrangement in Figure 1 was not similar enough to the specific log number line that children believe is correct. In Experiment 2, we address this concern by creating a logarithmic number line that conforms to the best-fitting empirical function that Siegler and Opfer (2003) obtained.

**Experiment 2: Comparison with Empirically Derived Number Lines**

The number lines of Experiment 1 were scaled so that the points filled the entire length of the line, as Figure 1 shows. However, the positions that children generated in previous number-line placement tasks do not usually reach to the upper end of the scale. Median positions of the higher numerals are compressed toward the center of the line, leaving the upper portion bare (e.g., Siegler & Opfer, 2003; Siegler & Booth, 2004). Perhaps these more compressed lines come closer to the way children think the positive integers are distributed than do the stimulus lines of the first experiment. In order to give the logarithmic lines their best chance, Experiment 2 replaces the logarithmic number line in Figure 1 with one that follows the best-fitting logarithmic function reported by Siegler and Opfer (2003). If children’s representation of the integers is similar to this log function, then they should prefer it to linear, exponential, and other functions.

Children’s preference for the linear number line in Experiment 1 could also be due to their favoring spatial symmetry rather than linearity. The linear line in that experiment was the only one that had obvious left-right symmetry (see Note 2). To evaluate this alternative explanation, we replaced one of the random lines from the earlier experiment with one in which the points converged and then diverged symmetrically from the line’s center (see the line labeled “symmetric” in Figure 2). Our own hypothesis,
however, is that neither the exact shape nor the asymmetry of the non-linear functions fully accounts for children’s preference for the linear distribution. Rather we conjecture that the reason for their choice is their belief that the positive integers 1-100 are equally spaced.

**Method**

This experiment replicates the procedure of Experiment 1, but substitutes a new set of number lines, those in Figure 2.

**Participants.** Twenty-four 5-year-olds ($M = 5.48$ years, Range = 5 years 1 month to 5 years 11 months) were recruited from the Evanston, Illinois community. We were able to verify with the children’s schools that the curriculum for eight of these children did not include instruction on either number lines or rulers. Formal information about the curriculum of the remaining children was not available. However, the ranked choices of the number lines for these two groups did not differ, $F(4, 88) < 1$.

**Materials and procedure.** In Experiment 2, we changed the logarithmic number line to one that duplicated the best-fitting log function to the empirical placement data, as reported by Siegler and Opfer (2003). For reasons of comparability, we also replaced the exponential line with one we created by left-right reversing the logarithmic line. As Figure 2 shows, this meant that the points on the log line ended, and those on the exponential line began, about a fourth of the way from the nearest end point. The lines used this spacing because it corresponded exactly to Siegler and Opfer’s best-fitting functions; that is, in that study, children’s average placements of numerals also occupied only about the first three-fourths of the number line. In addition, one of the random lines was replaced with a line whose spacing became more compressed, symmetrically, toward the center. The full set of lines appears in Figure 2. Participants saw each of the 10 possible pairs of these lines, and for each pair, they decided which line was more likely to show the way the numbers 1 through 100 go on the line.

Because neighboring points on the lines were sometimes more compressed than in Experiment 1, we presented the lines this time on larger poster-sized (43 x 33 in) paper so that the points were distinguishable. The lines themselves were 81 cm long. As in Figure 2, the points appeared in blue on red
lines. In addition to these changes, the experiment replaced Ernie and Johnny with two ostrich puppets who enjoy making pictures of numbers.

**Results and Discussion**

As in Experiment 1, participants had preferences among the lines. The children chose the linear number line on 67% of trials in which it appeared, the log line on 45%, and the exponential line on 31%. All these figures are somewhat lower than the comparable ones in Experiment 1, but they preserve the same preference ordering as in that experiment. By contrast, children chose the random line somewhat more often than before—on 50% of relevant trials—and the new symmetric line also attracted 56% of choices. The differences among the lines remained significant, however, according to a Friedman analysis of variance on ranks, $\chi^2(4) = 13.66$, $p = .008$. Planned comparisons showed that the children preferred the linear line over the logarithmic line ($\chi^2(1) = 5.00$, $p = .025$) and the linear line over the exponential line ($\chi^2(1) = 8.05$, $p = .005$). Recall that the logarithmic line followed the best-fitting equation that Siegler and Opfer (2003) obtained for the empirical distribution of number placements in their experiment. There is no evidence here, however, that participants found this line to be an especially good representation for the positive integers. Instead, participants favored linear over log spacing. One could argue that the best fitting-equation for the averaged data may not represent the best log function for an individual child. It is possible that if we had obtained preliminary number-line placement data from each child and then used these positions as the basis for an individually-tailored number line, we would have found that the same child preferred this line to others. Although this possibility is an interesting one, we do not pursue it here because of the difficulty of separating carry-over effects from true preferences for the number distributions.

In general, participants appeared less certain of the correct choice for the paired comparisons than in the preceding study, since the percentage of choices for the individual lines, just reported, tends to be closer to 50%. The shifts in percentages between experiments may be due, at least in part, to the presence of the new symmetric line. Because of the paired-comparison method, the popularity or unpopularity of one line will affect the percentage of times all other lines are chosen. Participants in this experiment...
tended to choose the linear line over the symmetric (but non-linear) line when they were pitted against each other, but the difference was not significant \((\chi^2(1) = 1.19, p = .275)\). Thus, it remains possible that part of the advantage of the linear line is due to the symmetry of the spacing rather than to its mathematical properties. Experiment 3 provides a test of this possibility.

**Experiment 3: Pure Preferences for Spacing**

Experiments 1 and 2 asked participants to pick the number line that best showed how numbers should go on the line. But, of course, the experiments provide no guarantee that the participants followed these instructions and responded to the lines’ mathematical properties rather than to their appeal as pure designs. Perhaps five- and six-year olds happen to think that evenly spaced circles are more pleasing than unequally spaced ones and selected the linear line for that reason. This interpretation would suggest, then, that children’s preferences would remain the same as those in the previous experiments if we simply ask them to choose the line they liked best (rather than the line that best represents the numbers). In this experiment, we test this interpretation using the lines from Experiment 2 (see Figure 2).

**Method**

This study used a procedure very similar to that of Experiment 1. Participants again learned about Ernie and Johnny, who liked placing stones on a line in their playground. However, the story made no mention of the stones representing numbers. Instead, the characters simply argued about how the stones should best be arranged. Following the instructions, a participant received ten forced-choice trials, corresponding to all possible pairs of lines from Figure 2. On each trial, the experimenter told the participant that one of the characters thought the stones should go in the way shown by the top line, whereas the second character thought the stones should go in the way shown by the bottom line. The participant was then asked, “Which drawing *do you* like best?”

**Participants.** Participants in this experiment were 20 five-year olds \((M = 5.42; \text{Range: 4 years 11 months to 6 years 0 months})\) recruited from Selinsgrove, PA area preschools and kindergartens.
Results and Discussion

When participants decided which lines they liked best (rather than which best represented the natural numbers), the percentage of choices for the individual lines fell close to chance and to each other. Participants chose the exponential line on 56% of the trials in which it appeared, the linear line on 54%, the logarithmic line on 49%, the symmetric (converging and diverging) line on 46%, and the random line on 45%. An analysis of variance on ranks found no differences among these preferences, \( \chi^2(4) = 2.33, p = .68 \). There is no evidence, then, that children favored linear spacing over log or exponential spacing in Experiments 1 and 2 because they liked evenly spaced points better than points that converged, diverged, or both. In fact, the ordering of preferences in this experiment for the exponential over the linear line is in the wrong direction to explain the consistent preference for linear over exponential lines in the preceding experiments.

Because Experiments 2 and 3 used the same line types, we can compare the results from these experiments to check whether participants responded differently to instructions to choose the line that represents numbers (Experiment 2) than to instructions to choose the line they liked best (Experiment 3). The Friedman rank test that we have used so far is limited to two-way analyses, but the comparison between experiments requires a three-way analysis for the effects of experiment, line type, and participant. We therefore used a general linear model applied to the ranked choices (Conover & Iman, 1981). The results of this analysis produced the expected interaction between type of line and experiment, \( F(4, 172) = 2.74, MS_e = 2.030, p = .03 \). Thus, children’s preferences for number representations do not mirror their preferences for designs.

We noted earlier that symmetry might have played a role in children’s choice of the linear number line. In Experiment 2, the two most popular items were the linear and the symmetric (converging-diverging) lines. But the data from the present study show that symmetry was not an especially attractive property in its own right. Participants selected a symmetric line (the linear or the converging-diverging line) over one of the remaining asymmetric lines (log, exponential, or random) on only 50% of trials that paired a symmetric with an asymmetric line. Although symmetry may have played a role in the results of
the preceding experiments, this was apparently not because children like symmetric lines better than asymmetric ones. That is, children may prefer symmetric lines to asymmetric ones when they believe that the points on the line represent numbers (as in Experiments 1 and 2) but not when the points do not stand for numbers. However, the present experiment rules out the most obvious alternative reason for the preference for linear lines in Experiments 1 and 2 (i.e., a pure preference for symmetric designs) and so puts pressure on the idea that participants in those experiments were responding to symmetry rather than to linearity.

Experiments 1 and 2 demonstrate that when we alter the number line task, removing the need for individual placement of specific numerals and probing for a more holistic representation of number, children do not prefer logarithmic models over linear ones. This suggests that log placements in earlier experiments were a product of children’s unfamiliarity with the larger number words, to the lines’ endpoints constraining positions of the numerals, or to other factors that we described earlier. The present results, like the earlier ones cited in the introduction, provide a reason for wariness in interpreting the number-placement data as a direct reflection of children’s mental representation of the positive integers.

Of course, the choice task of Experiments 1-3 imposes some constraints of its own. This task provides some insight into children’s thinking about the number line, while removing difficulties created by specific numeric symbols. However, it is a recognition task, whereas much of the data that have been interpreted as supporting a logarithmic mental number line depend on the production of numeral positions. In Experiments 4 and 5, we explore how children produce number distributions rather than choose among preconfigured distributions.

**Experiment 4: Number Placement without Numerals**

Earlier evidence for log-like distributions comes from studies, reviewed earlier, in which children indicate the position of specific numerals, such as “78,” on a number line. We have suggested that the children’s difficulty in locating the correct position may stem from their unfamiliarity with the meaning of the larger numerals (i.e., number symbols), and if so, they may be able to produce more linear distributions, provided we don’t require them to map individual numerals to particular positions. In this
experiment and the following one, children receive a set of 100 beads threaded on a length of string, and we ask them to show us how the numbers go on a number line by moving the beads to their correct positions on the string. No mention is made of specific numerals. If children prefer a logarithmic distribution of the integers, with the smaller numbers more widely separated than the larger ones, they should produce a compressed, log-like distribution, similar to those of earlier studies. However, if they prefer the integers to be evenly spaced, they should instead produce a more linear arrangement.

**Method**

**Participants.** Nineteen 4- and 5-year-old children ($M = 4.91$, Range = 4 years 2 months to 5 years 8 months) were recruited from the Selinsgrove, PA area. Data from one participant were lost after she returned beads to the starting position before the experimenter could remove the apparatus.

**Materials and procedure.** After spending time drawing and conversing with the experimenter, participants completed a Number Knowledge Test (Okamoto & Case, 1996). They then answered several open-ended questions about themselves and their school and community, including whether they had used a ruler at home or at school.

In the main portion of the experiment, we introduced the children to two penguin puppets in a story that paralleled those in Experiments 1 and 2. The experimenter explained that the puppets liked to think about numbers, and would often make pictures of numbers with beads on a string. This was demonstrated on a number line comprising a 43 cm long string with 13 beads. However (the experimenter explained), because the penguins often disagreed in what they said about numbers, the experimenter wanted to know what the child thought. The experimenter then removed the initial number line and brought out a larger number line consisting of a 100 cm string stretched on a wooden frame. The string contained 100 3-mm black beads. Participants were randomly assigned to one of three starting positions: For one group of participants, the beads were in a contiguous cluster at the left of the string; for a second group, they were clustered in the middle; and for a third group, they were clustered at the right.

The experimenter asked the child to pretend that each bead was a number, as the penguins did, and to “arrange the beads the way that numbers should go, so you can count with them, add them
together, and do all the other stuff that you can do with numbers.” The participants’ positioning of the beads was obviously constrained by the lengths of the beads and the string; for example, participants could not place the 30th bead in exactly the same position as the 31st. Note, however, that each bead occupied 3/1000ths of the length of the string, which left room for both linear and log arrangements. (See Experiment 5 for another approach to positioning.) Participants had as much time as they desired for the task. When the child indicated that she had finished, the experimenter removed the number line from the area and then debriefed the participant and parent.

Results and Discussion

We asked participants in this experiment to move beads on a string to show us how the numbers are arranged, and the central question is whether their placement comes closest to linear or logarithmic spacing. To find out, we measured the position of each of the 100 beads (from the string’s left-hand endpoint) in the participants’ final configurations. Figure 3 plots the median of these placements over participants (open points) as a function of the beads’ ordinal (left-to-right) position on the string (1-100). This graph shows no tendency of greater compression for larger numbers than for smaller ones, contrary to what one might expect if children believed that the natural numbers followed a logarithmic pattern.

If participants spaced the beads in a linear fashion on the string, we would expect the median obtained positions to be a linear function of the beads’ ordinal positions. (We use median positions here for consistency with earlier number-placement experiments.) But if they spaced the beads in a logarithmic fashion (greater compression at the high end than at the low end), we would expect the obtained positions to be a log function of the ordinal positions. To compare these possibilities, we performed regressions on the medians, using as predictors the ordinal positions of the beads and the logs of these positions. The best-fitting lines from these analyses appear in Figure 3: The solid line represents the linear predictions, and the dashed line, the log predictions. As one might expect, both regressions accounted for a significant portion of the variance (for the linear predictor, \( R^2 = .96, b = 1.17, SE = .025, t(98) = 47.56, p < .0001 \); for the log predictor, \( R^2 = .72, b = 31.8, SE = 1.98, t(98) = 16.03, p < .0001 \)). To evaluate the relative goodness of fit, we followed Siegler and Booth’s (2004) method of comparing the absolute values of the
residuals at each of the 100 points from the two analyses. This comparison found significantly smaller deviations (i.e., better fit) from the linear predictor \((M = 5.85 \text{ cm})\) than from the log predictor \((M = 15.33 \text{ cm})\), \(t(99) = 10.55, SE = .90, p < .0001\). These results suggest that when the symbol mapping difficulty is removed, children no longer show logarithmic number line placement.\(^4\,5\)

We can also look at the relative goodness of fit for the individual participants. For this purpose, we used the same linear and logarithmic regressions, but applied to the individual data, and we again compared the residuals. For 14 of the 18 participants, the linear model provided a significantly better fit than the logarithmic model; for three participants, the log model provided a significantly better fit than the linear model; and for one participant, there was no significant difference in the fits.

In this study, we asked the participants whether they had used a ruler at home or at school, on the chance that experience with a ruler might lead to more linear responses. All but two participants reported having used a ruler, but those with ruler experience had only slightly smaller residuals from the linear model than did those with no experience \((M = 6.8 \text{ cm vs. 9.9 cm})\). We also looked at whether the participants’ Number Knowledge score (Okamoto & Case, 1996) or age predicted linear preferences. Participants had a mean score of 8.61 \((SD = 3.82)\) on the Number Knowledge test (see Note 3). Mean age and age range appear in the Method section. To test the effect of these variables, we used logistic regression, where the dependent variable was whether the participant’s best-fitting function was linear (scored as 1) or not linear (scored as 0). However, no significant effect of either variable appeared: For the effect of age, \(b = 0.04, \text{Wald } \chi^2(1) = 0.311, p = .58\); for the effect of Number Knowledge, \(b = -0.08, \chi^2(1) = 0.336, p = .56\).

One potential obstacle in interpreting these results, however, has to do with the starting positions of the beads. The 100 beads initially appeared to participants in a contiguous group, at either the beginning, middle, or end of the string. There was no significant difference among the participants assigned to these starting positions in how well they conformed to the linear model: A comparison of the absolute residuals for the three starting positions produced \(F(2,15) < 1.\) The same was true of the absolute residuals from the log model. However, if participants never moved the beads or moved them in a
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minimal way, the result would be a linear arrangement, biasing the results in a linear direction. In fact, however, all but five participants made more than minimal adjustments. To see whether the five minimal adjusters were responsible for the findings, we repeated the regressions described earlier, excluding these five participants from the analyses. The results from these regressions still favored linear over log spacing: A comparison of the residuals for the two models again showed smaller absolute deviations for the linear ($M = 9.46$ cm) than for the log regression ($M = 18.60$ cm), $t(99) = 9.86$, $SE = .94$, $p < .0001$. The beads’ starting positions are therefore unlikely to explain the superiority of the linear fits.

Other strategies, however, may have had more of an impact on the results. For example, some participants split the beads into two groups, with one group moved to one end of the string and the other group moved to the other end. This pattern is likely responsible for the terracing that appears at the top and bottom portions of Figure 3. To discourage this strategy and similar ones, Experiment 5 adopts a different starting point for the beads, one that is neutrally spaced between a log and a linear function.

**Experiment 5: Number Placement from Neutral Starting Points**

In the present experiment, children again move beads on a string to demonstrate how the numbers go on the number line. But to rule out a possible influence of the beads’ starting positions on the children’s linear or logarithmic spacing, participants initially view an arrangement of the beads that is equally well-fit by linear and logarithmic models. The line with short dashes in Figure 4 shows this starting position. If children prefer that natural numbers be evenly spaced on an external line, they should adjust the beads in a more nearly linear direction, but if they prefer that the naturals be logarithmically spaced, they should move them in a more nearly logarithmic direction.

**Method**

We followed a procedure similar to that of Experiment 4, but with two main alterations: the starting positions of the beads on the number-line string and the order of the tasks.

**Participants.** Seventeen 4 and 5-year-olds ($M = 4.91$, Range = 4 years 1 month to 5 years 6 months) were recruited from the Selinsgrove, PA area. The initial configuration of beads appeared to nine
children in the order shown in Figure 4; the remaining eight participants saw the beads in an order that was left-right reversed.

**Materials and procedure.** At the start of each session, we arranged the beads on the string such that the distribution was fit equally well by logarithmic and linear equations. We also moved the number line task to the beginning of the session, so that the children would complete this task when they were least fatigued and most motivated.

After the coloring period, the experimenter began the number line task by introducing the penguin puppets and talking about how they enjoyed making snowballs and saving them for summertime snowball play. Each child was then shown two cups in which the snowballs (white and blue craft pompoms) were stored, each labeled with the name of a puppet. The experimenter asked the child to count the snowballs in each cup (2 and 3) and to determine who had made the most snowballs. Then the experimenter proceeded to explain how the puppets disagreed about number and requested the child’s help, as in Experiment 4.

**Results and Discussion**

We asked children to move the beads on the string to show where the numbers should go (“so you can count them, add them together, and do all the things you can do with numbers”). The initial bead position (line with short dashes in Figure 4) was equally well fit by log and linear functions, but children tended to move the beads toward a more linear arrangement. The points in Figure 4 show the medians of their final positions.

To determine whether linear or logarithmic functions provide the better account of the data, we performed regression analyses, using as predictors the ordinal bead positions and the logs of these positions, as we had in Experiment 4. Figure 4 shows the resulting linear predictions as the solid line and the log predictions as the long-dashed line. Both functions accounted for a significant portion of the data (for the linear function, $R^2 = .98$, $b = 0.89$, $SE = .013$, $t(98) = 67.40$, $p < .0001$; for the log function, $R^2 = .88$, $b = 26.48$, $SE = 0.98$, $t(98) = 26.91$, $p < .0001$). The mean absolute value of the deviations from the linear model was 2.84 cm, whereas the corresponding value for the log model was 7.64 cm. A $t$-test of
these residuals confirmed the impression from Figure 4 of a better fit for the linear model, $t(99) = 10.67$, $SE = 0.45, p < .0001$. These findings accord quite well with those of Experiment 4.

Once again, we evaluated fits for both linear and log distributions to each participant’s data. The linear function provided a significantly better fit for 14 of the 17 participants, and the log function provided a significantly better fit for 2. Neither function was significantly better than the other for the remaining participant. These results are also in good agreement with those of Experiment 4, and they suggest that the superiority of the linear model is not simply a function of the start positions of the beads. Instead, they confirm the idea that children believe that the natural numbers 1-100 occupy equally spaced points on an external number line.

We again checked whether participants who reported using rulers showed more nearly linear performance than those who didn’t. In this case, 8 of the 17 children said they had used rulers at home or in school. But the residuals from the linear model were this time slightly larger for the experienced participants ($M = 4.89$ cm) than for those without ruler experience ($M = 3.25$ cm). Of course, participants’ actual use of rulers may differ from what they reported. Nevertheless, we find no evidence that children’s exposure to external linear scales on rulers contributes to their understanding of number spacing. We also examined whether children’s age or their score on the Number Knowledge test ($M = 10, SD = 4.33$) predicted whether their placements followed a linear distribution. A logistic regression, similar to the one we performed in Experiment 4, again showed no significant effect of either variable: For age, $b = -0.011$, Wald $\chi^2(1) = 0.010, p = .92$; for Number Knowledge, $b = 0.052$, Wald $\chi^2(1) = 0.123, p = .73$.

**General Discussion**

We find that when children judge the positions of the numbers 1-100 on a number line, their performance indicates correct spacing of these numbers. As long as the children do not have to map specific number symbols onto the line, the bias toward a log-like distribution, seen in many earlier studies, evaporates. In Experiments 1 and 2, five- and six-year olds made judgments about number lines in which the positions of the numbers (but not their names) appeared as points, and the participants chose which line showed the way numbers should go onto the line. The participants displayed a preference for
linearly spaced number lines over logarithmic, exponential, or random lines. These decisions were not the result of children preferring equally-spaced lines for their pure design features, since this preference did not appear in Experiment 3, which directly asked participants to choose the line they liked best.

Experiments 4 and 5 varied the task by asking participants to move beads on a string to show how the numbers go on the line (rather than to select among pre-formatted lines, as in Experiments 1 and 2). The positions the children chose were best fit by linear models rather than by logarithmic ones. Although children of this age locate the positions of specific numerals in a log-like pattern during a typical number-line placement task, they prefer linear spacing both in recognizing (Experiments 1-2) and in producing (Experiments 4-5) number lines.

These findings agree with recent arguments suggesting that past results from the number-line placement task, as it is typically used, may not directly tap children’s mental representation of the natural numbers. One persuasive explanation is that the standard placement task may depend heavily on participants’ familiarity with specific number labels: Even college students who know the structure of the naturals produce log-like data when working with atypical numerals as anchors or targets (Chesney & Matthews, 2013; Hurst et al., 2014; Rips, 2013). Our studies complement this work; removing number words leads children to choose and to generate more uniform distributions.

To be sure, the present experiments show linear preferences only in the range 1-100. An open and important question is how much children of this age know about the natural numbers as a whole (Rips, Asmuth, & Bloomfield, 2006; Rips, Bloomfield, & Asmuth, 2008). Prior studies suggest that most younger five-year olds have not yet integrated 0 with the rest of the naturals (Wellman & Miller, 1986) and do not know that the naturals continue infinitely in the positive direction (Cheung, Rubenson, & Barner, 2017; Hartnett, 1992). However, the experiments reported here suggest that children’s recognition of the distribution of the first hundred positive integers is not seriously distorted, and this fact may simplify theories of later number learning for the integers. It may also have practical implications for teaching integer arithmetic in the early grades.
Relation to Earlier Number Line Studies

Current theories explain how participants could produce log-like number placements from an internal linearly-spaced representation of the numbers (e.g., Anobile et al., 2012; Cantlon et al., 2009; Cohen & Blanc-Goldhammer, 2011; Hurst et al., 2014; Rips, 2013). These theories can therefore provide a bridge between our results and those from the earlier number-placement tasks. Although children recognize that the naturals are evenly spaced on external number lines, their uncertainty about the meaning of the larger number words, together with truncation from the number-line boundaries, implies compression at the upper end of the 0-100 interval. The same theories can also account for why older children shift toward more linear placements: Older children become increasingly familiar with the meanings of words and symbols for larger numbers, reducing the uncertainty about their positions.

One might try to argue that the present tasks do not succeed in tapping children’s ideas about the natural numbers because they do not present individual numerals (e.g., “29” or “73”). According to this perspective, traditional number-line placement tasks come closer to assessing number knowledge by virtue of querying the positions of the numerals themselves. However, two points are worth noting about this argument. First, there is no dispute about the presence of gaps in the number knowledge of children of this age, as we mentioned earlier. Children are unclear about exactly where larger numerals (e.g., “88”) in the 0-100 range fall on the number line. The present idea is that this uncertainty may co-exist with a correct scaffolding for the numbers in a linear distribution that may aid children in acquiring the missing positional information and that may underlie correct knowledge of the naturals. In particular, children of this age already seem to understand—perhaps as an item of metacognition—that the natural numbers are evenly spaced on external number lines.

Second, there is no reason to suppose that the only true knowledge of the numerals consists in the ability to locate their absolute positions on a line. This idea seems to rest on the theory that the numerals get their meanings individually by association with a specific quantity. For example, the numeral for the natural number three may have as its meaning a quantity of three objects (the cardinality of a three-member set or a possibly continuous measure associated with such a set). According to this view, unless a
task can uncover children’s knowledge of these numeral-quantity pairings, it cannot get at the children’s understanding of the naturals at all. However, one immediate difficulty with this idea, taken as support for the conventional number-placement task, is that positioning a numeral on a number line is not in itself a straightforward test of children’s numeral-quantity pairings. Placing a numeral on a bare number line does not, in any obvious way, assess knowledge of cardinality. More important, however, the assumption that numerals’ meaning depends on their individual reference to particular cardinalities or associated magnitudes is itself subject to criticism. It seems very unlikely, for example, that adults’ understanding of large numbers (e.g., a trillion) depends on their ability to apprehend quantities of the appropriate size (e.g., sets of $10^{12}$ objects). Instead, contemporary structuralist theories of number (e.g., Parsons, 2008; Resnik, 1997; Shapiro, 1997) tie number meanings to the position these numbers occupy in the overall number system to which they belong (see Rips, 2015, and Rips et al., 2008, for the psychological relevance of these theories). The number-line tasks of the present experiments seem to be appropriate methods for studying children’s recognition of this structure.

On a more general level, one might object that *any* task that asks children for their explicit judgments about numbers is suspect on the grounds that these judgments are subject to biases. But it is unclear that explicit judgments about the distribution of numbers are problematic in a way that most other procedures are not. For example, it is very unclear why asking children about the spacing between numbers is inherently less trustworthy than asking them about the position of individual numbers (see the section *Issues with the Number-line Placement Task* for difficulties with the latter procedure). Like all methods in psychology, these procedures are subject to biases from unknown factors that may not have been controlled. But biases of this sort are empirical matters and can’t be assumed in advance. Furthermore, explicit judgments about numbers may be more relevant to children’s school-based arithmetic performance than implicit ones. When children learn mathematics, they often have to call on explicitly represented principles and facts to guide their thinking. In general, then, there appears to be no inherent reason to distrust explicit judgments of this subject matter and some advantages in examining them.
Limits to Linearity

The experiments we report here suggest that preschool and kindergarten children favor linear distribution of the integers from 1 to 100. Although knowledge of linearity is likely to be helpful in children’s learning of arithmetic in early grade school, we also note that linearity may pose difficulties when children advance to more complicated number systems and functions. After all, logarithmically and exponentially spaced numbers play an important role in mathematical and scientific contexts. In the case of rational and real numbers, students’ knowledge may be enhanced by understanding that equal spacing continues to hold for the real or rational counterparts of the whole numbers (Siegler et al., 2011)—for example, 2.0 – 1.0 = 3.0 – 2.0. But at the same time, children’s learning of the rationals and reals may be slowed if the children impose properties of the integers inappropriately on these new systems. Children may initially believe that all successive numbers are separated by discrete amounts of equal size and so have trouble learning about systems in which the numbers are dense (i.e., for any two distinct numbers $x$ and $y$, there is a number $z$, such that $x < z < y$), as in the case of the rationals and reals (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Hartnett & Gelman, 1998; Smith, Solomon, & Carey, 2005).

Even adults have difficulty seeing that linearity no longer applies in alternative number systems. In one series of experiments, college students rated several mathematical properties for their importance to number systems in general (Rips & Thompson, 2014). Among these properties was the feature, sometimes called trichotomy, that for any numbers $x$ and $y$ in the number system, $x < y$, $x = y$, or $x > y$. Participants gave trichotomy consistently high importance ratings (7.36 on a 1-to 8 scale in one experiment and 6.97 in another). Trichotomy was, in fact, the highest rated property in both experiments (e.g., outscoring associativity), even though it fails to hold in the complex numbers, with which the students were probably acquainted. Participants also consistently rated circular arrangements of numbers as unlikely to constitute number systems, despite the existence of modular number systems that have such a structure (e.g., Graham, Knuth, & Patashnik, 1994).

These earlier findings suggest that adults may continue to believe that linearity is a central organizing property for number systems, and the present results indicate that this belief may be present as
early as preschool. A bias for linear structures also appears in tasks that call for comparison of non-numeric items, where the actual structure is more complex (De Soto, 1961; Hayes-Roth & Hayes-Roth, 1975; Moeser, 1979). For this reason, instructors may need to take special care in introducing students to non-linear structures, such as the complex numbers, in order to overcome the students’ preconceptions.

**Origins of Linear Number Knowledge**

If five-year old children really do recognize that the first 100 positive integers are linearly related, where does this ability come from? One obvious possibility is that these children have already had enough exposure to numbers at home or in school to understand that whole numbers are equally spaced. For example, their use or observation of measuring devices, such as rulers or analog thermometers, or the presence of actual number lines in books or in the classroom, may have conveyed the idea of equal spacing. This experience could account for the fact that children prefer a linear number line in the present experiments, despite their possible uncertainty about the meaning of larger number words in the 1-100 range (and despite their use of an approximate number system in apprehending the cardinality of larger groups of objects).

Our attempts to find a link between children’s exposure to linearly-numbered devices and their preference for linear spacing consistently failed. The first-graders in Experiment 1 may have been more likely to encounter such devices than the kindergartners, but we found no significant difference in the tendency of these two groups to choose linear over other number lines. Similarly, children in Experiment 2 who had received no classroom instruction on number lines or rulers were no less likely than other children to prefer linear number lines. And those children in Experiments 4 and 5 who self-reported use of rulers were no more likely to place tokens for the numbers in linear positions. In fact, the deviation from linearity in the latter experiment was (non-significantly) in the wrong direction, with larger deviations for those with ruler experience. Of course, failure to find a link between exposure to linear devices and preference for linearity does not imply that no such connection exists. But we think it’s worth contemplating the possibility that children’s preference for linear structures is not simply the result of seeing numbers in external linear arrays.
However, the point of these experiments is not that children understand the correct spacing without the benefit of *any* relevant experience. Children may have encountered direct forms of instruction about the distribution of the naturals—for example, hints that the naturals are evenly spaced or that the difference between successive naturals is 1. In general, there is no reason to doubt that children’s experience with depictions of numbers or with instruction about number spacing may be part of the story of how they learn the correct distribution of the integers. Instead, as we mentioned in the Introduction, the present experiments help establish that children actually prefer equally spaced numbers, whatever the source of this preference. The importance of this finding lies in part in the fact that earlier experimental evidence from the number-placement tasks can be taken—incorrectly, we believe—as suggesting that no such preference exists.
References


Footnotes

1 Log-like responses from adults sometimes appear in tasks other than number-line placement. For example, when participants must produce “random” positive integers (or judge whether a given set of positive integers appears random) over an unbounded interval, their choices conform to compressive functions (power functions with exponents between 0 and 1). However, when the interval is bounded, their choices conform to a linear function (Banks & Coleman, 1981). These results could be taken as evidence that adults have both linear and log-like representations of the positive integers (see Siegler & Opfer, 2003). But like the number-placement results, the different functions in these experiments can also be viewed as the effect of task constraints (in this case, boundedness of the target interval).

2 The first random line in Figure 1 (Random 1) was constructed by generating 100 numbers from a uniform random distribution of numbers from 0 to 1 and linearly rescaling them to the 1-100 interval. The second random line (Random 2) was produced by generating 50 random numbers in the same way, rescaling to the 1-50 interval, and then (left-right) mirroring the same numbers to create points from 51 to 100. The intent was to use the second random line as a control for effects of symmetry (both the Linear and Random 2 lines are symmetric about the lines’ midpoints). However, the symmetry of Random 2 is difficult to perceive, and we therefore treat it in the same way as Random 1. Experiments 2 and 3 provide a more clear-cut test of the effects of symmetry.

3 The Number Knowledge Test (Okamoto & Case, 1996) consists of a preliminary question and four levels, numbered 0 through 3. The preliminary question (“count up to 10”), intended to be solved correctly by virtually all children, is designed to ease them into the test. Level 0 measures children’s ability to count and compare quantities up to 10 using physical objects (poker chips), and is aimed at the level of the average 4-year-old. Level 1, aimed at a 6-year-old level, dispenses with objects and asks children to count, compare quantities, add and subtract with numbers up to 10. Level 2, at the 8-year-old
level, measures the same skills with 2-digit numbers. Level 3, at the 10-year-old level, includes 3-digit numbers. Children are awarded 1 point for each item answered correctly, and they must answer all subparts correctly to get the whole item correct. A child must get a certain number of points at each level to go on to the next level. The total number of points the child earns is his or her raw score.

4 We also fit a one-cycle power model to the data of Experiment 4, since earlier investigations had found support for this model in the traditional number-placement task, as we mentioned in the introduction (Barth & Paladino, 2011; Rouder & Geary, 2014; Slusser et al., 2013; Sullivan et al., 2011). According to this model (Hollands & Dyre, 2000; Spence, 1990), the obtained position of the \( i \)-th bead, \( y_i \), should be a function of the true position \( x_i \) to a power \( \beta \). Scaled to a 0-to-100 line, the one-cycle power model is:

\[
y_i = \frac{100x_i^\beta}{x_i^\beta + (100 - x_i)^\beta}
\]

This model provides a very good fit to the obtained data: The mean absolute deviation was 4.34 cm and \( R^2 = .99 \) with one free parameter, \( \beta = 1.67 \). We note, however, that the obtained exponent is positive, rather than the negative exponent typically obtained in number-line placement. This difference corresponds to the fact that the positions of the first fifty beads tend to be underestimated in our task and the last fifty beads overestimated. (This can be seen by drawing in the line \( y = x \) in Figure 3.) The opposite pattern tends to hold in number placement tasks. We suggest, then, that the under-then-over pattern in our experiment may be the result of either the specific strategies of participants or the starting positions of the beads, at the beginning, middle, or end of the line. In accord with this possibility, the pattern does not reappear in Experiment 5, a similar task in which we use a different starting point (see Figure 4).

5 In addition, we looked at the fit of the linear and log models when the predicted functions were constrained to pass through 0, that is, when the model is fit with no intercept. With one fewer parameter, these models still do well. But the additional constraint does not change the relative standing of the linear
and log models. A comparison of the absolute values of the residuals again shows bigger residuals for the log model ($M = 22.59$ cm) than for the linear model ($M = 7.44$ cm), $t(99) = 17.53, p < .0001$.

If a participant made no adjustments at all in the spacing between the beads, they would continue to occupy a region of 30 cm on the string. In fact, the mean distance between the right-most and left-most beads in participants’ final configurations was greater than 59 cm (about double the initial distance) for 13 of the 18 participants. The distance for the remaining five minimal adjusters was less than 34 cm.
Figure 1. Number line stimuli for Experiment 1. The actual stimulus lines were 30 cm long, so that the points were distinct.
Figure 2. Number line stimuli for Experiment 2. The actual lines were 81 cm long, so that the points were distinct.
Figure 3. Median position of bead placement (open circles), Experiment 4. Solid line shows best-fitting linear predictor for the data; dashed line, best-fitting logarithmic predictor.
Figure 4. Median position of bead placement (open circles), Experiment 5. Solid line shows best-fitting linear predictor for the data; long dashed line, best-fitting logarithmic predictor. Short dashed line shows the beads’ start position.