Equilibrium Existence in Games with Ties

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Abstract

We provide conditions that operationalize Reny’s (1999) better-reply security for Bayesian games and use these conditions to prove the existence of equilibria for classes of games in which payoff discontinuities arise only at “ties.” These games include a general version of all-pay contests, first-prize auctions with common values, and Hotelling models with incomplete information.

We also show that in some applications, including all-pay contests with bid caps and first-price auctions with private values, equilibrium existence does not follow from existing results but can be established by approximation techniques.

1 Introduction

Games with discontinuous payoffs arise naturally in various settings. In an auction, if the highest bidders submit the same bid the prize is allocated according to a lottery but each tying bidder can obtain the prize with certainty by increasing her bid slightly. In a Hotelling model, firms situated at the same location split their set of customers but each firm can

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typically increase its share discretely by changing its location slightly. Additional examples abound.

Perhaps surprisingly, the existence of equilibria in many games with discontinuous payoffs, especially ones with incomplete information, is still an open question. Proving general existence results is difficult because the presence of discontinuous payoffs precludes a direct use of most fixed-point theorems, such as Kakutani’s theorem and its generalizations. In addition, the existence of an equilibrium may depend on subtle details of the setting. For example, in first-price auctions and all-pay auctions equilibrium may not exist if some bidder’s valuation for the prize is 0 with positive probability.\footnote{To see this, suppose there are two bidders. Bidder 1’s valuation is 0 and bidder 2’s valuation is drawn uniformly from the interval \([0, 1]\). If bidder 1 bids 0, then bidder 2 does not have a best response, both in a first-price and in an all-pay auction. But if bidder 1 bids more than 0 in a first-price auction she must lose with certainty, so low types of bidder 2 are not best responding. (This example is due to Lebrun (1996)). And in an all-pay auction bidder 1 must bid 0.} But this is the only reason that an equilibrium may not exist in a general class of contests that includes all-pay auctions, as we show in Section 6.1, whereas other distributional assumptions are required for equilibrium existence even in private-value first-price auctions, as we show at the end of Section 7.2.

One approach to proving the existence of equilibria is to approximate the original game by a sequence of games with a finite number of actions and types. However, a sequence of equilibria of the approximating games need not converge to an equilibrium of the original game. Reny (1999) suggested another approach,\footnote{See also McLennan, Monteiro, and Tourky (2011), Barelli and Meneghel (2014), and Reny (2016) who generalize Reny’s (1999) result.} which subsumed most existence results previously available for games with a continuum of actions and discontinuous payoffs. Instead of sequences of games with a finite number of actions and types, he studied sequences of games with continuous payoffs that approximate the original game with discontinuous payoffs. The results from Reny (1999) have been used subsequently by many authors. The two approaches complement one another, and which approach is more useful may depend on the details of the setting. For example, we show in Section 7.1 that equilibrium existence for all-pay auctions with bid caps does not follow from Reny’s (1999) result (or its generalizations), but can be established more generally for contests with bid caps by approximating them with finite
This paper provides tools for establishing equilibrium existence in distributional strategies for a general class of Bayesian “games with ties,” in which payoff discontinuities may arise only when players “tie,” for example, when winning bidders submit the same bid in an auction or when firms choose the same location in a Hotelling model. We first introduce a condition, *improving deviations*, and prove that this condition guarantees equilibrium existence. The condition says, roughly, that at any strategy profile with payoff discontinuities, the sum of players’ payoffs can be increased more by players deviating unilaterally than by simultaneously changing players actions and types slightly. We then specialize the condition to games with ties to obtain two other conditions, *favorable tie breaking* and *favorable tie breaking on average* that imply improving deviations. These conditions say, roughly, that the sum of payoffs can be increased more by players breaking ties unilaterally than by breaking ties simultaneously to maximize the sum of payoffs. Favorable tie breaking requires this for every strategy profile in which ties whose resolution affects at least one player’s payoff arise with positive probability, whereas favorable tie breaking on average requires this only for strategy profiles in which such ties arise “on average.” The former condition is, however, easier to check.

Our result that improving deviations implies equilibrium existence follows from Theorem 3.1 in Reny (1999). One may wonder why we need our result instead of applying Reny’s (1999) result directly. The reason is that Reny’s (1999) result requires *better-reply security*. This condition says that if a profile \(a\) of actions is not an equilibrium, but the profile \((a, u)\) of actions and payoffs belongs to the closure of the graph of the mapping from action profiles to payoff profiles, then some player has an action \(a_i'\) that gives her a strictly higher payoff than \(u_i\) against the action profiles of the other players in some open neighborhood of \(a_{-i}\). Checking for better-reply security is a demanding task in incomplete-information settings in which players use distributional strategies and the graph of the mapping from strategy profiles to payoff profiles is an infinite-dimensional object.

On the other hand, as we show in Section 6, our favorable tie breaking condition implies equilibrium existence in a straightforward manner in many settings with possibly incomplete

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3Distributional strategies were introduced by Milgrom and Weber (1985).
information, including general contests with many heterogeneous prizes and interdependent values and common-value first-prize auctions. Favorable tie breaking on average implies equilibrium existence in additional settings with incomplete information, such as Hotelling models in which customers and firms are located on an interval.

Other, related settings violate our conditions. These include private-value first-price auctions, contests with bid caps, and Hotelling models in which the range of firms’ locations (as opposed to customers’ locations) is exogenously constrained. In fact, these settings violate better-reply security and many of its generalizations. Nevertheless, we show in Section 7 that equilibria exist for contests with bid caps in which players’ prize valuations are strictly positive and for a class of private-value first-price auctions in which players’ valuations have discrete supports. We do this for contests with bid caps by approximating them with a sequence of games with a finite number of actions and types. We establish equilibrium existence for first-price auctions by approximating them with first-price auctions in which losing players pay a small cost and which satisfy favorable tie breaking. We believe that our arguments in both settings can be useful for proving existence in other settings.

1.1 Literature review

Early existence results for complete-information games with discontinuous payoffs were obtained by Dasgupta and Maskin (1986) and Simon (1987) by approximating the original game with a sequence of finite games. They were later subsumed by Reny’s (1999) results. Recently, several authors have generalized the work of Reny (1999) to incomplete-information settings. This line of research includes Lebrun (1996, 1999) for private-value first-price auctions, Jackson and Swinkels (2005), Monteiro and Page (2007), Prokopovych and Yannelis (2014), He and Yannelis (2016), Carbonell-Nicolau and McLean (2018), Carmona and Podczeck (2018), and Reny (2016). Reny (2019) surveys the literature on equilibrium existence in discontinuous games.

Carmona and Podczeck (2018) seems the most closely related to our paper. They study

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4 We show this for the first two settings in Section 7; this can be similarly shown for the third setting.

5 There is also more recent research on the existence of equilibria in games of complete information (for example, Carmona (2009)), which is only remotely related to the present paper.
games with sharing rules (in particular, first-price auctions and contests). They focus on providing conditions for the existence of a common equilibrium set for all possible sharing rules. Their results imply the existence of equilibria in some incomplete-information games with ties, for example, in a version of contests without a bid cap and with private values (see their Example 5). The most important added value of our favorable tie breaking condition, which guarantees equilibrium existence, in the context of contests and first-price auctions is that it applies to these games with interdependent and common values, respectively. Carmona and Podczeck’s (2018) results apply to a more restricted class of games with interdependent values, because their Φ-strong indeterminacy condition is violated in many such contests and first-price auctions.

Carbonell-Nicolau and McLean (2018) and He and Yannelis (2016) also apply their results to auctions. Carbonell-Nicolau and McLean (2018) show existence in some common-value auctions, but those auction must also satisfy a somewhat involved condition (Assumption C), which - as the authors point out - excludes settings in which whether a player prefers to win or lose a tie depends on other players’ types. He and Yannelis (2016) prove existence in all-pay auctions in which bidders assign a common value to the single prize. Our favorable tie breaking condition applies to a general class of contests with multiple prizes, which includes all-pay auctions, in which the prizes are assigned possibly different (and interdependent) values by different bidders.

We are able to cover a wider range of applications due to several factors. First, Carbonell-Nicolau and McLean (2018) and He and Yannelis (2016) find conditions on the primitives of incomplete-information games which imply that behavioral or distributional strategies satisfy both Reny’s (1999) payoff security and reciprocal upper semi-continuity; these conditions in turn imply better-reply security (according to one of Reny’s (1999) corollaries). In contrast, our conditions on games with ties imply better-reply security directly. Second, we noticed that (a) instead of assuming that the sum of players’ payoffs is upper semi-continuous, it is sufficient to assume that the sum of players’ payoffs is bounded by an upper semi-continuous function; and (b) the deviation (tie-breaking) actions can depend on the

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6Their paper contains a summary of the extensive previous literature on the existence of equilibria in auctions.
opponents’ strategies.

The paper by Allison, Bagh, and Lepore (2018) is also closely related to ours. Similarly to Carmona and Podczeck (2018), Allison, Bagh, and Lepore (2018) are concerned with the invariance of equilibria in some classes of games. They provide a sufficient condition for the invariance of equilibria, which they call *superior payoff matching*, and which is very similar to the existence of our tie breakers. In addition, their condition turns out to be sufficient for the existence of pure-strategy equilibria in some games of *complete* information with discontinuous payoffs. This games include some contests and oligopolies with endogenous choices of product qualities. Allison, Bagh, and Lepore (2018) also refer directly to Reny’s (1999) better-reply security and show existence even in some settings in which better-reply security fails.\footnote{We have been informed by B. Allison that they are currently working on an incomplete-information version of their results.}

Prokopovych and Yannelis (2019) address the problem of the existence of equilibria in first-price auctions with affiliated types and interdependent values. They provide sufficient conditions for the existence of sequences of monotone approximate equilibria whose limits are pure-strategy Bayes-Nash equilibria. Since these authors are concerned with monotone equilibria, their conditions are more restrictive than ours.

Finally, our results for private-value first-price auctions replicate those of Lebrun (1996) and Jackson and Swinkels (2005) in addition to providing novel sufficient conditions for equilibrium existence.

## 2 Main existence theorem

Consider a (possibly) incomplete-information game among \( n \) players whose type spaces are \( X_1, \ldots, X_n \), action spaces are \( B_1 = \cdots = B_n = B \), and payoff functions are \( u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) \) for \( i = 1, \ldots, n \). We assume that \( X_1, \ldots, X_n \) and \( B \) are compact metric spaces and

\[
0 \leq u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) \leq \beta
\]
for all $i$ and $x_1, ..., x_n, b_1, ..., b_n$. (This last assumption is equivalent to assuming that payoffs are bounded.) We denote by $F_i$ the distribution of player $i$’s type $x_i$ and assume that distributions $F_i$ are independent across players. Our aim is to show the existence of equilibria for a class of such games.

We will show the existence of equilibria in *distributional strategies* (introduced by Milgrom and Weber (1985)), which are defined as follows. A distributional strategy of player $i$, which we will denote by $\mu_i$ or $\sigma_i$, is a probability measure on $X_i \times B_i$ whose marginal on $X_i$ coincides with the distribution $F_i$ of player $i$’s type. We endow each player’s set of (distributional) strategies with the weak* topology and observe that the set of strategies is compact.\(^8\) We define the operations of addition and multiplication by a scalar in the usual way,\(^9\) so the strategy sets are also convex. Given a strategy profile $\mu = (\mu_1, \ldots, \mu_n)$, we also denote by $\mu = \mu_1 \times \cdots \times \mu_n$, with some abuse of notation, the product measure on the product space $X_1 \times \cdots \times X_n \times B^n$. Player $i$’s payoff given a strategy profile $\mu$ is $U_i(\mu) = \int u_i(x_1, ..., x_n, b_1, ..., b_n) d\mu$, that is, the expected value of $u_i$ taken with respect to all players’ strategies.\(^{10}\)

We first identify a sufficient condition for equilibrium existence, and then show that this condition is satisfied by many games in which payoff discontinuities arise only when multiple players take the same action. The condition says, roughly, that at any strategy profile with payoff discontinuities, the sum of players’ payoffs can be increased more by players deviating unilaterally than by simultaneously changing players actions and types slightly. We formalize this condition, which we call *improving deviations*, with the following definitions.

**Definition 1** A deviation plan specifies for every profile of strategies $\mu$, every player $i$, and every $\varepsilon > 0$ a measurable function $\tau_i^{\mu, \varepsilon} : X_i \times B_i \to B_i$ such that the deviation strategy $\mu_i^{\varepsilon}$,

\(^8\)The set of probability measures over $X \times B$ is compact, and it is straightforward to verify that the set of strategies is its closed subset.

\(^9\)Given a constant $\lambda$, two measures $\mu$ and $\nu$, and a measurable set $S$, we let $(\mu + \nu)(S) = \mu(S) + \nu(S)$, and $(\lambda \mu)(S) = \lambda \mu(S)$.

\(^{10}\)Throughout the paper we endow product spaces with the product topology, use the Borel $\sigma$-algebras, and assume that the payoff functions $u_i$ are measurable.
which prescribes for type \( x_i \) action \( \tau_i^{\mu,\varepsilon}(x_i, b_i) \) whenever \( \mu_i \) prescribes action \( b_i \), satisfies
\[
U_i(\mu) - \varepsilon \leq U_i(\mu_i^\varepsilon, \mu_{-i}).
\] (1)

A continuous deviation plan is a deviation plan such that every player \( i \)'s payoff \( U_i \) is continuous at \((\mu_i^\varepsilon, \mu_{-i})\) as a function of the strategies of all players other than \( i \).

**Definition 2** A payoff envelope is an upper semi-continuous function \( W(x_1, \ldots, x_n, b_1, \ldots, b_n) \) such that
\[
\sum_{i=1}^n u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) \leq W(x_1, \ldots, x_n, b_1, \ldots, b_n).
\] (2)

**Definition 3** A game has improving deviations if there is a payoff envelope \( W \) and a continuous deviation plan such that if the payoff \( U_i \) of some player \( i \) is discontinuous at \( \mu \) as a function of all players’ strategies, then for some \( \varepsilon > 0 \)
\[
\int W(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu < \sum_{j=1}^n U_j(\mu_j^\varepsilon, \mu_{-j}).
\] (3)

The definitions are easier to understand if we think about the payoff envelope \( W \) as an upper semi-continuous envelope of the sum of payoffs \( \sum u_i \), and about \( \tau_i^{\mu,\varepsilon}(x_i, b_i) \) as an action player \( i \) of type \( x_i \) could use to resolve a payoff discontinuity in her favor (or at least without losing much). We will see that in many applications, \( W \) and \( \tau_i^{\mu,\varepsilon}(x_i, b_i) \) are easy to find. The key condition, (3), says that when each player resolves the discontinuity in her favor the resulting sum of payoffs is higher than \( W \), that is, higher than when the discontinuity is resolved simultaneously for all players in a way that maximizes the sum of payoffs.

**Theorem 1** Every game with improving deviations has a Nash equilibrium in distributional strategies.

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11We can in fact define the payoff envelope \( W \) as the lowest upper semi-continuous envelope of the sum of payoffs, that is,
\[
W(x_1, \ldots, x_n, b_1, \ldots, b_n) = \inf_{\mathcal{O} \in \mathcal{O}} \sup_{x_1, \ldots, x_n, b_1, \ldots, b_n \in \mathcal{O}} \sum_{i=1}^n u_i(x_1, \ldots, \bar{x}_n, b_1, \ldots, \bar{b}_n),
\]
where \( \mathcal{O} \) is the set of open neighborhoods of \( x_1, \ldots, x_n, b_1, \ldots, b_n \). However, in some settings a different \( W \) may be easier to work with.
3 Proof of Theorem 1

To prove Theorem 1, we will use Reny’s (1999) Theorem 3.1. Since the game with distributional strategies is obviously compact and quasi-concave according to Reny’s (1999) terminology, his Theorem 3.1 guarantees that an equilibrium exists if the game is better-reply secure. To define better-reply security, we denote by $\Gamma$ the closure of the graph of the function that maps each profile of strategies to the vector of players’ payoffs. A game is better-reply secure if for every strategy profile $\mu^* = (\mu^*_1, \ldots, \mu^*_n)$ that is not an equilibrium and every vector $u^* = (u^*_1, \ldots, u^*_n)$ such that $(\mu^*, u^*)$ is in $\Gamma$, there is a player $i$, a strategy $\mu_i$, and a number $\eta > 0$ such that player $i$’s payoff from playing $\mu_i$ exceeds $u^*_i + \eta$ for every profile of strategies of the other players in some neighborhood of $\mu^*_{-i}$.

Consider $\mu^*$ and $u^*$ such that $\mu^*$ is not an equilibrium and $(\mu^*, u^*)$ is in $\Gamma$. Suppose first that the payoff of every player is continuous at $\mu^*$, so $u^*_i = U_i(\mu^*)$ for every player $i$. Take a player $i$ that has a profitable deviation $\mu_i$, so $U_i(\mu_i, \mu^*_{-i}) > U_i(\mu^*) + 3\eta$ for some $\eta > 0$. Because the game has improving deviations, there is a continuous deviation strategy $\mu^*_i$ of player $i$ such that $U_i(\mu^*_i, \mu^*_{-i}) > U_i(\mu^*) + 2\eta$. Since $U_i$ is continuous at $(\mu^*_i, \mu^*_{-i})$ as a function of the strategies of all players other than $i$, player $i$’s payoff from playing $\mu^*_i$ exceeds $u^*_i + \eta$ for every profile of strategies of the other players in some neighborhood of $\mu^*_{-i}$.

Now suppose that the payoff of some player $i$ is discontinuous at $\mu^*$. By definition of $\Gamma$, $\mu^k \to_k \mu^*$ and $u^k \to_k u^*$ for some sequence $(\mu^k, u^k)_{k=1}^\infty$ from the graph of the function that maps each profile of strategies to the vector of players’ payoffs (so $u^*_j = U_j(\mu^k)$ for every player $j$).

Since the payoff envelope $W$ is upper semi-continuous,

$$\limsup \int W d\mu^k \leq \int W d\mu^*.$$  

(See Billingsley (1995), Problem 29.1.) Thus, by (2),

$$u^*_1 + \cdots + u^*_n = \lim \int (u_1 + \cdots + u_n) d\mu^k \leq \int W d\mu^*.$$  

\[\text{This follows from the definition of a deviation strategy for } \varepsilon = \eta.\]
By (3) for \( \mu_{-j} = \mu_{-j}^* \), each player \( j \) has a deviation strategy \( \mu_j^* \) (for some \( \varepsilon > 0 \)) such that

\[
\int W d\mu^* < \sum_{j=1}^{n} U_j(\mu_j^*, \mu_{-j}^*),
\]

so

\[
u_1^* + \cdots + u_n^* < \sum_{j=1}^{n} U_j(\mu_j^*, \mu_{-j}^*).
\]

The last inequality implies that there is a player \( j \) such that \( U_j(\mu_j^*, \mu_{-j}^*) < u_j^* + 2\eta \) for some \( \eta > 0 \). Since \( U_j \) is continuous at \( (\mu_j^*, \mu_{-j}^*) \) as a function of all players other than \( j \), player \( j \)'s payoff from playing \( \mu_j^* \) exceeds \( u_j^* + \eta \) for every profile of strategies of the other players in some neighborhood of \( \mu_{-j}^* \).

### 4 Games with ties

Our focus is on games in which payoff discontinuities arise only when two or more players take the same action. We refer to such actions as ties and to such games as games with ties.

**Definition 4** Given an action \( b \) and an action profile \((b_1, \ldots, b_n)\) with \( b_i = b \) for two or more players \( i \), we say that action \( b \) is a tie and that the players \( i \) for whom \( b_i = b \) tie at \( b \).

**Definition 5** A game with ties is a game in which every player \( i \)'s payoff \( u_i \) is continuous at \((x_1, \ldots, x_n, b_1, \ldots, b_n)\) if player \( i \) does not tie at \( b_i = b \).

Not all ties necessarily lead to a payoff discontinuity. We refer to those ties that do as essential ties.

**Definition 6** (i) A tie \( b_i = b \) is essential for player \( i \) at \((x_1, \ldots, x_n, b_1, \ldots, b_n)\) if \( u_i \) is discontinuous at \((x_1, \ldots, x_n, b_1, \ldots, b_n)\) (as a function of all types and bids).

(ii) Strategy profile \( \mu \) has essential ties for player \( i \) if \( \mu \) assigns positive probability to the set \( T_i^* \) of profiles \((x_1, \ldots, x_n, b_1, \ldots, b_n)\) at which player \( i \) has essential ties.

(iii) Strategy profile \( \mu \) has essential ties if it has essential ties for some player.

We will show that for games with ties any deviation plan that avoids essential ties is continuous and the payoff of every player is continuous at any strategy profile that does not
have essential ties. We can therefore identify a class of games with ties that satisfies the assumptions of Theorem 1 as follows.

**Definition 7** A deviation plan without essential ties is a deviation plan $\tau$ such that for every profile of strategies $\mu$, every player $i$, and every $\varepsilon > 0$, strategy profile $(\mu^\varepsilon_i, \mu_{-i})$, where $\mu^\varepsilon_i$ is the corresponding deviation strategy, does not have essential ties for player $i$.

**Definition 8** A game satisfies favorable tie breaking if there is a payoff envelope $W$ and a deviation plan without essential ties such that for any strategy profile $\mu$ with essential ties (3) holds for some $\varepsilon > 0$.

We can now state our first equilibrium existence result for games with ties.

**Theorem 2** A game with ties that satisfies favorable tie breaking has improving deviations and therefore has a Nash equilibrium in distributional strategies.

For example, in all-pay auctions in which higher types have a higher value for the prize, $W$ can be defined as the sum of payoffs when the fair tie-breaking rule is replaced with the rule that gives the prize to the tying player with the highest value, and the bid $\tau^\varepsilon_i(x_i, b_i)$ can be defined as a bid slightly higher than $b_i$. By resolving a tie in her favor each player wins the prize, whereas only one player wins the prize when the tie is resolved simultaneously for all players, so (3) holds. We formalize and generalize this example in Section 6.1. We will see later that in other examples, different types $x_i$ of player $i$ may prefer breaking the same tie in different ways, which in addition can depend on the strategy profile $\mu$.

Favorable tie breaking requires (3) to be satisfied for all strategy profiles with essential ties. This requirement is easy to check, but may fail for some games with ties that has improving deviations. This is because a player’s payoff may be continuous at some strategy profiles that have essential ties for the player. With two players, for example, starting from a tie $b_i = b$, by bidding slightly more than $b$ player $i$’s utility may discontinuously increase if the type of player $j \neq i$ is high and discontinuously decrease if player $j$’s type is low, but change continuously in expectation over player $j$’s type. If (3) fails for such a strategy profile but holds for all strategy profiles at which players’ payoff are discontinuous, then Theorem 2
fails but Theorem 1 holds. This is the case in Section 6.3 below. We can therefore formulate a stronger version of Theorem 2 by using a weaker definition of essential ties.\footnote{This definition is indeed weaker in that a strategy profile that has essential ties on average for player \( i \) also has essential ties for player \( i \), but the reverse is not necessarily true.}

**Definition 9** (i) Given a strategy profile \( \mu_{-i} \), a bid \( b_i = b \) is an essential tie on average for player \( i \) at \( (x_i, b_i) \) if \( \int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\nu_{-i} \) is discontinuous at \( \mu_{-i} \) (as a function of \( \nu_{-i} \)).

(ii) A strategy profile \( \mu \) has essential ties on average for player \( i \) if \( \mu_i \) assigns a positive probability to the set \( T_i^* \) of type-bid pairs \( (x_i, b_i) \) at which player \( i \) has essential ties on average.

(iii) A strategy profile \( \mu \) has essential ties on average if it has essential ties on average for some player.

**Definition 10** A game satisfies favorable tie breaking on average if there is a payoff envelope \( W \) and a deviation plan without essential ties such that for any strategy profile \( \mu \) with essential ties on average (3) holds for some \( \varepsilon > 0 \).

**Theorem 3** A game with ties that satisfies favorable tie breaking on average has improving deviations and therefore has a Nash equilibrium in distributional strategies.

5 Proof of Theorems 2 and 3

By the definition of a game with improving deviations it suffices to prove that (i) in a game with ties a deviation plan without essential ties is a continuous deviation plan, and (ii) any strategy profile at which some player’s payoff is discontinuous as a function of all players’ strategies is a strategy profile with essential ties. Both (i) and (ii) immediately follow from the following result, whose proof is in the appendix.

**Lemma 1** Player \( i \)’s payoff is continuous as a function of all players’ strategies at any strategy profile \( \mu \) that does not have essential ties for player \( i \).
The only difference compared to Theorem 2 is in the definition of essential ties. The proof of Theorem 3 thus only requires a version of Lemma 1 under Definition 9 of essential ties on average. This is the content of the following result, whose proof is in the appendix.

**Lemma 2** Player i’s payoff is continuous as a function of all players’ strategies at any strategy profile \( \mu \) that does not have essential ties on average for player i.

## 6 Applications

We now demonstrate equilibrium existence in three settings of games with ties. Theorem 2 applies to the first two settings but not to the third setting, to which Theorem 3 applies.

The first setting is a model of perfectly discriminating contests, which generalizes multiprize all-pay auctions with complete and incomplete information and allows for private, common, and interdependent values. The key properties of this model are that (i) every player strictly prefers to win any tie at which not all tying players get the same prize, and (ii) the highest allowable bid is strictly dominated by some lower bid. This leads to a simple deviation plan in which every player increases her bid slightly if possible, and the highest possible bid is replaced with a dominating lower bid. Unilateral deviations at an essential tie correspond to all tying players winning the best prize associated with the tie, whereas simultaneously breaking the tie cannot award the best prize to all tying players, so (3) holds.

The second setting is a common-value first-price auction in which the value of the prize strictly increases in all players’ types. The key properties of this model are that (i) for any bid and any strategy profile of the players other than i, at most one type of player i can be indifferent between winning and losing, and (ii) the sum of payoffs is continuous (it is the value of the prize minus the highest bid), so the payoff envelope can be chosen to be the sum of payoffs. This leads to a simple deviation plan in which every player increases or decreases her bid slightly depending on whether in expectation, given her type and the other players’ strategies, she prefers to win or to lose the prize at that bid. Unilateral deviations at any essential tie strictly increase the payoff of at least one player (by property (i)) and do not lower the payoff of any other player, so (3) holds by property (ii). Property (ii) is lost by relaxing common values to interdependent or private values, and this can prevent unilateral
deviations from being better than simultaneous slight changes in players’ actions and types, and even lead to equilibrium non-existence.

The third setting is a Hotelling location model on the interval \([0,1]\) in which prices are exogenous and identical across firms, and firms have private information about their continuous location cost function. The key property of this model is that the sum of payoffs is continuous (it is the mass of consumers minus firms’ location costs), so the payoff envelope can be chosen to be the sum of payoffs. Theorem 2 does not apply to this setting because players’ payoffs may be continuous at a strategy profile with essential ties for which no deviation plan satisfies (3). However, if a player has an essential tie on average at some location, then a slightly higher or lower location strictly increases her payoff. This allows us to apply Theorem 3 by using the simple deviation plan in which every player increases or decreases her location slightly if in expectation, given the other players’ strategies, this strictly increases her payoff. Unilateral deviations at any strategy profile with essential ties on average strictly increase the payoff of at least one player and do not lower the payoff of any other player, so (3) holds by the fact that the payoff envelope is equal to the sum of payoffs.

6.1 Application 1: Multiprize contests with interdependent values

There are \(n\) players who compete for \(n\) prizes. Player \(i\)’s signal \(x_i \in X = [0,1]\) about the prize values is distributed according to a \(cdf F_i\) that does not have an atom at 0. We showed in footnote 1 that without this assumption equilibrium may not exist. The distributions need not be identical, but are commonly known and independent across players. Complete information is a special case, as are distributions that include atoms, gaps, and continuous

\(^{14}\)To see this, suppose there are three players with complete information, consumers are distributed uniformly on \([0,1]\), and consider the strategy profile in which players 1 and 2 locate at 1/2, and player 3 locates at 0 and 1 with equal probability. Then location 1/2 is an essential tie for players 1 and 2 at the location profile \((1/2, 1/2, 0)\) and at the location profile \((1/2, 1/2, 1)\), so the strategy profile has essential ties. But each player’s payoff continuously changes with her strategy. Thus, for sufficiently large (and continuous) costs of moving from players’ chosen locations in the strategy profile, no deviation plan satisfies (3).
components. Each prize is characterized by a number $y$, which represents its position in the prize ranking common to all players, and may be interpreted as an initial public signal about the prize’s value, obtained before players learn their private signals. We order the $n$ prizes so that $y_1 \leq y_2 \leq \cdots \leq y_n$.

Each player $i$ chooses a bid $b_i \in B = [0, 1]$, the player with the highest bid obtains prize $y_n$, the player with the second-highest bid obtains prize $y_{n-1}$, and so on. Ties are resolved by a fair lottery. The utility of player $i$ from obtaining prize $y$ is $\bar{u}_i (x_1, \ldots, x_n, b_1, \ldots, b_n, y)$, where $\bar{u}_i$ is a continuous function of $(x_1, \ldots, x_n) \in X^n$ and $(b_1, \ldots, b_n) \in B^n$ for all prizes $y$. In addition, $\bar{u}_i$ strictly increases in $y$ for all $(x_1, \ldots, x_n)$ and $(b_1, \ldots, b_n)$,\(^{15}\) with an exception for type $x_i = 0$ (type 0 is indifferent across all prizes).\(^{16}\) The strict monotonicity of $\bar{u}_i$ in $y$ makes this a (generalized) contest model: regardless of players’ types or bids it is better to win a higher prize. This is the case, for example, in an all-pay auction, but is not the case in a first-price auction (because at a bid higher than the prize’s value losing is better than winning).\(^{17}\) Notice that players’ utilities need not monotonically decrease in their bids, as they do in an all-pay auction.\(^{18}\) Notice also that the model accommodates private, common, and interdependent values and allows a player’s utility from a given prize to depend on other players’ bids.\(^{19}\)

In this application we assume that bids close to 1 are strictly dominated by a lower bid, and thus irrelevant. More precisely, we assume that for every player $i$ there is a bid $\tilde{b}_i < 1$ such that $\bar{u}_i (x_1, \ldots, x_n, (1, b_{-i}), y_n) < \bar{u}_i (x_1, \ldots, x_n, (\tilde{b}_i, b_{-i}), y_1)$ for all $(x_1, \ldots, x_n)$ and

\(^{15}\) Notice that we do not assume monotonicity in types or bids (of the player or of other players).

\(^{16}\) We could exclude type 0, and assume that $x_i \in [\underline{x}, 1]$ for some $\underline{x} > 0$. This would slightly simplify the analysis. We decided to include type 0 because it appears in most papers on contests with incomplete information.

\(^{17}\) This property also fails in Section 6.2 below and in Section 10.1 of the online appendix, which is why we restrict attention to private values and common values there.

\(^{18}\) For example, in a competition for a dominant market position based on advertising, a moderate level of advertising can increase the demand for the product, and thus the value of winning, by more than the cost of the advertising.

\(^{19}\) Returning to the advertising example from the previous footnote, each firm’s advertising may affect overall market demand, which in turn affects the winning firm’s profit from a dominant market position.
\[ b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n). \] We will relax this assumption in Section 7.1 and then interpret the constraint that \( b_i \leq 1 \) as a bid cap. Finally, \( u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) \) is the expected value of \( \bar{u}_i(x_1, \ldots, x_n, b_1, \ldots, b_n, y) \) given the bid profile \((b_1, \ldots, b_n)\).

These contests are clearly games with ties. We now show that they also satisfy favorable tie breaking, which guarantees equilibrium existence by Theorem 2. A tie \( b_i = b \) is essential for player \( i \) at \((x_1, \ldots, x_n, b_1, \ldots, b_n)\) if \( x_i > 0 \) and not all the players tying at \( b \) obtain equal prizes. Define the payoff envelope \( W \) as the sum of players’ payoffs when ties are broken in a way that maximizes this sum. More precisely, let \( W(x_1, \ldots, x_n, b_1, \ldots, b_n) \) be the sum of \( \bar{u}_i(x_1, \ldots, x_n, b_1, \ldots, b_n, \bar{y}_i((x_1, \ldots, x_n), (b_1, \ldots, b_n))) \) across all players \( i \), where \( \bar{y}_i((x_1, \ldots, x_n), (b_1, \ldots, b_n)) \) is determined as follows: (a) a player with a higher bid obtains a higher prize; and (b) if the bids of two or more players are equal, then among those players, prizes are allocated in any (measurable) way that maximizes the sum of the payoffs of the tying players. This guarantees that \( W \) is upper semi-continuous, and also that (2) holds.

We now describe a deviation plan. The idea is to increase bids slightly in order to win all essential ties, since winning a higher prize is always better. Bid \( b = 1 \) cannot be increased, but can be profitably replaced with a bid of \( \bar{b}_i \) (or a bid close to \( \bar{b}_i \) to avoid ties). Formally, we let the bid \( \tau_i^{\mu, \epsilon}(x_i, b_i) \) be a bid \( b'_i \) such that the marginal on \( B \) of each strategy \( \mu, j \neq i \), does not have an atom at \( b'_i \), and for \( b_i < 1 \) the bid \( b'_i \) satisfies \( b'_i > b_i \) and

\[
\bar{u}_i(x_1, \ldots, x_n, b_1, \ldots, b_n, y) - \bar{u}_i(x_1, \ldots, x_n, (b'_i, b_{-i}), y) < \epsilon
\] (4)

for all \((x_1, \ldots, x_n), b_{-i}, \) and \( y = y_1, \ldots, y_n \). The existence of such a \( b'_i \) follows from the fact that the marginal on \( B \) of each strategy has at most a countable number of atoms and from the uniform continuity of \( \bar{u}_i \) on \( X^n \times B^{n-1} \times \{b_i, 1\} \times \{y\} \) for \( y = y_1, \ldots, y_n \). For \( b_i = 1 \) let \( b'_i \) (in addition to the atom restriction) be such that \( \bar{u}_i(x_1, \ldots, x_n, (b'_i, b_{-i}), y_1) > \bar{u}_i(x_1, \ldots, x_n, (1, b_{-i}), y_1) \). (Recall that \( \bar{u}_i(x_1, \ldots, x_n, (\bar{b}_i, b_{-i}), y_1) > \bar{u}_i(x_1, \ldots, x_n, (1, b_{-i}), y_1) \).) We then have (1) because \( \bar{u}_i \) weakly increases in \( y \) for all \((b_1, \ldots, b_n) \) and \((x_1, \ldots, x_n) \). The deviation plan is without ties because, by Fubini’s Theorem, \( T_i^\mu \) has a positive measure only if the marginals on \( B \) of player \( i \)’s strategy and the strategy of another player have an atom at the same bid.

It remains to show that for any strategy profile \( \mu \) with essential ties (3) holds for some
For this, define function $V_i$ as player $i$'s payoff when ties are broken in her favor, that is, $V_i(x_1, ..., x_n, b_1, ..., b_n) = \bar{u}_i(x_1, ..., x_n, b_1, ..., b_n, \hat{y}_i(b_1, ..., b_n))$ where $\hat{y}_i(b_1, ..., b_n)$ is the prize player $i$ would win if she bids slightly above $b_i$ while every player $j \neq i$ bids $b_j$. Let $V$ be the sum of $V_i$ across all players $i$. Then 

$$V(x_1, ..., x_n, b_1, ..., b_n) \geq W(x_1, ..., x_n, b_1, ..., b_n),$$

with a strict inequality whenever a tie $b$ is essential at $(x_1, ..., x_n, b_1, ..., b_n)$ for two or more players (because $x_i > 0$ if the tie $b = b_i$ is essential for player $i$, and $\bar{u}_i$ strictly increases in $y$ whenever $x_i > 0$). Now, take a strategy profile $\mu$ with essential ties. Since for each player $i$ type $x_i = 0$ has probability 0, we have that 

$$\alpha = \int V(x_1, ..., x_n, b_1, ..., b_n) d\mu - \int W(x_1, ..., x_n, b_1, ..., b_n) d\mu > 0.$$ 

By (4) and the definition of $\tau^\mu_{i,\varepsilon}(x_i, b_i)$ for $b_i = 1$, we have 

$$\int V(x_1, ..., x_n, b_1, ..., b_n) d\mu - \sum_{j=1}^{n} U_j(\mu^\varepsilon_j, \mu^*_{-j}) < n\varepsilon,$$

where $\mu^\varepsilon_j$ is the deviation strategy associated with $\tau^\mu_{j,\varepsilon}$. Therefore, (3) holds for $\varepsilon < \alpha/n$.

### 6.2 Application 2: First-price auctions with common values

There are $n$ players who compete for one prize. Player $i$’s signal $x_i \in X = [0, 1]$ about the prize value is distributed according to a continuous cdf $F_i$.\(^{20}\) The distributions need not be identical, but are commonly known and independent across players. Each player $i$ submits a bid $b_i \in B = [0, 1]$, and the player with the highest bid wins the prize and pays her bid. Ties are resolved by a fair lottery. The value of the prize $v(x_1, ..., x_n)$ is common to all players, with $v(0, ..., 0) = 0$ and $v(1, ..., 1) < 1$,\(^{21}\) strictly increasing in each signal, and continuous as a function of the entire profile of signals. The utility of player $i$ is 

$$\bar{u}_i(x_1, ..., x_n, b_i) = \begin{cases} v(x_1, ..., x_n) - b_i & \text{if } i \text{ wins the prize}, \\ 0 & \text{if } i \text{ does not win the prize}. \end{cases}$$

\(^{20}\)Reny (1999) also allows for multi-dimensional types, i.e., $x_i \in [0, 1]^m$ for $i = 1, ..., n$. Our existence result can be generalized to this setting.

\(^{21}\)That $v(1, ..., 1) < 1$ guarantees that any equilibrium remains an equilibrium if bidders can place any non-negative bid.
Finally, $u_i(x_1, ..., x_n, b_1, ..., b_n)$ is the expected value of $\bar{u}_i(x_1, ..., x_n, b_i)$ given the bid profile $(b_1, ..., b_n)$.

Various versions of common-value auctions have been studied by several other authors, and as pointed out in the introduction, some existence results have already been established. The added value of our result in this context is that we do not make additional assumptions that restrict its range of applications. In particular, whether a player wants to win or lose a particular tie may depend on the signals of the other players. Relatedly, this application illustrates the importance of the feature of our existence result that the deviation plan $\tau_i^{\mu, \varepsilon}$ is allowed to depend on the profile of strategies $\mu$.

These auctions are clearly games with ties. We now show that they also satisfy favorable tie breaking. A tie $b_i = b$ is essential for player $i$ at $(x_1, ..., x_n, b_1, ..., b_n)$ if $b$ is the winning bid and $b_i \neq v(x_1, ..., x_n)$. Define the payoff envelope $W$ as the sum of players’ payoffs, that is,

$$W(x_1, ..., x_n, b_1, ..., b_n) = \sum_{i=1}^{n} u_i(x_1, ..., x_n, b_1, ..., b_n) = v(x_1, ..., x_n) - \max\{b_1, ..., b_n\},$$

so $W$ is continuous and (2) holds as an equality.

We now describe a deviation plan. The idea is at every essential tie $b$ to either increase or decrease the bid slightly, depending on whether (in expectation, conditional on tying at $b$) winning is better than losing. Notice that whether winning is better than losing at a particular bid depends on the other players’ strategies, so unlike with contests the deviation plan will depend on the profile of strategies (beyond avoiding ties). More precisely, consider for each player $i$ and bid $b_i$ the event $E_i(b_i) \subset X_{-i} \times B_{-i}$ in which $b_i$ is a winning bid and at least one other player bids $b_i$. If $\mu_{-i}(E_i(b_i)) > 0$, let $\tau_i^{\mu, \varepsilon}(x_i, b_i)$ be a bid slightly higher or lower than $b_i$, depending on whether, conditional on $E_i(b_i)$, player $i$ of type $x_i$ prefers winning the prize and paying the winning bid, or losing and paying nothing.\footnote{Of course, ex-post, which of the two options player $i$ prefers also depends on the types of the other players, but $\tau_i^{\mu, \varepsilon}$ is allowed to depend only on the type and bid of player $i$. Thus, by a preferred option we mean the option that gives player $i$ the higher expected payoff (over the strategies of the other players) conditional on $E_i(b_i)$.} Note that player $i$ strictly prefers one of the two options, except possibly at a single type $x_i$, because
$v(x_1, ..., x_n)$ strictly increases in $x_i$. Let $\tau_{i,x}^*(x_i, b_i) = b_i$ for the (at most one) type $x_i$ that is indifferent between the two options, and for all types $x_i$ when $\mu_{-i}(E_i(b_i)) = 0$. Notice that at $b_i = 1$ player $i$ strictly prefers losing to winning because $v$ increases in all players’ signals and $v(1, ..., 1) < 1$.

By choosing $\tau_{i,x}^*(x_i, b_i)$ sufficiently close to $b_i$ (whenever $\tau_{i,x}^*(x_i, b_i) \neq b_i$), we have $U_i(\mu) \leq U_i(\mu_i^\varepsilon, \mu_{-i})$, where $\mu_i^\varepsilon$ is the corresponding deviation strategy, with a strict inequality whenever $\mu$ has essential ties for player $i$. This is because if $\mu$ has essential ties for player $i$ then (by Fubini’s theorem) the marginal of $\mu_i$ on $B$ assigns positive probability to some $b_i$ for which $E_i(b_i) > 0$, and by choosing $\tau_{i,x}^*(x_i, b_i)$ sufficiently close to $b_i$ the expected payoff of player $i$ of every (except at most one) type $x_i$ from playing $\tau_{i,x}^*(x_i, b_i)$ strictly increases relative to playing $b_i$. This guarantees that (1) holds, and also that (3) holds. In addition, whenever $\tau_{i,x}^*(x_i, b_i) \neq b_i$ we choose $\tau_{i,x}^*(x_i, b_i)$ to be different from any bid at which the marginal on $B$ of any strategy $\mu_j$, $j \neq i$, has an atom. Thus, the deviation plan is without essential ties, which completes the demonstration that these auctions are games with ties that satisfy favorable tie breaking.

### 6.3 Application 3: Hotelling models

A finite number of firms compete for a unit mass of customers, each with unit demand. The customers are distributed with a positive density on the interval $[0,1]$. Each firm chooses a location in $[0,1]$, and then each customer chooses the firm closest to his or her location. We assume that prices are fixed, equal across firms, and normalized to 1. Variable production costs are negligible, so each firm’s payoff is equal to the share of customers buying from the firm minus the cost of locating at the firm’s chosen location. Firms face different costs of choosing different locations. Firm $i$’s type $x_i \in [0,1]$ is distributed according to a cdf $F_i$, and the location cost function $c_i(x_i, b_i)$ is a continuous function of the firm’s type $x_i$ and location $b_i$. We assume that each firm must choose some location.

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23 The fact that the marginal of $\mu_i$ on $B$ assigns positive probability to $b_i$ together with the assumption that $F_i$ is continuous guarantee the strict increase for a positive measure of types $x_i$.

24 The additive separability of the customer share, which does not depend on $x_i$, and the cost, which depends on $x_i$, is important for the analysis.
The existence of mixed-strategy equilibria in the complete-information setting was established by Simon (1987), whose result allows for multi-dimensional locations (for example, customers can be distributed on, and firms can choose their locations from, the cube $[0, 1]^m$). We conjecture that our existence result generalizes to settings with multi-dimensional locations. However, finding sharp conditions for existence would probably require writing a separate paper, given the complexity of Simon’s (1987) analysis of complete information settings.\(^{25}\) Our objective here is not to generalize Simon’s (1987) result but to demonstrate that Theorem 3 can be used to prove equilibrium existence in some Hotelling models with incomplete information.

The Hotelling model with incomplete information is clearly a game with ties. We now show that it also satisfies favorable tie breaking on average. Define the payoff envelope $W$ as the sum of players’ payoffs (shares minus location costs), that is,

$$W(x_1, \ldots, x_n, b_1, \ldots, b_n) = 1 - \sum_{i=1}^{n} c_i(x_i, b_i),$$

so $W$ is continuous and (2) holds as an equality.

We now describe a deviation plan. The idea is to shift a player’s location slightly if doing so leads to a discrete payoff increase. More precisely, consider for each player $i$ and location $b_i$ the event $E_i(b_i) \subset X_{-i} \times B_{-i}$ in which at least one other player locates at $b_i$. If $\mu_{-i}(E_i(b_i)) > 0$, let $\tau_i^{\mu_{-i}}(x_i, b_i)$ be a location slightly to the left or to the right of $b_i$ (different from any location at which the marginal on $B$ of any strategy $\mu_j$, $j \neq i$, has an atom) if, conditional on $E_i$, an infinitesimal shift in this direction discretely increases player $i$’s (expected) market share, that is, discretely increases player $i$’s payoff.\(^{26}\) Notice that

\(^{25}\)Multi-dimensional versions of Hotelling’s model are settings in which Reny’s (1999) approach, and our Theorem 1, are able to guarantee the existence of an equilibrium, while existence may be difficult or impossible to establish by approximating the original game with a sequence of games with a finite number of actions and types. This is because profitable deviations of different types at different locations may require shifting locations in different and specific directions.

\(^{26}\)By an infinitesimal shift from a location $b$ in a direction we mean the same location $b$ but with a different market sharing rule: Customers in the direction of the shift find the firm’s shifted location closer than location $b$, and customers in the opposite direction of the shift find the firm’s shifted location farther away than location $b$. 

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whether an infinitesimal shift in a particular direction is beneficial for a player depends on the strategies $\mu_{-i}$ of the other players but is independent of the player’s type, since types only affect location costs. Notice also that the improving infinitesimal shift must be to the right for $b_i = 0$ and to the left for $b_i = 1$. If no infinitesimal shift discretely increases player $i$’s payoff let $\tau_i^{\mu, \varepsilon}(x_i, b_i)$ be a location close to or at $b_i$ different from any location at which the marginal on $B$ of any strategy $\mu_j$, $j \neq i$, has an atom. If $\mu_{-i}(E_i(b_i)) = 0$, let $\tau_i^{\mu, \varepsilon}(x_i, b_i) = b_i$ for all types $x_i$. Similarly to Section 6.2, (1) holds and the deviation plan is without essential ties.

Since the payoff envelope $W$ is equal to the sum of players’ payoffs, to apply Theorem 3 it suffices to show that if a strategy profile $\mu$ has essential ties on average for player $i$, then $U_i(\mu) < U_i(\mu_i^*, \mu_{-i})$, where $\mu_i^*$ is the corresponding deviation strategy. By construction, $U_i(\mu) < U_i(\mu_i^*, \mu_{-i})$ whenever there is a positive $\mu_i$-measure of bids $b_i$ for which an infinitesimal shift in player $i$’s location discretely increases player $i$’s market share. Thus, to conclude the proof it suffices to show that if $\mu$ has essential ties on average for player $i$ then there is a positive $\mu_i$-measure of bids $b_i$ for which an infinitesimal shift in player $i$’s location discretely increases player $i$’s market share. We will show that if no infinitesimal shift from location $b_i$ discretely increases player $i$’s market share, then location $b_i$ is not an essential tie on average for player $i$ (recall that whether a bid is an essential tie for a player is independent of the player’s type).

Consider a strategy profile $\mu$ and a location $b_i$ for which no infinitesimal shift discretely increases player $i$’s market share. Suppose that player $i$ bids $b_i$. If no other player has an atom at $b_i$, then player $i$’s payoff is continuous as a function of the other players’ strategies, because discontinuities arise only at ties. It cannot be that two or more other players have an atom at $b_i$, because then an infinitesimal shift in player $i$’s location would discretely improve player $i$’s payoff (she discretely increases her share on the side to which she is moving, so at least one side is strictly profitable). And if only one other player $j$ has an atom at $b_i$, then if that player moves slightly (which is what it means for his strategy to be weak$^*$-close to $\mu_j$),

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27 Thus, unlike in Section 6.2, it is possible that for some locations $b_i$ with $\mu_{-i}(E_i(b_i)) > 0$ no infinitesimal shift is profitable for player $i$. This prevents the use of Theorem 2, as discussed in the description of the Hotelling application at the beginning of Section 6.
that is very similar to player $i$ moving slightly - but that does not increase player $i$’s payoff because an infinitesimal shift does not change player $i$’s payoff. We formalize this argument with the following two claims, whose proofs are in the appendix.

**Claim 1** If no infinitesimal shift from location $b_i$ increases player $i$’s market share, then the marginal on $B$ of the strategy of only one player $j \neq i$ can have an atom at $b_i$.

**Claim 2** If (a) for no player $j \neq i$ the marginal on $B$ of the strategy of player $j$ has an atom at $b_i$, or (b) the marginal on $B$ of the strategy of only one player $j \neq i$ has an atom at $b_i$, and no infinitesimal shift from location $b_i$ discretely increases player $i$’s market share, then $b_i$ is not an essential tie at any $(x_i, b_i)$.

### 7 Beyond favorable tie breaking

As we demonstrated in Section 6, Theorems 2 and 3 can be used to establish equilibrium existence in many games with ties. These results should be used with care, however, since slight changes to such games can lead to games with ties to which Reny’s (1999) existence result (and therefore Theorems 1, 2, and 3), as well as other equilibrium existence results in the literature, do not apply, even though an equilibrium exists. We now present two applications that illustrate this claim. The approximation methods we use to prove equilibrium existence may prove useful for additional applications.

The first application is (general) contests with bid caps. The difficulty here is that if players tie at the cap, they can break the tie only by reducing their bids, which may lower their payoff if the value of the prize exceeds the cap, whereas slightly and simultaneously changing all players’ actions and valuations can increase the sum of payoffs.28 To see this, suppose two players with valuations drawn uniformly from the interval $[1, 3]$ compete in an all-pay auction with a bid cap of 1, and consider the strategy profile $\mu$ in which both players bid 1. Both players’ payoffs are discontinuous at $\mu$, and the value of any payoff envelope at

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28 A similar issue appears in at least two other cases: (i) if a bid cap is imposed in a first-price auction, and (ii) in the Hotelling model if customers are distributed on $\mathbb{R}$ with a substantial mass of customers below 0 (or above 1, or both) but firms can choose their locations only on the interval $[0, 1]$. 
\( \mu \) is no lower than the sum of payoffs when the fair tie-breaking rule is replaced with giving the prize to the tying player with the highest value, that is, no lower than 1/3.\(^{29}\) However, when the other player bids 1, the payoff from any strategy is at most 1/8,\(^{30}\) so the sum of payoffs from unilateral deviations is at most 1/4. Thus, there is no deviation plan for which (3) holds.\(^{31}\) Our proof of equilibrium existence, which will cover this example, requires only that players’ valuation distributions do not have an atom at 0.\(^{32}\) We will view contests with bid caps as limits of a particular sequence of contests with bids caps and a finite number of types and possible bids, and show that the limit of the equilibria along the sequence is an equilibrium of the original contest.\(^{33}\) The key to our proof is the property that if some player bids at the cap, then no other player wants to bid close to, but less than, the cap (see Claim 6 below).\(^{34}\)

The second application is first-price auctions with private values. The issue with using our existence results is that while unilateral tie breaking can increase the sum of payoffs, this increase may not exceed the increase generated by slightly and simultaneously changing

\(^{29}\)The probability that a player with type \( x \) is awarded the prize at \( \mu \) under this tie-breaking rule is \((x - 1)/2\), so the sum of payoffs is

\[
2 \left( \int_1^3 x \frac{x - 1}{2} \, dx - 1 \right) = 1/3.
\]

\(^{30}\)The best response for types lower than 2 is 0 and for types higher than 2 is 1.

\(^{31}\)Reny’s (1999) better-reply security also fails in this example. To see this, denote by \( \sigma^\varepsilon \) a strategy that assigns higher bids to higher types with all bids in \([1 - \varepsilon, 1]\), and notice that the strategy profile \((\sigma^\varepsilon, \sigma^\varepsilon)\) converges in weak* to \( \mu \) as \( \varepsilon \) approaches 0. When the players play \((\sigma^\varepsilon, \sigma^\varepsilon)\), each player’s payoff approaches 1/6 as \( \varepsilon \) approaches 0, so \((1/6, 1/6)\) is a limiting payoff vector. Even though \( \mu \) is not an equilibrium, no player can get more than 1/8 when the other player bids 1 (as explained in the text).

Reny’s (2016) point security also fails in this example. For small \( \varepsilon > 0 \), each player’s payoff when the players play \((\sigma^\varepsilon, \sigma^\varepsilon)\) exceeds the payoff from any strategy \( \sigma \) when the other player bids 1.

\(^{32}\)Footnote 1 in the introduction shows that equilibrium may not exist if a player has an atom at 0.

\(^{33}\)The equilibria along the sequence induce (in a straightforward way) \( \varepsilon \)-equilibria of the original contest, so our proof can be viewed as showing that a specific sequence of \( \varepsilon \)-equilibria converges to an equilibrium.

\(^{34}\)This is an important difference between contests and, for example, Hotelling models. In the Hotelling model of Section 6.3, if some players choose an extreme location (0 or 1), then other players may have an incentive to choose different locations that are close to it.
all players’ actions and valuations. To see this, suppose there are two players with known valuations \(0 < x_1 < x_2\), and consider the strategy profile in which both players bid \(b = x_1\). The tie \(b\) is essential at \((x_1, x_2, b, b)\) (player 2’s payoff is discontinuous there), and by upper semi-continuity the value of any payoff envelope at \((x_1, x_2, b, b)\) is at least \(x_2 - x_1\), which corresponds to breaking the tie in favor of player 2. But \(u_2(x_1, x_2, b, b) < x_2 - x_1\) for any bid \(b_2\), and \(u_1(x_1, x_2, b_1, b) \leq 0\) for any bid \(b_1\), so there is no deviation plan for which (3) holds.\(^{35,36}\) For our proof of equilibrium existence, which covers this example, we will view first-price auctions as limits of a sequence of first-price auctions with costly bids as the cost approaches 0. In a first-price auction with costly bids losing players pay a (possibly bid-dependant) cost and only the winner pays her bid. Equilibrium existence for first-price auctions with costly bids is established in Section 10.1 of the online appendix, and requires only that players’ valuation distributions do not have an atom at 0. We will provide sufficient conditions on players’ valuation distributions (in addition to no atoms at 0) for the limit of the equilibria along the sequence to be an equilibrium of the first-price auction.\(^{37}\) Some of these sufficient conditions replicate known results from the literature while others are new.

We also give an example that demonstrates that the conditions we provide are in a certain sense necessary.

\(^{35}\)Reny’s (1999) better-reply security also fails in this example: \((b, b)\) (for \(b = x_1\)) is not an equilibrium but there is no bid of player 2 that gives her a payoff strictly higher than \(x_2 - x_1\) when player 1 bids \(x_1\), and there is no bid of player 1 that gives her a payoff higher than 0 when player 2 bids \(x_1\).

Reny’s (2016) point security also fails in this example. Player 1 cannot get more than 0 whenever player 2 bids at least \(x_1\). To get more than 0 when player 1 bids in some open neighborhood of \(x_1\), player 2 must bid strictly more than \(x_1\). Consider \(x_2^* = b > x_1\). Consider a pair \(y = (x_1, y_2)\) for \(y_2 \in (x_1, x_2^*)\) and a pair \(x' = (x_1, x_2')\). The payoff of player 2 from the pair \((x_1, x_2')\) is less than his payoff from \(y\), so point security fails if player 2 uses pure strategies. A very similar argument shows that point security fails even if player 2 uses a mixed strategy \(x_2^*\).

\(^{36}\)Other results on equilibrium existence, including, for example, those in Section 5 of Reny (2019) likely fail as well, since it is also not the case that the limit of every sequence of \(\varepsilon\)-Nash equilibrium is a Nash equilibrium. Going back to the example, consider bidder 1 bidding \(x_1\) and bidder 2 bidding \(x_1 + \varepsilon\). This is an \(\varepsilon\)-Nash equilibrium, but the limit is a tie at \(x_1\), which is not a Nash equilibrium.

\(^{37}\)Footnote 1 in the introduction shows that equilibrium may not exist if a player has an atom at 0.
7.1 Contests with bid caps

Contests were defined in Section 6.1, in which we assumed that bids are restricted to $B = [0, 1]$ but no player wants to bid 1 or slightly less than 1. In this section, we are concerned with cases in which players may want to bid 1 or more, but such bids are not allowed by the contest rules. We interpret this restriction as a bid cap. (Our analysis will, however, apply to all contests, whether or not players may want to bid 1 or more). The only restriction we impose is that type $x = 0$, which is indifferent between all prizes, does not want to bid 1.

To prove equilibrium existence we consider a sequence of approximating discrete contests. For each $m$, let $J^m = \{0, 1/2^m, 2/2^m, ..., 1\}$ and consider a contest in which players’ types are drawn from $J^m$. The probability that player $i$’s type is $k/2^m$ is $F_i(k/2^m) - F_i((k - 1)/2^m)$. Players are restricted to bids in $J^m$. Such contests have equilibria (possibly in mixed strategies) by the Kakutani fixed-point theorem, and the equilibrium strategies $\sigma_1^m, ..., \sigma_N^m$ can be identified with probability measures on $X \times B$. Let $\sigma = (\sigma_1, ..., \sigma_n)$ be the weak* limit of a subsequence of a sequence $(\sigma^m)_{m=1}^\infty$ of equilibria, where $\sigma^m = (\sigma_1^m, ..., \sigma_n^m)$. This subsequence can with no loss of generality be assumed to be the entire sequence. We will show that $\sigma$ is an equilibrium of the original contest.

It follows directly from the definition of weak* convergence in Billingsley (1995) that the marginal of $\sigma_i$ on $X$ coincides with distribution $F_i$. So, it remains to check that no player can profitably deviate to another strategy.

Suppose (to the contrary) that in the original contest player $i$ has a profitable deviation that increases her payoff by some $C > 0$ relative to playing $\sigma_i$ against $\sigma_{-i}$. We first show that player $i$ also has a profitable deviation of a particular form. For this, observe that any measurable function $h : X \to J^m$, together with the condition that the marginal of a strategy on $X$ is equal to $F_i$, uniquely determine a strategy for player $i$ in the original contest.

**Claim 3** For sufficiently small $\varepsilon > 0$, if $m$ is sufficiently large, there exists a measurable function $h : X \to J^m$ such that: (i) the strategy defined by $h$ for player $i$ is a profitable

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38 We assumed that for every player $i$ there is a bid $\bar{b}_i < 1$ such that $\bar{u}_i(x_1, ..., x_n, (1, b_{-i}), y_n) < \bar{u}_i(x_1, ..., x_n, (\bar{b}_i, b_{-i}), y_1)$ for all $(x_1, ..., x_n)$ and $b_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_n)$.

39 Formally, for every player $i$ and all type profiles $x_{-i}$ and bid profiles $b_{-i}$ of the other players there is a bid $\bar{b}_i < 1$ such that $\bar{u}_i((0, x_{-i}), (1, b_{-i}), y_n) < \bar{u}_i((0, x_{-i}), (\bar{b}_i, b_{-i}), y_1)$. 25
deviation in the original contest that increases player $i$’s payoff relative to playing $\sigma_i$ against $\sigma_{-i}$ by at least by $C/2$; and (ii) for any $j \neq i$, the marginal on $B$ of strategy $\sigma_j$ has no atom larger than $\varepsilon$ at any bid in $h(X) - \{1\}$.

**Proof.** Denote by $A \subseteq B\{1\}$ the set of bids lower than 1 at which the marginal of one or more of the strategies $\sigma_j$, $j \neq i$, has an atom of size larger than $\varepsilon$. Let $K^m = J^m\setminus A$. We will define a function $h$ with range $K^m$ that will satisfy part (i) of the claim. Notice that because $A$ is finite, the distance between any two consecutive points in $K^m$ approaches 0 as $m$ grows large.

We first argue that if $m$ is sufficiently large, then player $i$ has a profitable deviation that increases her payoff relative to playing $\sigma_i$ at least by $C/2$, and in which each type $x$ of player $i$ chooses a bid in $K^m$ with probability 1. Indeed, for every type $x$ there exists a bid $t$ such that the expected payoff of type $x$ from bidding $t$ against $\sigma_{-i}$ is lower by at most $C/4$ than the expected payoff from bidding any other $t'$. And if $m$ is sufficiently large, bidding $h(x) = \tau$ such that $\tau' < t \leq \tau$, where $\tau'$ and $\tau$ are consecutive bids in $K^m$, reduces the expected payoff of type $x$, compared to bidding $t$, by at most $C/4$. Moreover, the same $m$ can be chosen for all types $x$, because $U_i$ is continuous given $y$ and the spaces of types and bids are compact.

In addition, the bids $h(x)$ can be chosen such that the function $h$ is measurable. Indeed, for each $t$ in $K^m$, let $X_t$ consist of all the types $x$ whose expected payoff from bidding $t$ against $\sigma_{-i}$ is at least as high as the expected payoff from bidding any other $t'$ in $K^m$. Then, the set $X_t$ is closed (by continuity of $U_i$ in $x$). So, we can, for example, define $h(x)$ as the largest $t$ such that $x$ is in $X_t$. Therefore, the strategy determined by $h$ is a profitable deviation, which increases the payoff relative to playing $\sigma_i$ by at least $C/2$, which establishes part (i) of the claim. $\blacksquare$

The following simple lemma, whose proof is in the appendix, will be useful in proving the claims that follow.

**Lemma 3** For any player $j$, strategy $\nu_j$ in the original contest, bid $t$, and number $\eta > 0$, there exists a number $\delta > 0$ such that

$$\nu_j(X \times ((t - \delta, t) \cup (t, t + \delta))) < \eta.$$  \hspace{1cm} (5)
That is, player $\tau$ bids in $(t - \delta, t + \delta) - \{t\}$ with an arbitrarily low probability.

Given an $\varepsilon > 0$, let $m$ be such that Claim 3 holds for player $i$, and let $\mu_i$ be the strategy defined by a function $h: X \to J^m$ with properties (i) and (ii) from Claim 3. Note that $h$ is also a function from $X$ to $J^m'$ for any $m' > m$.

**Claim 4** For every sufficiently large $m$, player $i$'s payoff from playing $\mu_i$ against $\sigma_i^m$ is lower than the payoff of playing $\mu_i$ against $\sigma_{-i}$ by no more than $D\varepsilon$, where $D$ is a constant that is independent of $\varepsilon$ and $m$.

**Proof.** The claim is true (with, for example, $D = 1$) for $\mu_i$ conditional on the set of types $x$ such that $h(x) = 1$. Indeed, by the result in Billingsley (1995), problem 29.1, for sufficiently large $m$, the probability that the marginal of $\sigma_j^m (j \neq i)$ on $B$ assigns to $t = 1$ can only be slightly higher than that assigned by the marginal of $\sigma_j$. Thus, in what follows we consider $\mu_i$ conditional on the set of types $x$ such that $h(x)$ takes some fixed value $t < 1$.

For this bid $t$, take a $\delta$ such that (5) from Lemma 3 is satisfied for $\eta = \varepsilon$, $\nu_j = \sigma_j$, and all $j \neq i$. Player $i$'s payoff from bidding $t$ against $\sigma_{-i}$ is the sum of the following two components: (a) the payoff from bidding $t$ against $\sigma_{-i}$ contingent on the bids of all players $j \neq i$ belonging to $[0, t - \delta/2) \cup (t + \delta/2, 1]$; (b) the payoff of bidding $t$ against $\sigma_{-i}$ contingent on the bids of at least one player $j \neq i$ belonging to $[t - \delta/2, t + \delta/2]$. The payoff of bidding $t$ against $\sigma_{-i}^m$ is the sum of two analogous components.

First, we argue that for sufficiently large $m$ player $i$'s payoff from bidding $t$ against $\sigma_{-i}^m$ is lower by at most $\varepsilon$ than her payoff from bidding $t$ against $\sigma_{-i}$, contingent on the bids of all players $j \neq i$ (under $\sigma_{-i}^m$ and $\sigma_{-i}$, respectively) belonging to $[0, t - \delta/2) \cup (t + \delta/2, 1]$.\(^{40}\)

The event that the bids of all players $j \neq i$ belong to $[0, t - \delta/2) \cup (t + \delta/2, 1]$ can be represented as the union of $2^{n-1}$ events of the form

$$\prod_{j \neq i} X \times I_j,$$

where $I_j = [0, t - \delta/2)$ or $I_j = (t + \delta/2, 1]$ for each $j \neq i$. Define a function $f: X^{n-1} \times B^{n-1} \to \mathbb{R}$ by setting its value on each such event in the union to be the payoff of player $i$ from

\(^{40}\)We stress that by the payoff of player $i$ we mean her expected payoff from bidding $t$, where the expectation is taken across the types $x$ with $h(x) = t$. 27
bidding \( t \) and obtaining the \((k + 1)\)-st highest prize, where \( k \) is the number of \( j \)'s such that \( I_j = (t + \delta/2, 1] \). On the complement of all the \( 2^{n-1} \) components, set \( f \) to equal 0. Note that the set \( X \times \{ [0, t - \delta/2) \cup (t + \delta/2, 1] \} \) is open, so our function \( f \) is lower semi-continuous because we assumed that the payoffs are no lower than zero. Thus, by the result in Billingsley (1995), problem 29.1, the integral of this function with respect to measure \( \sigma_{\mu}^m \) for sufficiently large \( m \), can be lower at most by \( \varepsilon \) than the same integral with respect to measure \( \sigma_{-i} \). But by bidding \( t \) player \( i \) outbids all players \( j \) who bid in \([0, t - \delta/2) \), and is outbid by all players \( j \) who bid in \((t + \delta/2, 1] \). Thus, the former integral is player \( i \)'s payoff when she bids \( t \) against \( \sigma_{-i}^m \), and the latter integral is her payoff when she bids \( t \) against \( \sigma_{-i} \).

Now suppose that the bids of some player \( j \neq i \) belong to \([t - \delta/2, t + \delta/2) \). By property (ii) from Claim 3 and (5) (recall that \( \eta = \varepsilon \)), the probability that \( \sigma_{-i} \) assigns to the event that the bid of some player \( j \neq i \) belongs to \([t - \delta/2, t + \delta/2) \) is lower than \( 2\varepsilon(n - 1) \).

Thus, by Theorem 29.1 in Billingsley (1995), for sufficiently large \( m \) the probability assigned by \( \sigma_{-i}^m \) to the event that the bid of some player \( j \neq i \) belongs to \([t - \delta/2, t + \delta/2) \) is also lower than \( 2\varepsilon(n - 1) \), because the event is a closed set. This implies that the payoff of player \( i \) against \( \sigma_{-i} \) and the payoff of player \( i \) against \( \sigma_{-i}^m \), both contingent on the bid of some player \( j \neq i \) belonging to \([t - \delta/2, t + \delta/2) \) are lower than \( 2\varepsilon(n - 1)\beta \), because we assumed that the payoffs are bounded by \( \beta \).

Notice that \( \mu_i \) is a strategy in the original contest but not necessarily in the \( m \)-th approximating contest (or in any of the other approximating contests) because while its marginal on \( B \) puts probability 1 on \( J^m \), its marginal on \( X \) is \( F_i \). We therefore define strategy \( \mu_i^m \) in the \( m \)-th approximating contest by letting type \( k/2^m \) bid an \( h^m(x) \) that maximizes this type’s expected payoff against \( \sigma_{-i} \) over all bids \( h(x) \) for \( x \in ((k - 1)/2^n, k/2^n] \). Claim 4 guarantees that for large enough \( m \) the difference between the payoff of \( \mu_i \) against \( \sigma_{-i}^m \) and the payoff of \( \sigma_i \) against \( \sigma_{-i} \) is positive and bounded away from 0. By continuity of player \( i \)'s payoff in \( x \), if \( m \) is sufficiently large, then the difference between the payoff of \( \mu_i^m \) against \( \sigma_{-i}^m \) and the payoff of \( \sigma_i \) against \( \sigma_{-i} \) is positive and bounded away from 0. Thus, it suffices to show that the payoff of \( \sigma_i^m \) against \( \sigma_{-i}^m \) converges to that of \( \sigma_i \) against \( \sigma_{-i} \). This is the most

\footnote{For each player \( j \neq i \), one \( \varepsilon \) is for \([t - \delta/2, t) \cup (t + \delta/2, 1] \) and one \( \varepsilon \) is for a possible atom at \( t \).}
challenging and nonstandard part of the proof. To show this part, we need to establish some properties of strategy profiles $\sigma^m$ and $\sigma$.

Given a strategy profile $\nu = (\nu_1, ..., \nu_n)$, we say that a bid $t$ is a tie if the strategies of two or more players have an atom at $t$, and we refer to those players as the tying players (at $t$). If, in addition, the prizes that the tying players receive are not all equal, then we say that $t$ is an essential tie.

**Lemma 4** At any strategy profile $\nu$ such that player $i$ is not a tying player at any essential tie, player $i$’s payoff is continuous (as a function of all players’ strategies).

This lemma follows directly from Lemma 1 in Section 5.

The next two claims establish a property of contests with bid caps that is key in our proof of equilibrium existence. We came up with the idea for this property and the proof of equilibrium existence by considering the following example of a two-player all-pay auction with complete information, analyzed in Che and Gale (1998).

**Example 1** Consider a two-player all-pay auction. Player 1 values the object at $v_1 = 4/3$, and player 2 values the object at $v_2 = 8/3$. Recall that no player is allowed to bid more than 1.

In the unique equilibrium, player 1 bids 0 with probability 1/2, bids uniformly on $[0, 2/3]$ with probability 1/4, and bids 1 with probability 1/4. Player 2 bids uniformly on $[0, 2/3]$ with probability 1/2 and bids 1 with probability 1/2. The equilibrium strategies have an essential tie only at the bid cap 1, and no player bids close to the cap.

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42 For example, this does not hold in an asymmetric two-player complete-information first-price auction with a fine grid of bids: it is an equilibrium for the player with the lower valuation to bid the highest bid weakly lower than her valuation and for the other player to bid the next highest bid, but this player’s payoff drops discretely in the limit (whereas the payoff of the player with the lower valuation remains 0).

43 In particular, it follows from Fubini’s Theorem that player $i$ ties at $\nu$ if and only if for some $j \neq i$ the set $T_{ij} = \{(x_1, b_1), \ldots, (x_n, b_n)) \in (X \times B)^n : b_i = b_j \}$ has strictly positive $\nu$-measure.

44 In particular, player $i$ is a tying player at an essential tie if and only if for some $j \neq i$ the subset $T_{ij}^E$ of $T_{ij}$ that consists of the type-bid profiles at which player $i$’s payoff is not continuous as a function of her bid has positive $\nu$-measure. (Notice that those type-bid profiles are those at which player $i$’s payoff is not continuous as a function of the entire type-bid profile.)
Claim 5 The strategy profile $\sigma$ has no essential tie at any bid $t < 1$.

Proof. If there was an essential tie at some $t < 1$, then for sufficiently large $n$ each of the tying players would be bidding in an arbitrarily small interval around $t$ with a probability bounded away from 0. But then at least one of the players could increase the probability of winning a better prize by an amount bounded away from zero by bidding slightly above this interval, instead of bidding in this interval. ■

Claim 6 One of the following two conditions holds: (a) profile $\sigma$ has no essential tie at $t = 1$; (b) there is a $\delta > 0$ such that for sufficiently large $m$ no player bids in interval $(1 - \delta, 1)$ with positive probability, i.e., $\sigma_j^m(X \times (1 - \delta, 1)) = 0$ for all $j$.

Proof. Suppose that $\sigma$ has an essential tie at $t = 1$, and denote by $T_\sigma$ the set of tying players at 1. Let $a = \Pi_{j \in T_\sigma} a_j$, where $a_j$ is the size of the atom of the marginal of $\sigma_j$ at $t = 1$, be the product of the sizes of all the atoms at $t = 1$.

Since the payoffs are continuous in types and type 0 of every player $i$ strictly prefers bidding some $\bar{b}_i < 1$ to bidding 1, there is an $\underline{x} > 0$ and $\bar{b} < 1$ such that for all types $x \leq \underline{x}$ of every player, bidding more than $\bar{b}$ is strictly dominated by bidding less than $\bar{b}$. Let $\delta \in (0, 1 - \bar{b})$ be such that for all type profiles $x_{-i}$ and bid profiles $b_{-i}$ of the other players

$$\frac{a}{2^m} u_i((x_i, x_{-i}), (1, b_{-i}, y_{l+1}) + (1 - \frac{a}{2^m}) u_i((x_i, x_{-i}), (1, b_{-i}, y_l) > u_i((x_i, x_{-i}), (1 - \delta, b_{-i}, y_l))$$

(6)

for all types $x_i > \underline{x}$ of player $i$ whenever $y_{l+1} > y_l$. Because the tie at 1 is essential, (6) implies that each player of type $x > \underline{x}$ would be willing to increase her bid from $1 - \delta$ to 1 to tie with the players who tie at $t = 1$, even if the marginal of the strategy of each tying player $j$ had an atom at $t = 1$ only of size $a_j/2$.

For any given $m$ and each tying player $j$ in $T_\sigma$, let $t_j^m$ be the highest $t$ such that the marginal of $\sigma_j^m$ assigns to the interval $[t, 1]$ at least probability $a_j/2$. For sufficiently large $m$, by weak* convergence of $\sigma_j^m$ to $\sigma_j$, we have that $t_j^m > 1 - \delta$ and, in addition, $\sigma_j^m$ assigns to the interval $(1 - \delta, 1]$ at least probability of $3a_j/4$. We claim that then $t_j^m = 1$ for all players $j$ in $T_\sigma$. Indeed, consider player $j$ with the lowest $t_j^m$ and suppose that $t_j^m < 1$. Then, by (6) player $j$ could profitably deviate by replacing the bids from $(1 - \delta, t_j^m]$ with bid $t = 1$, because in $\sigma$ the tie at $t = 1$ is essential.
However, if $t_j^m = 1$ for all $j$ in $T_\sigma$, then no player bids in $(1 - \delta, 1)$ with positive probability. This is because types $x \leq \bar{x}$ do not bid more than $\bar{b}$, and types $x > \bar{x}$ are better off of replacing any bids in $(1 - \delta, 1)$ with bid $t = 1$, by (6) and because the tie at $t = 1$ is essential. ■

We can now show that player $i$’s payoff from playing $\sigma_i^m$ against $\sigma_{-i}^m$ converges to that from playing $\sigma_i$ against $\sigma_{-i}$. If condition (a) in Claim 6 is satisfied, then the convergence follows from Claim 5 and Lemma 4. So, suppose that condition (b) is satisfied. Then player $i$’s payoff from playing $\sigma_i^m$ against $\sigma_{-i}^m$ for sufficiently large $m$, as well as the payoff from playing $\sigma_i$ against $\sigma_{-i}$, is equal to the sum of two payoffs: (i) the payoff contingent on player $i$ bidding $t = 1$; and (ii) the payoff contingent on player $i$ bidding in $[0, 1 - \delta]$.

In case (i) the payoff from playing $\sigma_i^m$ against $\sigma_{-i}^m$ converges to the payoff from playing $\sigma_i$ against $\sigma_{-i}$. This is because for every player $j$ the probability assigned to $t = 1$ by the marginal of $\sigma_j^m$ converges to the probability assigned to $t = 1$ by the marginal of $\sigma_j$. This follows from applying the definition of weak* convergence to any continuous function $f : X \times B \to [0, 1]$ that takes value 1 on $X \times \{1\}$ and value 0 on $X \times [0, 1 - \delta]$. To see why this convergence implies the payoff convergence, notice that by bidding $t = 1$ player $i$ ties with all players who bid $t = 1$, and outbids all players who bid $t < 1$.

Finally, in case (ii) the payoff from playing $\sigma_i^m$ against $\sigma_{-i}^m$ converges to the payoff from playing $\sigma_i$ against $\sigma_{-i}$. To see why, denote by $\sigma_i^m|I[0, 1 - \delta], \sigma_i|[0, 1 - \delta], \sigma_{-i}^m|[0, 1 - \delta], \sigma_{-i}|[0, 1 - \delta]$ the strategies and profiles $\sigma_i^m, \sigma_i, \sigma_{-i}^m, \sigma_{-i}$ conditional on bids in $[0, 1 - \delta]$. Then $\sigma_i^m|[0, 1 - \delta]$ converges to $\sigma_i|[0, 1 - \delta]$ and $\sigma_{-i}^m|[0, 1 - \delta]$ converges to $\sigma_{-i}|[0, 1 - \delta]$ because $\sigma_i^m$ converges to $\sigma_i$ and $\sigma_{-i}^m$ converges to $\sigma_{-i}$ and no player bids in $(1 - \delta, 1)$ with positive probability for large $m$.\footnote{To see the convergence, take any continuous function $f : X \times [0, 1 - \delta] \to [0, 1]$ and show that its integral with respect to $\sigma_i^m|[0, 1 - \delta]$ converges to its integral with respect to $\sigma_i|[0, 1 - \delta]$ by extending $f$ to a continuous function on $X \times B$ whose value is 0 on $X \times \{1\}$ and applying weak* convergence of $\sigma_i^m$ to $\sigma_i$.} Claim 5 and Lemma 4, along with the fact that by bidding in $[0, 1 - \delta]$ player $i$ loses to all players who bid $t = 1$, imply that the payoff from playing $\sigma_i^m$ contingent on bidding in $[0, 1 - \delta]$ against $\sigma_{-i}^m$ converges to the payoff from playing $\sigma_i$ contingent on bidding in $[0, 1 - \delta]$ against $\sigma_{-i}$. This is because these payoffs are the sum of the payoffs contingent on the other players bidding in $[0, 1 - \delta]$ or bidding $t = 1$. 
7.2 First-price auctions with private values

Consider a first-price auction with \( n \) players. Player \( i \)'s prize valuation \( x_i \in X = [0, 1] \) is distributed according to a cdf \( F_i \) that does not have an atom at 0. The distributions need not be identical, but are commonly known and independent across players. Each player \( i \) submits a bid \( b_i \in B = [0, \bar{b}] \), where \( \bar{b} > 1 \),\(^{46}\) and the player with the highest bid wins the prize and pays her bid. Every other player pays 0. Ties are resolved by a fair lottery. The utility of a player \( i \) is

\[
\bar{u}_i(x_1, ..., x_n, b_i) = \begin{cases} 
    x_i - b_i & \text{if } i \text{ wins the prize,} \\
    0 & \text{if } i \text{ does not win the prize;}
\end{cases}
\]

finally, \( u_i(x_1, ..., x_n, b_1, ..., b_n) \) is the expected value of \( \bar{u}_i(x_1, ..., x_n, b_i) \) given the bid profile \( (b_1, ..., b_n) \).

We will establish equilibrium existence under some novel conditions on the distribution of players' valuations. Jackson and Swinkels (2005) showed that an equilibrium may fail to exist only if players' valuation distributions have atoms. Lebrun (1996, 1999) provided sufficient conditions for existence that allow for some atoms but exclude, for example, valuation distributions with discrete supports. Our result applies to such distributions.

**Proposition 1** A first-price auction with private values in which players' valuation distributions do not have an atom at 0 has an equilibrium in distributional strategies if the support \( \text{Supp}(F_i) \) of every \( F_i \) is discrete, and \( \text{Supp}(F_i) \cap \text{Supp}(F_j) = \emptyset \) whenever \( i \neq j \).

Proposition 1 follows from Proposition 2 in Section 10.2 of the online appendix.\(^{47}\)

Our proof approach is as follows. First, we define first-price auctions with costly bids. In such an auction each player submits a bid, and the player with the highest bid wins the prize and pays her bid. Every other player pays a cost that is continuous in her bid, equal to 0 for a bid of 0, and positive for positive bids. Equilibrium existence for these games is established in Section 10.1, and follows from Theorem 2. We then take a first-price auction and consider

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\(^{46}\)This guarantees that any equilibrium remains an equilibrium if bidders can place any non-negative bid.

\(^{47}\)Proposition 2 also delivers the equilibrium existence results of Lebrun (1996) and Jackson and Swinkels (2005) for private-value first-price auctions, and also provides a novel sufficient condition for existence for two-player auctions.
a sequence of first-price auctions with costly bids that have the same number of players and the same valuation distributions as in the first-price auction. We refer to the $m$-th auction with costly bids in the sequence as “the $m$-th game” and to the original first-price auction as “the first-price auction.” The sequence approximates the first-price auction by having losing players’ cost of bidding $b$ in the $m$-th game be $b/m$. Let $\mu^m = (\mu^m_1, \ldots, \mu^m_n)$ be an equilibrium of the $m$-th game, and let $\mu = (\mu_1, \ldots, \mu_n)$ be the weak* limit of a subsequence of the sequence $(\mu^m)^\infty_{m=1}$. This subsequence can with no loss of generality be assumed to be the entire sequence. We show that under the condition in Proposition 1, the limit $\mu$ is an equilibrium of the first-price auction. Because $\mu$ is a strategy profile in the first-price auction, it is enough to check that no player can profitably deviate to another strategy.

The idea is to show that the existence of a profitable deviation for some player $i$ in the first-price auction given strategy profile $\mu$ implies the existence of a profitable deviation for player $i$ in the $m$-th game for large $m$. This is not difficult to show if player $i$’s payoff at the limit profile $\mu$ in the first-price auction does not discontinuously decrease relative to the limit of her payoffs along the sequence of equilibria. But essential ties for player $i$ at $\mu$ can precipitate such a discontinuity, in which case $\mu$ may not be an equilibrium of the first-price auction. Our proofs shows that the condition in Proposition 1 guarantees that $\mu$ does not have essential ties, which implies that $\mu$ is an equilibrium of the first-price auction.

We now give an example that illustrates the approximation idea underlying the proof of Proposition 1 and shows that the condition in Proposition 1 is in some sense necessary.

**Example 2** Consider a two-player first-price auction. Player 1’s valuation is $v > 0$ and player 2’s valuation is $v$ with probability $p$ and $V > v$ with probability $1 - p$. If $p = 0$, then the condition in Proposition 1 is satisfied; if $p = 1$, then both players’ valuation is $v$. Thus, in both cases an equilibrium exists.49

But for $p$ in $(0, 1)$ the condition in Proposition 1 is not satisfied. Appendix 9.2 shows that in the unique equilibrium of the $m$-th game (first-price auction with costly bids in which

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48 It follows directly from the definition of weak* convergence that the marginal of $\mu_i$ on $X$ coincides with distribution $F_i$.

49 In the former case, an equilibrium is for player 2 to bid $v$ and for player 1 to bid uniformly on $[v - \varepsilon, v]$ for some small $\varepsilon > 0$. In the latter case, an equilibrium is for both players to bid $v$. 

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the losing player’s cost of bidding \( b \) is \( b/m \) player 1 bids continuously on \((0, v]\) and has an atom at bid \( b = 0 \), type \( v \) of player 2 bids continuously on \([0, \hat{b}]\), and type \( V \) of player 2 bids continuously on \([\hat{b}, v]\) for some \( \hat{b} < v \). As \( m \) grows large \( \hat{b} \) approaches \( v \), the size of player 1’s atom at 0 approaches 0, and for every bid \( x \leq \hat{b} \) the probability that player 1 bids at most \( x \) and the probability that type \( v \) of player 2 bids at most \( x \) approach 0. The limit of the sequence of equilibria is therefore both players bidding \( v \) with probability 1. This is an essential tie and is not an equilibrium of the first-price auction, since type \( V \) of player 2 would be better of by increasing her bid slightly.

In fact, this first-price auction does not have an equilibrium.

**Claim 7** The first-price auction in Example 2 does not have an equilibrium.

**Proof.** Suppose in contradiction that there is an equilibrium, and consider players’ equilibrium strategies. We will show that player 1 bids \( v \) with probability 1, to which type \( V \) of player 2 does not have a best response. We first show that player 1 bids less than \( v \) with probability 0.

Suppose to the contrary that player 1 bids less than \( v \) with positive probability. Then the payoffs of player 1 and both types of player 2 are positive. To see why, observe that type \( v \) of player 2 also bids less than \( v \) with positive probability, since by bidding slightly less than \( v \) she wins with positive probability, which gives her a positive payoff, whereas bidding \( v \) or higher gives her a nonpositive payoff. Since player 2 bids less than \( v \) with positive probability, the same argument shows that player 1’s payoff is positive. And the payoff of type \( V \) of player 2 is also positive since she can bid as if she were type \( v \). But it is not possible for all three payoffs to be positive. To see this, consider the infimum \( b \) of the union of the best response sets across players and types. If neither player has an atom at \( b \), then by bidding \( b \) and sufficiently close to \( b \) a player’s payoff approaches 0. It also cannot be that both players have an atom at \( b \), since they would both benefit from increasing their bid a little bid and outbidding the other (since all three payoffs are positive). And if only one player has an atom at \( b \), then this player loses with probability 1 by bidding \( b \), which contradicts the positivity of all three payoffs. Thus, player 1 bids less than \( v \) with probability 0.
It remains to show that player 1 bids strictly more than \( v \) with probability 0. Suppose that player 1 bids strictly more than \( v \) with positive probability. If by doing so player 1 wins with positive probability, then she would be better off deviating to bidding 0. If by doing so she loses for sure, then type \( v \) of player 2 must be outbidding her, so type \( v \) of player 2 would be better off deviating to 0. This concludes the proof. ■

Notice that this failure of equilibrium existence can be viewed as a failure of both lower- and upper-hemicontinuity of the equilibrium correspondence. The former is seen by considering values of \( p \) that approach 0 or 1, and the latter is seen by considering, for a fixed \( p \) in \((0, 1)\), the \( m \)-th game as \( m \) grows large, in which the losing player’s cost of bidding \( b \) is \( b/m \). Notice also that if we increased or decreased player 1’s valuation slightly (or the valuation of the low type of player 2) then condition 3 of Proposition 1 would apply and equilibrium existence would be restored.

8 Concluding remarks

This paper studies Bayesian games with ties, and introduces improving deviations, favorable tie breaking, and favorable tie breaking on average, which are sufficient conditions for equilibrium existence in distributional strategies. We apply these conditions to obtain novel equilibrium existence results for multiprize contests with interdependent values, first-price auctions with common values, and Hotelling models. We then study contests with bid caps and first-price auctions with private values, for which the conditions fail. We prove equilibrium existence for these games via approximation techniques. The sufficient conditions and approximation techniques we present may be useful for other applications.
9 Appendix

9.1 Proofs

9.1.1 Proof of Lemma 1

Consider a sequence \((\mu^l)_{l=1}^{\infty}\) of strategy profiles that converges to \(\mu\). We will show that for any \(\delta > 0\), if \(l\) is large enough, then \(|U_i(\mu^l) - U_i(\mu)| < \delta\). Since \(\mu\) does not have essential ties, \(\mu(T^*_i) = 0\). Let \(T^*_i\) be an open subset of \(X_1 \times ... \times X_n \times B^n\) such that \(T^*_i \subset T^*_i\) and \(\mu(T^*_j) < \varepsilon\). Let \(u^*_i : X_1 \times ... \times X_n \times B^n \rightarrow R\) be a continuous function that coincides with \(u_i\) on the complement \((T^*_i)^C\) of \(T^*_i\), and is bounded by \(\beta\) on \(T^*_i\). Let \(V^*_i(\mu^l) = \int u^*_i d\mu^l\) and \(V^*_i(\mu) = \int u^*_i d\mu\). Since

\[
|U_i(\mu) - U_i(\mu^l)| \leq |U_i(\mu) - V^*_i(\mu)| + |V^*_i(\mu) - V^*_i(\mu^l)| + |V^*_i(\mu^l) - U_i(\mu^l)|,
\]

it suffices to show that for small enough \(\varepsilon > 0\) and all large enough \(l\) each of the three terms on the right is smaller than \(\delta/3\). Since \(\mu(T^*_i) < \varepsilon\), we have that \(|U_i(\mu) - V^*_i(\mu)| \leq \int_{T^*_i} |u_i - u^*_i| d\mu < \varepsilon \beta < \delta/3\) for small enough \(\varepsilon\). By the definition of weak* convergence, \(|V^*_i(\mu^l) - V^*_i(\mu)| < \delta/3\) for large enough \(l\).

To estimate the last term, recall that \(u^*_i = u_i\) on the complement \((T^*_i)^C\) of \(T^*_i\), and both functions \(u^*_i\) and \(u_i\) are continuous on \((T^*_i)^C\). Therefore, since \(X_1 \times ... \times X_n \times B^n\) is compact, \(|u^*_i - u_i| < \delta/6\) on some open set \(G\) that contains \((T^*_i)^C\). This yields that \(\int_G |u^*_i - u_i| d\mu^l < \delta/6\) for all \(l\). Further, since the set \(G^C\) is closed, \(\mu^l(G^C) < \mu(G^C) + \delta/12\beta\) for large enough \(l\), by Theorem 29.1 in Billingsley (1995). Since \(G^C \subset T^*_i\), we have that \(\mu(G^C) < \varepsilon\). This yields that

\[
|V^*_i(\mu^l) - U_i(\mu^l)| = \int_G |u^*_i - u_i| d\mu^l + \int_{G^C} |u^*_i - u_i| d\mu^l < \delta/6 + 2\beta \mu^l(G^C) < \delta/3
\]

for small enough \(\varepsilon\). This completes the proof.

9.1.2 Proof of Lemma 2

Consider a sequence \((\mu^l)_{l=1}^{\infty}\) of strategy profiles that converges to \(\mu\). We will show that for any \(\delta > 0\), if \(l\) is large enough, then \(|U_i(\mu^l) - U_i(\mu)| < \delta\). Given any \(\varepsilon > 0\), take an open subset \(T^*_i\) of \(X_i \times B\) such that \(T^*_i \subset T^*_i\) and \(\mu_i(cT^*_i) < \varepsilon\), where \(cT^*_i\) denotes the
closure of $T_i^\varepsilon$. We can in addition assume that $\mu_i^l(clT_i^\varepsilon) < \varepsilon$ for sufficiently large $l$ (see Billingsley (1995), Theorem 29.1). For every $(x_i, b_i)$ from the complement of $T_i^\varepsilon$, we have that $\int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\nu_{-i}$ is a continuous function of $\nu_{-i}$ at $\mu_{-i}$. So, there is an $\overline{l}$ such that

$$\left| \int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu_{-i}^l - \int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu_{-i} \right| < \delta/2$$

(7)

if $l \geq \overline{l}$. Thus, there is also an $\overline{l}$ that is common for all $(x_i, b_i)$ from a set $S \subset X_i \times B - clT_i^\varepsilon$ whose $\mu_i$-measure is higher than $1 - \varepsilon$. We can assume that the set $S$ is closed, because all measurable sets contain closed subsets of arbitrarily close measure. By Theorem 29.1 of Billingsley (1995), we can also assume that $\mu_i^l(S) > 1 - \varepsilon$ for sufficiently large $l$. By (7),

$$\left| \int_S \left( \int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu_{-i}^l \right) d\mu_i - \int_S \left( \int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu_{-i} \right) d\mu_i^l \right| < \delta/2$$

(8)

if $l \geq \overline{l}$ for such a common $\overline{l}$. Actually, inequality (8) holds without restricting integration to $S$, provided that $\varepsilon$ is sufficiently small compared to $\delta$, because the measures $\mu_i^l$ and $\mu_i$ of the complement of $S$ are smaller than $\varepsilon$ and we assume that functions $u_i$ are bounded.

Notice now that $\int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu_{-i}$ is a continuous function of $(x_i, b_i)$ on $S$. Therefore,

$$\left| \int \left( \int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu_{-i} \right) d\mu_i - \int \left( \int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu_{-i} \right) d\mu_i^l \right| < \delta/2$$

(9)

for sufficiently large $l$. To see why, multiply $\int u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) d\mu_{-i}$ by a continuous function that takes value 1 on $S$ and value 0 on $clT_i^\varepsilon$. This product is a continuous function, so by the definition of weak$^*$-convergence, its expected value with respect to measures $\mu_i^l$ converges to its expected value with respect to measure $\mu_i$. And since the measures $\mu_i^l$ and $\mu_i$ of the complement of $S$ are smaller than $\varepsilon$, we obtain (9) provided that $\varepsilon$ is sufficiently small compared to $\delta$. (Recall that we assume that functions $u_i$ are bounded.)

Inequalities (8) and (9) imply that $|U_i(\mu^l) - U_i(\mu)| < \delta$ for sufficiently large $l$. 

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9.1.3 Proof of Lemma 3

Proof. Let $G_k = X \times (t - 1/k, t + 1/k)$, and let $G = X \times \{t\}$. Since the sequence $(G_k)_{k=1}^\infty$ is descending and

$$\bigcap_{k=1}^\infty G_k = G,$$

we have that $\nu_j(G_k \setminus G) < \eta$ for sufficiently large $k$. And $X \times ((t - \delta, t) \cup (t, t + \delta)) = G_k \setminus G$ for $\delta = 1/k$. ■

9.1.4 Proof of Claim 1

Given a profile $b_{-i}$ of locations of other players, let $\Delta_l(b_{-i})$ and $\Delta_r(b_{-i})$ be the difference in the market share of player $i$ whose current location is $b_i$ made by the infinitesimal shifts to the left and to the right, respectively. Notice that

$$\Delta_l(b_{-i}) + \Delta_r(b_{-i}) \geq 0$$

for any $b_{-i}$, and the inequality is strict if $b_j = b_i$ for more than one $j \neq i$. Indeed, an infinitesimal shift to each direction increases by a factor of $k + 1$ the share of customers in this direction, where $k$ is the number of other players $j \neq i$ such that $b_j = b_i$. Thus, the sum of the expected values of $\Delta_l(b_{-i})$ and $\Delta_r(b_{-i})$ is positive if the marginals on $B$ of the strategy of two or more other players have an atom at $b_i$.

9.1.5 Proof of Claim 2

Suppose that $b_i^k \to b_i$ and $\mu_j^k \to_k \mu_j$, $j \neq i$. (We disregard $x_i$, because it affects only the fixed costs of choosing a location, and the fixed costs are continuous in it.) Suppose first that condition (a) is satisfied. Then, the marginal of each $\mu_j$ assigns only an arbitrarily small probability to $clW_i$ for a small neighborhood $W_i$ of $b_i$, where $clW_i$ is the closure of $W_i$. By Billingsley (1995), Theorem 29.1, the marginal of $\mu_j^k$ for sufficiently large $k$ also assigns only an arbitrarily small probability to $clW_i$. Let $W_i$ be another neighborhood of $b_i$ such that $clW_i \subset W_i$. Thus, the only substantial difference between the market share of player $i$ located at $b_i^k$ when the other players play $\mu_{-i}^k$ and the market share of player $i$ located at $b_i$ when the other players play $\mu_{-i}$ can come from the strategies of other players contingent on the
locations in the complement of $clW_i$. However, the share of player $i$ is a continuous function of the locations selected by all players if player $i$ is located in $V_i$, and the other players are located in the complement of $V_i$.

So, the difference between the market share of player $i$ located at $b^k_i$ when the other players play $\mu^k_{-i}$ and the market share of player $i$ located at $b_i$ when the other players play $\mu_{-i}$, contingent on the locations in the complement of $clW_i$ must also be small for sufficiently large $k$. To see why multiply the share of players $i$ by a continuous function that is equal to 0 on $clV_i$, and is equal to 1 on the complement of $W_i$. The difference between the expected value of this product and the market share of player $i$ contingent on the locations of the other players in the complement of $clW_i$ is small for both $\mu_{-i}$ and $\mu^k_{-i}$, because the marginals of all measures in $\mu_{-i}$ and $\mu^k_{-i}$ assign small probability to $clW_i$. So, we obtain the required property directly from the definition of weak*-convergence.

Suppose now that condition (b) is satisfied. Denote by $j^*$ the $j$ such that the marginal of $\mu_j$ has an atom at $b_i$. Then, the marginal of $\mu_j$ for all $j \neq i, j^*$ assigns only an arbitrarily small probability to $clW_i$ for a sufficiently close neighborhood $W_i$ of $b_i$, and so does the marginal of $\mu^k_j$ for sufficiently large $k$. Moreover, the marginal of $\mu^k_{j^*}$ for sufficiently large $k$ assigns to $clV_i$, for a neighborhood $V_i$ of $b_i$, a probability arbitrarily close to that assigned by $\mu_{j^*}$. To see why, consider another neighborhood $V'_i$ such that $clV_i \subset V'_i$, and such that the marginal of $\mu_{j^*}$ assigns to $V'_i$ a probability arbitrarily close to that assigned by the marginal of $\mu_{j^*}$ to $clV_i$. Then, by Theorem 29.1 from Billingsley (1995), the probability assigned to $V'_i$ by the marginal of $\mu^k_j$ for sufficiently large $k$ cannot be much smaller that assigned by the marginal of $\mu_{j^*}$, and the probability assigned to $clV_i$ by the marginal of $\mu^k_{j^*}$ for sufficiently large $k$ cannot be much greater that assigned by the marginal of $\mu_{j^*}$.

With no loss of generality, assume that $clV_i \subset W_i$. Replacing $W_i$ with a smaller $W'_i$ such that $clV_i \subset W'_i$ if necessary, we can assume that $\mu_{j^*}$ and all $\mu^k_{j^*}$ (for sufficiently large $k$) assign an arbitrarily small probability to $clW_i - clV_i$. Thus, the only substantial difference between the market share of player $i$ located at $b^k_i$ when the other players play $\mu^k_{-i}$ and the market share of player $i$ located at $b_i$ when the other players play $\mu_{-i}$ can come from: (i) the strategies of all players $j \neq i$ choosing locations in the complement of $clW_i$, or (ii) the strategy of player $j^*$ choosing a location in $clV_i$ and the strategies of all players $j \neq i, j^*$.
choosing locations in the complement of $cW_i$.

However, the market share of player $i$ is a continuous function of the locations selected by all players if player $i$ is located in $V_i$ and the other players are located in the complement of $V_i$. So, the difference in the market shares of player $i$ must also be small for sufficiently large $k$ in case (i). It must also be small (for sufficiently large $k$) in case (ii), if $cW_i$ is chosen sufficiently close to $b_i$ by the assumption that the infinitesimal shifts from location $b_i$ (to the right and to the left) have no effect on player $i$’s market share.\footnote{More precisely, we assume that infinitesimal shifts do not increase player $i$’s expected payoff, but in this case a simple accounting argument shows that such shifts also do not decrease player $i$’s expected payoff.}

### 9.2 The example in Section 7.2

Because a losing player who bids $b$ pays $b/m$, which increases in $b$, the equilibria of these games share some features with the equilibria of all-pay auctions. As a preliminary observation, notice that in equilibrium no player bids more than her value (she would be better off bidding 0), and at all bids up to her value a player strictly prefers winning to losing. We now describe features of any equilibrium of the $m$-th game, and give some intuition for them; the formal arguments are standard in the all-pay auction literature.\footnote{For example, they are almost identical to those in Siegel (2014).}

First, no equilibrium strategy has an atom at any positive bid. This is because if player $i$ had an atom at a positive bid $x$, then bids slightly lower than $x$ would not be best responses for the other player (she would be better off bidding 0 is $x$ is higher than her value, and better off bidding slightly more than $x$ and winning if $x$ is not higher than her value). But then player $i$ would be better off bidding less than $x$ because by doing so she would be winning with the same probability. Second, every positive bid up to the supremum of the union of players’ best response sets across players and types is a best response for player 1 and some type of player 2. To see why, notice that fixing the other player’s equilibrium strategy, each player’s payoff is continuous as a function of her bid $x > 0$, since players’ strategies do not have atoms at positive bids. This means that if some bid $x > 0$ is not a best response for a player, then a neighborhood $N(b)$ of bids is not a best response for the player. But this means that the other player does not have best responses in $N(b)$ (for any bid in $N(b)$ a slightly lower bid is...}
better). And this means that the supremum of $N(b)$ is not a best response for either player. Thus if $x > 0$ is not a best response for some player, then no bid higher than $x$ is a best response for either player. Third, every best response of type $V$ of player 2 is weakly higher than every best response of type $v$ of player 2. This is because, fixing player 1’s strategy, if type $v$ of player 2 weakly prefers a bid $x$ to a lower bid $x'$, then type $V$ strictly prefers $x$.

This implies that players’ best response sets are characterized by two intervals, $[a^l, a^h]$ and $[A^l, A^h]$, where $a^l \leq a^h = A^l \leq A^h \leq v$, such that every bid $b$ in $(a^l, a^h]$ is a best response for player 1 and type $v$ of player 2, and every bid $b$ in $[A^l, A^h]$ is a best response for player 1 and type $V$ of player 2. In addition, $a^l = 0$. Indeed, if neither player has an atom at $a^l$, then by bidding $a^l$ and sufficiently close to $a^l$ a player loses with a probability that approaches 1. It also cannot be that both players have an atom at $a^l$, since they would both benefit from increasing their bid a little bit and outbidding the other. And if only one player has an atom at $a^l$, then this player loses with probability 1 by bidding $b$. This argument also shows that the equilibrium payoff of some type of some player is 0. In fact, the equilibrium payoff of player 1 is 0. Indeed, if the payoff of player 1 were positive, then so would be the payoffs of types $v$ and $V$ of player 2: type $v$ of player 2 gets a payoff no lower than that of player 1 by bidding slightly above the highest best response of player 1, and type $V$ gets a payoff no lower than that of type $v$ by imitating type $v$. But we have argued that not all three payoffs can be positive.

That player 1’s payoff is 0 means that $A^h = v$, which in turn implies that the payoff of type $V$ of player 2 is $V - v$. We now use the characterization of players’ best response intervals and the payoffs of player 1 and type $V$ of player 2 to construct the unique equilibrium (and in the process pin down $a^h = A^l$ and the payoff of type $v$ of player 2). We denote by $F_1(\cdot)$ the cdf of player 1’s equilibrium strategy and by $F_2(\cdot, v)$ and $F_2(\cdot, V)$ the cdfs of player 2’s equilibrium strategies for her two types. Notice that $a^h > 0$, so $A^l > 0$, otherwise $a^l = a^h = 0$, which would imply that type $v$ of player 2 bids 0 with probability 1, so player 1’s payoff would be positive. But we argued that player 1’s payoff is 0.

Consider a bid $x$ in the interval $[A^l, A^h] = [A^l, v]$, which are best responses for player 1 and type $V$ of player 2, and recall that no player has an atom at $x$, since $x > 0$. Since player
1’s payoff is 0, we have that

\[ 0 = (v - x)(p + (1 - p)F_2(x, V)) - \frac{x}{m}(1 - (p + (1 - p)F_2(x, V))) \\]

\[ F_2(x, V) = \frac{\frac{x}{m}(1 - p) - (v - x)p}{(v - x)(1 - p) + \frac{x}{m}(1 - p)}. \]

Since \( F_2(x, V) \) ranges from 0 to 1 for \( x \) in \([A^l, v] \) we have that \( A^l = mpv/(1 - p + mp) \). Since the payoff of player 2’s type \( V \) is \( V - v \), we have that

\[ V - v = (V - x)F_1(x) - \frac{x}{m}(1 - F_1(x)) \Rightarrow \]

\[ (V - x + \frac{x}{m})F_1(x) = V - v + \frac{x}{m} \Rightarrow F_1(x) = \frac{V - v + \frac{x}{m}}{V - x + \frac{x}{m}}. \]

Thus, \( F_1(x) \) ranges from \((V - v + pv/(p(m - 1) + 1))/((V - vp(m - 1)/(p(m - 1) + 1)) to 1 for \( x \) in \([mpv/(1 - p + mp), v] \).

Consider bids \( x \) in the interval \([a^l, a^u] = (0, mpv/(1 - p + mp)] \), which are best responses for player 1 and type \( v \) of player 2, and recall that no player has an atom at \( x \), since \( x > 0 \).

Since player 1’s payoff is 0, we have that

\[ 0 = (v - x)(pF_2(x, v)) - \frac{x}{m}(1 - pF_2(x, v)) \Rightarrow \]

\[ ((v - x)p + \frac{px}{m})F_2(x, v) = \frac{x}{m} \Rightarrow F_2(x, v) = \frac{x}{(v - x)p + \frac{px}{m}}, \]

and \( F_2(x, v) \) ranges from 0 to 1 for \( x \) in \((0, mpv/(1 - p + mp)] \).

It remains to identify \( F_1(x) \) for \( x \) in \((0, mpv/(1 - p + mp)] \). The payoff of player 2’s type \( v \) is his payoff when bidding any \( x \) in \((0, mpv/(1 - p + mp)] \),

\[ (v - x)F_1(x) - \frac{x}{m}(1 - F_1(x)). \]

Substituting \( x = mpv/(1 - p + mp) \) and \( F_1(x) = (V - v + pv/(p(m - 1) + 1))/((V - vp(m - 1)/(p(m - 1) + 1)) \) we obtain

\[ (v - \frac{mpv}{1 - p + mp}) \frac{V - v + \frac{pv}{p(m - 1) + 1}}{V - v} - \frac{pv}{1 - p + mp}(1 - \frac{V - v + \frac{pv}{p(m - 1) + 1}}{V - v}) = \]

\[ \frac{v(V - v)(1 - p)}{V(1 - p) + pv + mp(V - v)} \]
We therefore have that

\[
\frac{v(V-v)(1-p)}{V(1-p) + pv + mp(V-v)} = (v-x)(F_1(x)) \quad \text{for all } x \in [0, mpx/(1-p+mp)].
\]

and

\[
F_1(x) = \frac{v(V-v)(1-p)}{V(1-p) + pv + mp(V-v)} + \frac{x}{m}.
\]

and \( F_1(x) \) ranges from \((V - v)(1 - p)/(V(1 - p) + pv + mp(V - v))\) to \((V - v + pv((p(m - 1) + 1))/(V - v p(m - 1)/(p(m - 1) + 1))\) for \( x \) in \([0, mpx/(1 - p + mp)]\). In particular, \( F_1 \)

has an atom at 0.

In summary,

\[
F_1(x) = \begin{cases} 
\frac{v(V-v)(1-p)}{V(1-p) + pv + mp(V-v)} + \frac{x}{m} & \text{if } x \in [0, mpx/(1-p+mp)] \\
\frac{V-x + \frac{m}{m}}{V-x + \frac{m}{m}} & \text{if } x \in [mpx/(1-p+mp), v]
\end{cases}
\]

\[
F_2(x, v) = \frac{z}{(v-x)p + \frac{px}{m}} \quad \text{for } x \in [0, mpx/(1-p+mp)],
\]

and

\[
F_2(x, V) = \frac{z(1-p) - (v-x)p}{(v-x)(1-p) + \frac{px}{m}(1-p)} \quad \text{for } x \in [mpx/(1-p+mp), v].
\]

As \( m \) grows large \( mpx/(1 - p + mp) \) approaches \( v \), so \( F_2(\cdot, V) \) converges to an atom of size 1 at \( v \). And for every \( x < mpx/(1 - p + mp) \to v \), \( F_1(x) \) and \( F_2(x, v) \) approach 0, so \( F_1(\cdot) \) and \( F_2(\cdot, v) \) also converge to an atom of size 1 at \( v \).
10 Online Appendix

10.1 First-price auctions with private values and costly bids

We establish equilibrium existence for some private-value first-price auctions in which bids are costly, so losing players pay a cost and only the winner pays her bid. We use this result in Section 7.2 to establish equilibrium existence for some first-price auctions with private values (and costless bids), which are games with ties for which Theorems 1, 2, and 3 do not apply. First-price auctions with costly bids are also of independent interest. For example, a player may need to secure a loan that will only be used in the event of winning but securing the loan entails a fee that could depend on the size of the loan. We model this feature by adding a payment function that captures the cost of bidding in the event of a loss. Such auctions can also be used to model contests with conditional investments, as in Siegel (2010), in which all contestants bear some costs and only the winner pays the entire cost of her bid.52

The key property of this model is that whenever a player is indifferent between winning and losing, she strictly prefers to bid 0. This leads to a simple deviation plan in which every player increases her bid slightly if she prefers to win at that bid, and otherwise lowers her bid to 0. Unilateral deviations at an essential tie correspond to each of the tying players either winning the prize or bidding 0, whichever is better, whereas simultaneously breaking the tie awards the prize to one player and all other players pay a cost if the tie is at a bid $b > 0$. This guarantees (3), and relies on values being private - with interdependent values unilateral deviations may not be better than simultaneous tie breaking.53 In addition, the fact that

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52 An example of such a contest is a design contest based on prototypes, in which each contestant $i$ submits a design $b_i$ for a product (where $b_i$ is the cost of producing the product) and incurs the cost $p_i(b_i)$ of producing a prototype. Only the winner has to construct the actual product (and therefore pay the additional cost $b_i - p_i(b_i)$ of developing the complete product).

53 To see this, suppose there are three players - player 1’s value is $a > 0$ or $3a$ with equal probabilities, player 2’s value is $2a$ plus player 1’s value, and player 3’s value is $4a$. Consider bid profile in which players 2 and 3 bid $4a - \varepsilon$ for some positive $\varepsilon < a$ and player 1 bids less than $4a - \varepsilon$. Ignoring the bidding cost of losing, the sum of players’ expected payoffs at this bid profile is $\varepsilon$. With unilateral tie breaking this sum can be increased by at most $\varepsilon$, but by increasing player 2’s bid slightly when player 1’s value is $3a$ and decreasing player 2’s bid slightly when player 2’s value is $a$ the sum of payoffs can be increased by close to $a$, so for
losing players pay a cost is crucial - in a private-value first-price auction, equilibrium may not exist (see Section 7.2).

There are \( n \) players. Player \( i \)'s prize valuation \( x_i \in X = [0,1] \) is distributed according to a cdf \( F_i \) that does not have an atom at 0. The distributions need not be identical, but are commonly known and independent across players. Each player \( i \) submits a bid \( b_i \in B = [0, \bar{b}] \), where \( \bar{b} > 1 \), and the player with the highest bid wins the prize and pays her bid. Every other player \( i \) pays \( p_i(b_i) \), where payment function \( p_i \) is continuous, \( p_i(0) = 0 \), and \( p_i(b_i) > 0 \) for \( b_i > 0 \). Ties are resolved by a fair lottery. The utility of a player \( i \) is

\[
\bar{u}_i(x_i, b_i) = \begin{cases} 
  x_i - b_i & \text{if } i \text{ wins the prize,} \\
  -p_i(b_i) & \text{if } i \text{ does not win the prize;}
\end{cases}
\]

finally, \( u_i(x_1, \ldots, x_n, b_1, \ldots, b_n) \) is the expected value of \( \bar{u}_i(x_i, b_i) \) given the bid profile \((b_1, \ldots, b_n)\). Notice that these auctions are not covered by the contest application in Section 6.1. The reason is that for some bids lower than \( \bar{b} \) losing may be preferable to winning, which violates the assumption in that application that \( \bar{u}_i \) strictly increases in \( y \) for all \((x_1, \ldots, x_n)\) with \( x_i > 0 \) and all \((b_1, \ldots, b_n)\).

These auctions are clearly games with ties. We now show that they also satisfy favorable tie breaking. A tie \( b_i = b \) is essential for player \( i \) at \((x_1, \ldots, x_n, b_1, \ldots, b_n)\) if \( b \) is the winning bid and \( x_i - b_i \neq -p_i(b_i) \). Define the payoff envelope \( W \) as in Section 6.1, that is, as the sum of players’ payoffs when ties are broken in a way that maximizes this sum. We define a deviation plan by increasing bids slightly in order to win all essential ties as long as the bid does not exceed the prize value, and otherwise lowering the bid to 0 (or close to 0 to avoid ties). This is beneficial because in the former case winning is strictly better than losing and in the latter case bidding 0 is strictly better than both winning and losing. Formally, we let the bid \( \tau_{i}^{b,\varepsilon}(x_i, b_i) \) be a bid \( b'_i \) such that the marginal on \( B \) of each strategy \( \nu_j, j \neq i \), does not have an atom at \( b'_i \) (so by Fubini’s theorem the deviation plan is without ties), and for \( b_i \leq x_i \) the bid \( b'_i \) satisfies \( b'_i > b_i \) and

\[
\max\{p_i(b'_i) - p_i(b_i), b'_i - b_i\} < \varepsilon. \tag{10}
\]

small bidding costs of losing (3) fails for any payoff envelope \( W \).

\footnote{This guarantees that any equilibrium remains an equilibrium if players can place any non-negative bid.}

\footnote{For example, if \( p_i(b_i) = b_i/2 \), then for types \( x_i < 1/2 \) bidder \( i \) prefers to lose at bids close to 1.}
For $b_i > x_i$ let $b'_i$ be such that $\max\{b'_i, p_i(b'_i)\} < \varepsilon$. This guarantees (1).

It remains to show that for any strategy profile $\mu$ with essential ties (3) holds for some $\varepsilon > 0$. For this, define function $\bar{V}_i$ as player $i$’s payoff when ties are broken in her favor if $b_i \leq x_i$ and 0 if $b_i > x_i$. That is,

$$
\bar{V}_i(x_1, \ldots, x_n, b_1, \ldots, b_n) = \begin{cases} 
  x_i - b_i & \text{if } b_i = \max\{b_1, \ldots, b_n\} \text{ and } b_i \leq x_i \\
  -p_i(b_i) & \text{if } b_i < \max\{b_1, \ldots, b_n\} \text{ and } b_i \leq x_i \\
  0 & \text{if } b_i > x_i 
\end{cases}.
$$

Let $\bar{V}$ be the sum of $\bar{V}_i$ across all players $i$. Then

$$
\bar{V}(x_1, \ldots, x_n, b_1, \ldots, b_n) \geq W(x_1, \ldots, x_n, b_1, \ldots, b_n),
$$

with a strict inequality whenever a tie $b$ is essential for some player at $(x_1, \ldots, x_n, b_1, \ldots, b_n)$ and $x_i > 0$ for every player $i$ (because there are at least two tying players, winning is strictly better than losing for player $i$ if $b_i \leq x_i$ and $x_i > 0$, and bidding 0 is strictly better than bidding $b_i$ if $b_i > x_i$). Now, take a strategy profile $\mu$ with essential ties. Since for each player $i$ type $x_i = 0$ has probability 0,

$$
\alpha = \int \bar{V}(x_1, \ldots, x_n, b_1, \ldots, b_n)d\mu - \int W(x_1, \ldots, x_n, b_1, \ldots, b_n)d\mu > 0.
$$

By (10) and the definition of $\tau_i^{\mu, \varepsilon}(x_i, b_i)$ for $b_i > x_i$, we have

$$
\int \bar{V}(x_1, \ldots, x_n, b_1, \ldots, b_n)d\mu - \sum_{j=1}^{n} U_j(\mu_j^\varepsilon, \mu_j^*) < n\varepsilon,
$$

where $\mu_j^\varepsilon$ is the deviation strategy associated with $\tau_j^{\mu, \varepsilon}$. Therefore, (3) holds for $\varepsilon < \alpha/n$.

**10.2 Proof of Proposition 1**

We will prove the following result, which shows equilibrium existence under four conditions, the last of which corresponds to Proposition 1. To state these conditions, we denote by $x_i$ the lower bound of the support of player $i$’s valuation distribution $F_i$.

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56 The support is the smallest closed set to which $F_i$ assigns measure 1.
Proposition 2 A first-price auction with private values in which players’ valuation distributions do not have an atom at 0 has an equilibrium in distributional strategies if any of the following conditions hold:

1. Every player $i$’s valuation distribution $F_i$ is atomless.

2. There is some $\underline{x} \geq 0$ such that for every player $i$, $\underline{x}_i = \underline{x}$ and $F_i$ does not have an atom at $\underline{x}$.

3. There are two players ($n = 2$) and if $\underline{x}_j \leq \underline{x}_i$, where $j \neq i$, then $F_j$ does not have an atom at $\underline{x}_i$.

4. The support $\text{Supp}(F_i)$ of every $F_i$ is discrete, and $\text{Supp}(F_i) \cap \text{Supp}(F_j) = \emptyset$ whenever $i \neq j$.

Condition 1 in Proposition 2 is a special case of Jackson and Swinkels (2005), and condition 2 is the one in Lebrun (1996). Conditions 3 and 4 are novel sufficient conditions.

We begin with some preliminaries. For any strategy profile $\sigma$, denote by $u_{\sigma,m}^i$ player $i$’s payoff in the $m$-th game and by $u_{\sigma}^i$ player $i$’s payoff in the first-price auction when the strategy profile is $\sigma$. We first observe that players’ payoffs along the sequence of games converge uniformly (across strategy profiles) to their payoff in the first-price auction.

Claim 8 For any player $i$ and any $\delta > 0$, for sufficiently large $m$ we have that $|u_{\sigma,m}^i - u_{\sigma}^i| < \delta$ for all strategy profiles $\sigma$.

Proof. Since the prize is awarded to the highest player and ties are broken randomly both in the $m$-th game and in the first price auction, and $p_{\sigma}^m(b) = b/m$, we have that $|u_{\sigma,m}^i - u_{\sigma}^i| \leq \bar{b}/m$. ■

Next, we define for the first-price auction a deviation plan without essential ties by increasing each bid $b < \bar{b}$ slightly and reducing a bid of $\bar{b}$.\footnote{More precisely, let $\tau_{\mu,m}(x_i, b_i)$ be a bid $b'_i$ such that $b'_i \in (b_i, b_i + \varepsilon)$ if $b_i < \bar{b}$ and $b'_i < \bar{b} - \varepsilon$ if $b'_i = \bar{b}$; in addition, the marginal of $\mu_j$, $j \neq i$, does not have an atom at $b'_i$.} When we later refer to deviation strategies given some strategy profile, they will correspond to this deviation plan. Without loss of generality assume that $u_{\mu,m}^i$ converges as $m$ grows large (recall that $\mu^m$ is the...
equilibrium of the \( m \)-th game), and denote its limit by \( u_i^\mu \). A tie \( b_i = b \) is essential for player \( i \) at \((x_1, \ldots, x_n, b_1, \ldots, b_n)\) if \( b \) is the winning bid and \( x_i \neq b_i \).

We first show that if players’ payoffs do not discontinuously decrease at \( \mu \), then \( \mu \) is an equilibrium of the first-price auction.

**Claim 9** If \( u_i^\mu \geq u_i^* \) for every player \( i \), then \( \mu \) is an equilibrium of the first-price auction.

**Proof.** Suppose that \( \mu \) is not an equilibrium of the first-price auction and that player \( i \) has a profitable deviation \( \sigma_i \). We will show that for large enough \( m \) player \( i \) has a profitable deviation in the \( m \)-th game given the equilibrium strategy profile \( \mu^m \).

Let \( \delta = u_i^{(\sigma_i, \mu_{-i})} - u_i^\mu \) be player \( i \)'s gain from the profitable deviation \( \sigma_i \) in the first-price auction. Denote by \( \sigma_i^\varepsilon \) player \( i \)'s deviation strategy associated with strategy profile \((\sigma_i, \mu_{-i})\) for \( \varepsilon = \delta/4 \) (the deviation plan was defined immediately after the proof of Claim 8). We will show that for large enough \( m \) strategy \( \sigma_i^\varepsilon \) is a profitable deviation for player \( i \) in the \( m \)-th game given the strategy profile \( \mu^m \). By (1),

\[
|u_i^{(\sigma_i, \mu_{-i})} - u_i^{(\sigma_i^\varepsilon, \mu_{-i})}| < \delta/4.
\]

Since \((\sigma_i, \mu)\) has no essential ties for player \( i \) and \( \mu^m \) converges to \( \mu \), by Lemma 1 we have that for large enough \( m \),

\[
|u_i^{(\sigma_i^\varepsilon, \mu_{-i})} - u_i^{(\sigma_i^\varepsilon, \mu^m_{-i})}| < \delta/4.
\]

By Claim 8, we have that for large enough \( m \),

\[
|u_i^{(\sigma_i^\varepsilon, \mu^m_{-i})} - u_i^{(\sigma_i^\varepsilon, \mu^m_{-i}), m}| < \delta/4.
\]

The last three inequalities imply that for large enough \( m \)

\[
|u_i^{(\sigma_i, \mu_{-i})} - u_i^{(\sigma_i^\varepsilon, \mu^m_{-i}), m}| < 3\delta/4.
\]

Since \( \delta = u_i^{(\sigma_i, \mu_{-i})} - u_i^\mu \), the last inequality implies that

\[
u_i^{(\sigma_i^\varepsilon, \mu^m_{-i}), m} - u_i^\mu > \delta/4.
\]

\footnote{Otherwise, we can take the appropriate subsequences for all players.}
Since $u_i^{\mu^{m},m}$ converges to $u_i^*$ and $u_i^{\mu} \geq u_i^*$, the previous inequality implies that for large enough $m$

$$u_i^{(\sigma^*, \mu^{m}), m} > u_i^{\mu^{m}, m},$$

which concludes the proof. □

The next claim shows that since payoff discontinuities arise only because of essential ties, if $\mu$ does not have essential ties then it is an equilibrium of the first-price auction.

**Claim 10** If $\mu$ does not have essential ties, then $\mu$ is an equilibrium of the first-price auction.

**Proof.** By Claim 9, it suffices to show that $u_i^{\mu} = u_i^*$ for every player $i$. This is equivalent to showing that for every $\delta > 0$ and large enough $m$

$$|u_i^{\mu} - u_i^{\mu^{m}, m}| < \delta. \quad (11)$$

Because $\mu$ does not have essential ties for player $i$, Lemma 1 implies that

$$|u_i^{\mu} - u_i^{\mu^{m}}| < \delta/2$$

for large enough $m$. By Claim 8, we have

$$|u_i^{\mu^{m}} - u_i^{\mu^{m}, m}| < \delta/2$$

for large enough $m$. The last two inequalities imply that (11) holds for large enough $m$. □

To prove Proposition 2 we show that each of the conditions in Proposition 2 implies that $\mu$ does not have essential ties. To this end, suppose that $\mu$ has essential ties (see the definition of essential ties immediately preceding Lemma 4).

**Lemma 5** Suppose that strategy profile $\mu$ has essential ties for player $i$. Then there exists a bid $b^*$ with the following properties.

1. Player $i$ and at least one other player bid $b^*$ with positive probability, that is, the marginals on $B$ of $\mu_i$ and of $\mu_j$ for some $j \neq i$ have an atom at $b^*$.

2. No player bids strictly more than $b^*$ with probability 1, that is, $\mu_j(X \times (b^*, \bar{b})) < 1$ for every player $j$, so a player tying at $b^*$ does not lose for sure.

3. A $\mu_i$-positive measure of types $x_i > b^*$ of player $i$ bid $b^*$ with positive probability, that is, $\mu_i((b^*, 1] \times \{b^*\}) > 0$. 

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4. Any player \( j \neq i \) who bids \( b^* \) with positive probability does so only when her type is \( b^* \), that is, \( \mu_j(X \{b^*\} \times \{b^*\}) = 0 \). In particular, \( b^* > 0 \).

5. For every player \( j \neq i \) who bids \( b^* \) with positive probability there is some player \( k \neq j \) who bids \( b^* \) with positive probability and bids strictly less than \( b^* \) with probability 0, that is, \( \mu_k(X \times [0, b^*)) = 0 \). This player \( k \) has valuations strictly lower than \( b^* \) with probability 0, that is, \( x_k \geq b^* \). If only two players bid \( b^* \) with positive probability, then this is player \( i \), that is, \( k = i \).

The following observation will be used repeatedly in the remaining proofs, including in the proof of Lemma 5.

**Observation 1** In the equilibrium \( \mu^m \) of the \( m \)-th game no player \( j \) bids more than her value, that is, \( \mu^m_j([0, x_j] \times (x_j, \bar{b})] = 0 \), because any bid \( b > x_2 \) is strictly dominated by a bid of 0 for every type \( x_j \).

We now prove Lemma 5.

**Proof.** Properties 1 and 2 follow immediately from Fubini’s theorem and the definition of essential ties. In addition \( \mu_j(X \{b^*\} \times \{b^*\}) > 0 \), because if we had \( \mu_j(X \{b^*\} \times \{b^*\}) = 0 \) then the fact that \( \mu_j(X \times \{b^*\}) > 0 \) would imply that the tie at \( b^* \) is not essential. To obtain property 3 it thus suffices to show that \( \mu_j([0, b^*) \times [b^*, \bar{b}] = 0 \). Let \( A^k = [0, b^* - 1/k] \times (b^* - 1/k, \bar{b}] \subseteq X \times B \), so \( \mu^m_j(A^k) = 0 \) by Observation 1. Since \( A^k \) is open in \( X \times B \), Theorem 29.1 in Billingsley (1995) implies that \( \mu_j(A^k) = 0 \). Since \( [0, b^*) \times [b^*, \bar{b}] \subseteq \bigcup_{k=1}^{\infty} A^k \), we have \( \mu_j([0, b^*) \times [b^*, \bar{b}] = 0 \) for every player \( j \). This demonstrates property 3 and also shows that for the first part of property 4 it is enough to show that \( \mu_j((b^*, 1] \times \{b^*\}) = 0 \) for a player \( j \neq i \) who bids \( b^* \) with positive probability.

Suppose that \( \mu_j((b^*, 1] \times \{b^*\}) > 0 \) for such a player \( j \neq i \), and recall that \( \mu_i((b^*, 1] \times \{b^*\}) > 0 \) by property 3. Thus, for some \( \varepsilon > 0 \), some \( \delta > 0 \), and all \( \gamma > 0 \),

\[
\mu_j(A^{\varepsilon, \gamma}) > \delta \text{ and } \mu_i(A^{\varepsilon, \gamma}) > \delta,
\]

where \( A^{\varepsilon, \gamma} = (b^* + \varepsilon, 1] \times (b^* - \gamma, b^* + \gamma) \). By Theorem 29.1 in Billingsley (1995) we have

\[
\mu^m_j(A^{\varepsilon, \gamma}) > \delta \text{ and } \mu^m_i(A^{\varepsilon, \gamma}) > \delta
\]  (12)
for large enough \( m \). Notice that for any \( \beta > 0 \) and all small enough \( \gamma > 0 \), every type in \((b^* + \varepsilon, 1]\) strictly prefers to increase her bid from \( b^* - \gamma \) to \( b^* + \gamma \) if by doing so she increases her probability of winning by at least \( \beta \). Given strategy profile \( \mu^m \), in the \( m \)-th game consider the set of best responses from \((b^* - \gamma, b^* + \gamma)\) of the types from \((b^* + \varepsilon, 1]\) of player \( i \) (when the other players bid according to \( \mu^m_i \)), and consider the infimum of the union of these sets. Consider the corresponding infimum for player \( j \), and denote by \( b^*_m \) the minimum of these two infima.

It cannot be that for small enough \( \gamma \) player \( i \) or \( j \) has at atom at \( b^*_m \) when they have types in \((b^* + \varepsilon, 1]\), that is, we must have \( \mu^m((b^* + \varepsilon, 1] \times \{b^*_m\}) \neq \mu^m((b^* + \varepsilon, 1] \times \{b^*_m\}) = 0 \), because, by (12), bidding \( b^* + \gamma \) instead of \( b^*_m \) increases such a player’s probability of winning by some \( \beta > 0 \). But (12) and the lack of an atom at \( b^*_m \) implies that by bidding \( b^* + \gamma \) instead of bids sufficiently close to \( b^*_m \) players \( i \) and \( j \) can increase their probability of winning by some \( \beta > 0 \). Thus, neither player \( i \) nor player \( j \) have best responses close to \( b^*_m \), contradicting the definition of \( b^*_m \). This shows that \( \mu_j((b^*, 1] \times \{b^*\}) = 0 \) and demonstrates the first part of property 4. For the second part of property 4, observe that \( \mu_j(\{b^*\} \times \{b^*\}) > 0 \) also implies that \( b^* > 0 \), since by assumption \( F_j(0) = 0 \).

For property 5, take a player \( j \neq i \) who bids \( b^* \) with positive probability, so \( \mu_j(\{b^*\} \times \{b^*\}) > 0 \) by property 4. Suppose that every player \( k \neq j \) bids strictly less than \( b^* \) with positive probability, that is, \( \mu_k(X \times [0, b^*)) > 0 \). This implies that \( \mu_k(X \times [0, b^* - \varepsilon)) > 0 \) for some \( \varepsilon > 0 \) and every \( k \neq i \). Thus, the probability that all players other than \( j \) bid strictly less than \( b^* - \varepsilon \) is greater than some \( \delta > 0 \). Theorem 29.1 in Billingsley (1995) then implies that for large enough \( m \) the probability that in \( \mu^m \) all players other than \( j \) bid less than \( b^* - \varepsilon \) is also greater than \( \delta \). Consequently, for large enough \( m \), by bidding \( b^* - \varepsilon \) in the \( m \)-th game when the other players bid according to \( \mu^m_j \) player \( j \) wins with at least probability \( \delta \), which gives every one of her types \( x_j > b^* - \varepsilon/2 \) a payoff greater than \( \delta \varepsilon/4 \). This implies that for large \( m \) types \( x_j \) in \((b^* - \delta \varepsilon/8, b^* + \delta \varepsilon/8)\) do not bid in \((b^* - \delta \varepsilon/8, b^* + \delta \varepsilon/8)\), since by doing so they get less than \( \delta \varepsilon/4 \) even if they win with probability 1. Thus, \( \mu_j^m((b^* - \delta \varepsilon/8, b^* + \delta \varepsilon/8) \times (b^* - \delta \varepsilon/8, b^* + \delta \varepsilon/8)) = 0 \) for large enough \( m \), so

\[
\mu_j((b^* - \delta \varepsilon/8, b^* + \delta \varepsilon/8) \times (b^* - \delta \varepsilon/8, b^* + \delta \varepsilon/8)) = 0
\]
by Theorem 29.1 in Billingsley (1995). This implies that \( \mu_j(\{b^*\} \times \{b^*\}) = 0 \), a contradiction. Thus, there is a player \( k \neq j \) who bids strictly less than \( b^* \) with probability 0. By property 2 player \( k \) bids \( b^* \) with positive probability. Finally, because in the \( m \)-th game no player bids more than her value (Observation 1), \( \mu_k(X \times [b^*, \bar{b}]) = 1 \) implies that \( x_k \geq b^* \).

We conclude the proof of Proposition 2 by showing that each of the conditions in Proposition 2 implies that no bid \( b^* \) has the properties listed in Lemma 5.

**Lemma 6** Under each of the conditions in Proposition 2 there is no bid \( b^* \) with the properties listed in Lemma 5.

**Proof.** Condition 1 precludes the existence of such a \( b^* \) because property 4 implies that the valuation distribution of at least one player has an atom at \( b^* \). Condition 2 precludes the existence of such a \( b^* \) because, under condition 2, property 4 implies that \( b^* > x \), but condition 2, property 5, and Observation 1 imply that \( b^* = x \).

Suppose that condition 3 holds. If \( x_1 = x_2 \), then condition 3 implies that condition 2 holds. Suppose that \( x_1 > x_2 \) (the case \( < \) is symmetric). Suppose that \( \mu \) has essential ties for player \( i \) and consider the corresponding bid \( b^* > 0 \) from Lemma 5. By property 3, a \( \mu_i \)-positive measure of types \( x_i > b^* \) of player \( i \) bid \( b^* \) with positive probability; by property 5, \( x_i \geq b^* \). By properties 1 and 4 player \( j \neq i \) bids \( b^* \) with positive probability and only when his valuation is \( x_j = b^* \). Thus, \( x_j \leq b^* \), so \( x_j \leq b^* \leq x_i \), which implies that \( i = 1 \) and \( j = 2 \) (because \( x_1 > x_2 \)). Moreover, player 2’s valuation distribution \( \mu_2 \) has an atom at \( b^* \) (since she bids \( b^* \) only when her valuation is \( x_2 = b^* \)). We therefore have that \( x_2 \leq b^* < x_1 \), where the strict inequality follows because \( \mu_2 \) does not have an atom at \( x_1 \) (condition 3).

Let \( \delta > 0 \) be such that player 2 bids \( b^* \) with probability higher than \( \delta \). By Theorem 29.1 in Billingsley (1995), for every \( \varepsilon \) in \((0, b^*)\), in the \( m \)-th game for large enough \( m \) player 2 chooses bids in the interval \((b^* - \varepsilon, b^* + \varepsilon)\) with probability higher than \( \delta \), that is, \( \mu_2^m(X \times (b^* - \varepsilon, b^* + \varepsilon)) > \delta \). Thus, for large enough \( m \), by bidding \( b^* + \varepsilon \) instead of \( b^* - \varepsilon \) the payoff of type \( x_1 \) of player 1 in the \( m \)-th game increases by more than \( \delta(x_1 - b^* - \varepsilon) - 2\varepsilon \). This is strictly greater than 0 for every type \( x_1 \) of player 1 when \( \varepsilon < \delta(x_1 - b^*)/(2 + \delta) \). Thus, for large enough \( m \), the bid \( b^* - \varepsilon \) is not a best response for player 1 in the \( m \)-th game when player 2 bids according to \( \mu_2^m \). But, just like in all-pay auctions, in the \( m \)-th game every
positive bid is a best response for at least two players, a contradiction. Thus, condition 3 precludes the existence of such a bid $b^*$.

Suppose that condition 4 holds, $\mu$ has essential ties for player $i$, and $b^* > 0$ is the bid from Lemma 5. Properties 1 and 4 and the assumption that the supports of players’ valuation distributions do not intersect imply that only one other player $j \neq i$ bids $b^*$ with positive probability, and this player $j$ is the only player whose discrete support includes $b^*$. Property 5 implies that for any $\varepsilon > 0$ player $i$ chooses bids in $(b^* - \varepsilon, \bar{b})$ with probability 1, that is, $\mu_i(X \times (b^* - \varepsilon, \bar{b})) = 1$. By Theorem 29.1 in Billingsley (1995), $\mu_i^m(X \times (b^* - \varepsilon, \bar{b})) = 1$ for large enough $m$. Thus, for any $\varepsilon > 0$ and large enough $m$, any player $k \neq i$ who bids $x \leq b^* - \varepsilon$ in the $m$-th game loses with probability 1 when the other players bid according to $\mu_{-k}^m$. By property 2 and Theorem 29.1 in Billingsley (1995), there is some $\delta > 0$ such that for any $\varepsilon > 0$ and large enough $m$, $\mu_l^m(X \times [0, b^* + \varepsilon)) > \delta$ for every player $l$, so by bidding $b^* + \varepsilon$ any player can win at least with some probability $\eta > 0$ (which is independent of $\varepsilon$ and $m$). These two observations, the fact that player $j$ is the only one whose support includes $b^*$, and the assumption that the supports of players’ valuations are finite imply that for some $\varepsilon > 0$ and large enough $m$, in the $m$-th game bid $b^* - \varepsilon$ is not a best response of any player $k \neq i, j$ (for any type): types lower than $b^*$ do not have best responses above their value (Observation 1) and types higher than $b^*$ prefer bidding $b^* + \varepsilon$ to bidding $b^* - \varepsilon$.

Since in the $m$-th game $b^* - \varepsilon$ is a best response for at least two players (as discussed in

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59The key property of the $m$-th game is that a bid of 0 dominates a bid equal to a player’s valuation, at which the player is indifferent between winning and losing. This implies that in equilibrium no player’s strategy has an atom at a bid greater than 0, since the player must be winning with positive probability at such a bid, so no other player bids slightly below this bid (it is better to bid slightly higher than the bid). This in turn implies that every positive bid up to the supremum $b_{\text{sup}}$ of the union of players’ best response sets (across all types) is a best response for at least two players: the marginals of players’ strategies on $B$ are continuous above 0, so if a bid is not a best response for some player, neither is a neighborhood of the bid. Thus, a positive bid cannot be a best response for a single player, since the player can decrease her bid and win with the same probability. And if a positive bid $b < b_{\text{sup}}$ is not a best response for any player, then $b$ must be equal to $b_{\text{sup}}$, because otherwise the marginal on $B$ of some player’s strategy must have an atom at the lowest bid $b' > b$ in order to make some players willing to bid $b'$ instead of $b$. But, as we have argued, there are no such atoms. This type of argument appears more formally in many papers on all-pay auctions and contests (see, for example, Siegel (2014)).
the previous paragraph), it must be a best response for players $i$ and $j$. But an argument almost identical to the one used in the previous paragraph for condition 3 shows that $b^* - \varepsilon$ is not a best response for player $i$ in the $m$-th game for large enough $m$, since all of player $i$’s types are strictly greater than $b^*$. \footnote{The only modification needed is to observe that by property 2 of Lemma 5, and the fact that $\mu_k(X \times \{b^*\}) = 0$ for every $k \neq i, j$, we have that $\mu_k(X \times [0, b^* - \varepsilon)) \geq \beta$ for some $\beta > 0$, every $\varepsilon > 0$, and all large enough $m$, so by bidding $x_i > b^* - \varepsilon$ player $i$ outbids all players other than $j$ at least with some probability $\alpha > 0$ (which is independent of $\varepsilon$ and $m$).} This concludes the proof. 

11 References


