On convergence of sequences in complete lattices*

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Abstract

We generalize the famous Tarski result by showing that: if $X$ is a complete lattice, and $f : X \to X$ is an order-preserving mapping, then for all points $x \in X$, the limit superior and the limit inferior of the (possibly transfinite) sequence of iterations $x, f(x), f^2(x),\ldots, f^\alpha(x),\ldots$ are fixed points of $f$. These limits are the sharp fixed-point bounds between which sufficiently large transfinite iterations are located.

1 Introduction

The celebrated Tarski (1935) fixed-point theorem\textsuperscript{1} says that an increasing (or order-preserving) mapping $f$ on a complete lattice has a fixed point. Moreover, the set of fixed points is also a complete lattice. The lowest fixed point is the “limit” of the sequence of iterations of the lowest element of the lattice, and the highest fixed point is the “limit” of the sequence of iterations of the highest element of the lattice. In the general case, these sequences have to be transfinite, but in applications, it often suffices to study sequences indexed by natural numbers.

The Tarski theorem has numerous applications in studying simple discrete dynamic processes. In economics, it has also been used to prove equilibrium existence in supermodular games (see Topkis (1979) and Vives (1990)), to prove the existence of stable matchings (Adachi (2000) and Fleiner (2003)), and to solve certain types of discontinuous generalized variational inequalities (see Nishimura and Ok (2012) and Li and Ok (2012)\textsuperscript{2}).

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\textsuperscript{1}This result is also known as the Knaster-Tarski theorem. See Knaster and Tarski (1928). In economics, authors typically refer to it as the Tarski theorem.

\textsuperscript{2}The first of these two papers also provides several applications to optimization and games.
However, the theorem is not fully satisfying for the following reason. It says nothing about the sequences of iterations of \( x \), when \( x \) is neither the lowest nor the highest element of a lattice. Such sequences often appear in the analysis of dynamic processes, e.g., the best-response dynamics in games.

For an arbitrary \( x \), a result similar to that for the lowest or the highest \( x \) is not possible. It is easy to give examples of diverging sequences of iterations. However, we will show in this note that the properly defined limit superior and limit inferior of the sequences of iterations are fixed points of \( f \) for all \( x \). These limits are the sharp fixed-point bounds between which all sufficiently large iterations are located. When the two limits coincide, the sequence of iterations converges to a fixed point.

Our result generalizes the Tarski theorem to all \( x \). As the Tarski theorem itself, our result has to allow for transfinite sequences, but in applications, it often suffices to study sequences indexed by natural numbers. At the end of the note, we will suggest some applications.

2 Results

Throughout the paper \((X, \leq)\) is always a complete lattice. The meet and join operations are denoted by \( \land \) and \( \lor \), respectively. Completeness postulates that for all subsets \( Y \subseteq X \) there exist an infimum (the greatest lower bound) and a supremum (the least upper bound). They will be denoted by \( \land Y \) and \( \lor Y \), respectively.

For every \( x \in X \), every order-preserving mapping \( f : X \rightarrow X \), and every ordinal number \( \alpha \), define two transfinite sequences \( (x_\beta)_{\beta < \alpha} \) and \( (x^\beta)_{\beta < \alpha} \) as follows:

\[
x_\beta = \begin{cases} 
  x & \text{if } \beta = 0 \\
  f(x_{\beta - 1}) & \text{if } \beta \text{ has a predecessor } \beta - 1 \\
  \lor_{\gamma < \beta} \land_{\gamma \leq \delta < \beta} x_\delta & \text{if } \beta \text{ is a limit ordinal}; 
\end{cases}
\]

\[
x^\beta = \begin{cases} 
  x & \text{if } \beta = 0 \\
  f(x^{\beta - 1}) & \text{if } \beta \text{ has a predecessor } \beta - 1 \\
  \land_{\gamma < \beta} \lor_{\gamma \leq \delta < \beta} x_\delta & \text{if } \beta \text{ is a limit ordinal}. 
\end{cases}
\]

(1) (2)

Note that each sequence: \( (x_\beta)_{\beta < \alpha} \) and \( (x^\beta)_{\beta < \alpha} \) is uniquely determined by \( x \) and \( f \). The definitions of sequences \( (x_\beta)_{\beta < \alpha} \) and \( (x^\beta)_{\beta < \alpha} \) are illustrated by the following two examples.

Example 1 Given a limit ordinal number \( \lambda \), let \( X = \{ \beta \leq \lambda : \beta \text{ is an ordinal number} \} \). Next, let \( f : X \rightarrow X \) be defined by letting \( f(\beta) = \beta + 1 \) for \( \beta < \lambda \), and \( f(\lambda) = \lambda \). Then, for \( x = 0 \), \( x_\beta = x^\beta = \beta \) for all \( \beta \leq \lambda \).

Example 2 Let \( X = [0,1]^2 \) be the unit square, and let \( \leq \) be the coordinate-by-coordinate ordering of \( X \), i.e., \( (x'_1, x'_2) \leq (x''_1, x''_2) \) if \( x'_1 \leq x''_1 \) and \( x'_2 \leq x''_2 \). Let \( f : X \rightarrow X \) be the reflection across the diagonal \( D = \{(x_1, x_2) \in X : x_1 = x_2\} \), that is, \( f(x_1, x_2) = (x_2, x_1) \) for all \( (x_1, x_2) \in X \).

Then \( X \) is a complete lattice, and \( f \) is an order-preserving mapping. If \( x \in D \), then \( x_\beta = x^\beta = x \) for all \( \alpha \) and \( \beta < \alpha \). If \( x = (x_1, x_2) \notin D \), then \( x_k = x^k = x \) for all even natural numbers \( k = 2, 4, \ldots \), and
$x_k = x^k = f(x) = (x_2, x_1)$ for all odd natural numbers $k = 1, 3, \ldots$. So, $x_\beta = (\min\{x_1, x_2\}, \min\{x_1, x_2\})$ for $\omega \leq \beta < \alpha$ and $x^\beta = (\max\{x_1, x_2\}, \max\{x_1, x_2\})$ for $\omega \leq \beta < \alpha$.

One method of proving Tarski’s fixed-point theorem (referred to in the introduction) is by studying the iterations $x_k = x^k$, $k = 0, 1, \ldots$, of the lowest or the highest $x \in X$. If the mapping $f$ is continuous (i.e., the image of the supremum of an increasing sequence is equal to the supremum of images, and the image of the infimum of a decreasing sequence is equal to infimum of images), then: (a) the supremum of the sequence $x_k = x^k$, $k = 0, 1, \ldots$, where $x$ is the lowest element of the lattice, is a fixed point of $f$; and (b) the infimum of the sequence $x_k = x^k$, $k = 0, 1, \ldots$, where $x$ is the highest element of the lattice, is a fixed point of $f$. However, if $f$ is not continuous, the theorem can be established only by studying transfinite iterations $(x^\beta)_{\beta < \alpha}$ and $(x_\beta)_{\beta < \alpha}$. Then, the supremum of $(x^\beta)_{\beta < \alpha}$ and the infimum of $(x_\beta)_{\beta < \alpha}$ for sufficiently large ordinals $\alpha$ are guaranteed to be fixed points of $f$.

To generalize Tarski’s result to all $x \in X$, we will assume that $\alpha$ is a regular cardinal number. A regular cardinal number $\alpha$ is defined by the following property: No set of cardinality $\alpha$ can be represented as the union of a family of subsets such that each subset from the family has a cardinality smaller than $\alpha$, and the family itself is of a cardinality smaller than $\alpha$.

**Theorem 1** Suppose that $(X, \leq)$ is a complete lattice, and $f : X \to X$ is an order-preserving mapping. Let $\alpha > |X|$ be a regular cardinal number. Then, for any $x \in X$, there exist $\beta, \overline{\beta} < \alpha$ such that $x_\beta = x_{\overline{\beta}}$ for all $\underline{\beta} \leq \beta < \alpha$, and $x^\beta = x^{\overline{\beta}}$ for all $\underline{\beta} \leq \beta < \alpha$. In particular, $x_\beta$ and $x^\beta$ are fixed points of $f$.

**Proof.** We will prove the result for the sequence $(x^\beta)_{\beta < \alpha}$; the proof for the sequence $(x_\beta)_{\beta < \alpha}$ is analogous.

**Step I.** Let $Y$ be the subset of $X$ defined by: $x \in Y$ if the supremum of the set $\{\beta < \alpha : x^\beta = x\}$ is equal to $\alpha$. By the definition of $Y$, we have that $|\{\beta < \alpha : x^\beta = x\}| < \alpha$ for any given $x \in X - Y$. Obviously,

$$\{\beta < \alpha : x^\beta \in X - Y\} = \bigcup_{x \in X - Y} \{\beta < \alpha : x^\beta = x\}.$$

Thus, since $|X - Y| < \alpha$ and $\alpha$ is regular, $|\{\beta < \alpha : x^\beta \in X - Y\}| < \alpha$. It follows that there is an ordinal $\alpha^1 < \alpha$ such that $x^\beta \in Y$ for $\beta \geq \alpha^1$.

**Step II.** There is an ordinal $\alpha^2 < \alpha$ such that $\{x^\beta : \alpha^1 \leq \beta \leq \alpha^1 + \alpha^2\} = Y$ and $x^{\alpha^1} = x^{\alpha^1 + \alpha^2}$. Indeed, for every $x \in Y$, take a $\beta_x \geq \alpha^1$ such that $x^{\beta_x} = x$, and then define $\alpha^2$ as the lowest ordinal number such that $\beta_x \geq \alpha^1 + \alpha^2$ for all $x \in Y$ and $x^{\alpha^1} = x^{\alpha^1 + \alpha^2}$.

**Step III.** Let $\alpha^3 = \alpha^1 + \alpha^2 + \omega^3$. Note that $\alpha^3$ is a limit ordinal. By Step II, and the definition of the sequence $(x^\beta)_{\beta < \alpha}$, we have that $\{x^\beta : \alpha^1 + \alpha^2 n \leq \beta \leq \alpha^1 + \alpha^2 (n + 1)\} = Y$ for $n = 1, 2, \ldots$. For every $\gamma < \alpha^3$ there is an $n = 1, 2, \ldots$ such that $\gamma \leq \alpha^1 + \alpha^2 n$. Thus, $(x^\gamma)_{\gamma \leq \delta < \alpha^3}$ contains all elements of $Y$. In addition, if $\gamma \geq \alpha^1$, then $(x^\gamma)_{\gamma \leq \delta < \alpha^3}$ contains no $x \notin Y$. Thus, if $\alpha^1 \leq \gamma < \alpha^3$, then the supremum of $(x^\gamma)_{\gamma \leq \delta < \alpha^3}$ is

\footnote{Intuitively, $\alpha_3$ is the ordering obtained by putting the ordering $\alpha_1$ first, followed by the ordering $\alpha_2$, the ordering $\alpha_2^i$, $\ldots$, the ordering $\alpha_2^i$, $\ldots$, where $\alpha_2^i = \alpha^i$ for $i = 1, 2, \ldots$. See Jech (2002).}
equal to the supremum of $Y$. It follows that

$$x^{\alpha^3} = \bigwedge_{\gamma < \alpha^3} \bigvee_{\gamma \leq \delta < \alpha^3} x^{\delta} = \bigwedge_{\alpha^3 \leq \gamma < \alpha^3} \bigvee_{\gamma \leq \delta < \alpha^3} x^{\delta}$$

is the supremum of $Y$.

**Step IV.** Since $x^{\alpha^3+1} \in Y$ by Step II, $x^{\alpha^3+1} \leq x^{\alpha^3}$ by Step III. We will prove by transfinite induction that the sequence $(x^\gamma)_{\alpha^3 \leq \gamma < \alpha}$ is nonincreasing. Suppose that for some $\beta < \alpha$ and for all $\beta' < \beta$ the sequence $(x^\gamma)_{\alpha^3 < \gamma < \beta'}$ is nonincreasing. Then, the sequence $(x^\gamma)_{\alpha^3 < \gamma < \beta}$ is nonincreasing. Suppose first that $\beta$ is a limit number. To make the inductive step in this case, we must show that $x^\beta \leq x^\gamma$ if $\alpha^3 \leq \gamma < \beta$. By the inductive assumption, the supremum of $(x^\delta)_{\gamma \leq \delta < \beta}$ is equal to $x^\gamma$. Thus,

$$x^\beta = \bigwedge_{\gamma < \beta} \bigvee_{\gamma < \delta < \beta} x^\delta = \bigwedge_{\alpha^3 \leq \gamma < \beta} \bigwedge_{\gamma \leq \delta < \beta} x^\delta = \bigwedge_{\alpha^3 \leq \gamma < \beta} x^\gamma$$

Hence $x^\beta \leq x^\gamma$ for all $\gamma$ such that $\alpha^3 \leq \gamma < \beta$.

Suppose now that $\beta$ has a predecessor $\beta - 1$. To make the inductive step, we must show that $x^\beta \leq x^{\beta - 1}$. By the inductive assumption, the infimum of the sequence $(x^\delta)_{\alpha^3 \leq \delta < \beta}$ is $x^{\beta - 1}$. Thus, since $f$ is order-preserving,

$$x^\beta = f(x^{\beta - 1}) \leq \bigwedge_{\alpha^3 \leq \delta < \beta - 1} f(x^\delta) = x^{\beta - 1}.$$

**Step V.** Let $\overline{\beta} := \alpha^3$. Recall that $\alpha^3 \geq \alpha^1$, so $x^\overline{\beta} \in Y$. If $x^\beta \neq x^\overline{\beta}$ for some $\overline{\beta} < \beta < \alpha$, then by Step IV, $x^\gamma \leq x^\beta < x^\overline{\beta}$ for all $\beta < \gamma < \alpha$. This would imply that $x^\overline{\beta} \notin Y$. Thus, $x^\beta = x^\overline{\beta}$ for all $\overline{\beta} < \beta < \alpha$. In particular, $f(x^\overline{\beta}) = x^{\overline{\beta} + 1} = x^\overline{\beta}$, so $x^\overline{\beta}$ is a fixed point of $f$. ■

We will now make precise the claim that $x_{\overline{\beta}}$ and $x^\overline{\beta}$ are sharp fixed-point bounds between which sufficiently large transfinite iterations of $x$ are located.

**Definition 1** Let $x$ be any point of a complete lattice $X$, $f : X \to X$ be an order-preserving mapping, and $\alpha > |X|$ be a regular cardinal number. Let $(x_\beta)_{\beta < \alpha}$ and $(x^{\beta})_{\beta < \alpha}$ be the sequences defined by (1) and (2), respectively. Define $\liminf f x$ as $x_{\overline{\beta}}$ such that $x_{\overline{\beta}} = x_{\overline{\beta}}$ for all $\overline{\beta} \leq \beta < \alpha$, and define $\limsup f x$ as $x^\overline{\beta}$ where $x^\beta = x^\overline{\beta}$ for all $\overline{\beta} \leq \beta < \alpha$.

Theorem 1 guarantees the existence of such $x_{\overline{\beta}}$ and $x^\overline{\beta}$. Note that $x_{\overline{\beta}}$ does not depend on the choice of $\overline{\beta}$ such that $x_{\overline{\beta}} = x_{\overline{\beta}}$ for all $\overline{\beta} \leq \beta < \alpha$, and $x^\overline{\beta}$ does not depend on the choice of $\overline{\beta}$ such that $x^\beta = x^\overline{\beta}$ for all $\overline{\beta} \leq \beta < \alpha$. Note also that $x_{\overline{\beta}}$ and $x^\overline{\beta}$ are independent of the choice of the cardinal number $\alpha$. (Of course, $\alpha$ must be regular and greater than $|X|$.) Note finally that $\liminf f x$ and $\limsup f x$ coincide with $\liminf x_k$ and $\limsup x_k$ defined in the “traditional way,” via the natural numbers as the index set, whenever $\liminf x_k$ and $\limsup x_k$ are fixed points of $f$.

We obtain the following result as an immediate corollary from Theorem 1.

**Corollary 1** Suppose that $(X, \leq)$ is a complete lattice, and $f : X \to X$ is an order-preserving mapping. Then, $\liminf f x$ is the greatest fixed point $x$ of $f$ with the property that $x \leq x_{\beta}$ for sufficiently large $\beta < \alpha$, and $\limsup f x$ is the smallest fixed point $\overline{x}$ of $f$ with the property that $x^\beta \leq \overline{x}$ for sufficiently large $\beta < \alpha$. 
Finally, we would like to suggest some potential ways of applying our results. Corollary 1 provides a tool for studying adaptive learning in games (and in other settings such as matching). In some situations, players may not learn playing an equilibrium if they learn about playing the game through the best-response dynamics. Then, Corollary 1 gives bounds on what they will learn. For example, Gordon et al. (2020) provide such bounds for the Crawford and Sobel (1982) model of information transmission. They compute the bounds directly, but their method coincides with that suggested in Corollary 1.

Another potential way of applying our results is to proving that fixed points may not exist in some situations. More specifically, integral and differential equations can be turned into fixed-point equations for operators on \( L^p \)-spaces. An ordered \( L^p \)-space becomes a complete lattice by attaching a top element 1 and a bottom element 0. Corollary 1 says that any fixed point that is comparable to some element \( x \) must be greater than or equal to the \( \limsup\, f\, x \), or smaller than or equal to the \( \liminf\, f\, x \). If the \( \limsup\, f\, x \) turns out to be 1 and the \( \liminf\, f\, x \) turns out to be 0, then a certain region in the \( L^p \)-space is free of fixed points for the operator, and hence free of solutions for the related equations.\(^4\)

In applications, we typically do not have to compute the transfinite sequences \((x^\beta)_{\beta<\alpha}\) and \((x^\beta)_{\beta<\alpha}\), but only the sequences \((x_k)_{k=1}^\infty = (x^k)_{k=1}^\infty\).

3 References


\(^4\)The author is grateful to an anonymous referee for pointing out this application.