The Persuasion Duality*

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Abstract

We present a unified duality approach to Bayesian persuasion. The optimal dual variable, interpreted as a price function, is shown to be a supergradient of the concave closure of the objective function at the prior belief. Under regularity conditions, our general duality result implies known results for the case when the objective function depends only on the expected state. We apply our approach to the multi-dimensional moment persuasion problem, showing how to construct prices for moments when the objective is differentiable, deriving conditions for optimality of convex-partitional signals, and characterizing the optimal persuasion scheme in the case when the state is two-dimensional and the objective is quadratic.

Keywords: Bayesian persuasion, information design, duality, prices

JEL codes: D82, D83

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1 Introduction

Kamenica and Gentzkow (2011) show that the optimal signal in a Bayesian persuasion problem concavifies the objective function in the space of posterior beliefs over the state (see Bergemann and Morris, 2019 and Kamenica, 2019 for excellent overviews of the burgeoning literature on Bayesian persuasion). Although conceptually attractive, concavification is not always a tractable approach. Thus, several recent papers (see Kolotilin, 2018, Dworczak and Martini, 2019, Dizdar and Kováč, 2020, Galperti et al., 2021, and Kolotilin et al., 2022) used duality theory to characterize the optimal signal.

In this paper, we present a unified duality approach to the Bayesian persuasion problem. Our approach builds on and extends the geometric duality of Gale (1967). We show that the optimal dual variable is a supergradient of the concave closure of the objective function at the prior belief (Section 3). Thus, strong duality holds if and only if the concave closure is superdifferentiable at the prior. This regularity condition is always satisfied when the state space is finite and the prior has full support (because every finite-dimensional concave function is superdifferentiable at every interior point). But superdifferentiability may not hold when the state space is infinite—additional assumptions are needed for strong duality to hold. We show that strong duality holds if (and only if) the concave closure has bounded steepness at the prior. A sufficient condition is that the objective function is Lipschitz.

In the special case when the objective function depends only on a finite set of moments of the posterior distribution (for example, the expected state), we show that the price function on the space of moments is a convex function that is greater than the objective function (Section 4). Thus, our results generalize the duality results established by Kolotilin (2018), Dworczak and Martini (2019), Dizdar and Kováč (2020), and Kolotilin et al. (2022) and extend them to the multi-dimensional case. A contemporaneous paper Malamud and Schrimpf (2021) has also made progress on analyzing multi-dimensional moment persuasion, using different tools. While some of our results in Section 4.3 overlap with theirs, we believe the two approaches to be complementary: for example, Malamud and Schrimpf allow the state space to be non-compact, while we cover cases when optimal signals are non-deterministic. We comment on the precise relationship to these papers in Section 4 and Appendix C.2.

We illustrate the usefulness of the generalized duality approach by deriving a sufficient condition for a convex-partitional signal (an extension of the one-dimensional notion of a

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1 Using the theory of real analytic functions, Malamud and Schrimpf establish a remarkably powerful result that, under a regularity condition on the prior and the objective function, there exists an optimal deterministic signal. This result forms the foundation of their analysis. Furthermore, relying on metric geometry and the theory of the Hausdorff dimension, they show that optimal signals correspond to low-dimensional manifolds.
monotone-partitional signal) to be optimal in a multi-dimensional moment persuasion problem. We demonstrate that prices can be derived explicitly from the conjectured support of the posterior moments, assuming the objective function is differentiable. We use these tools to characterize the optimal signal in the classical case of Rayo and Segal (2010) when the state is two-dimensional and the objective function is a quadratic form (Section 5).

2 Model

2.1 Preliminaries

Let $(\Omega, \rho)$ be a compact metric space, where $\rho$ is a metric on $\Omega$. The set of Lipschitz functions on $\Omega$, denoted by $L(\Omega)$, is the set of functions $p : \Omega \to \mathbb{R}$ such that

$$\|p\|_L := \sup \left\{ \frac{|p(\omega) - p(\omega')|}{\rho(\omega, \omega')} : \omega, \omega' \in \Omega, \omega \neq \omega' \right\} < \infty.$$ 

A function $p \in L(\Omega)$ is $L$-Lipschitz if $\|p\|_L \leq L$. Let $L_1(\Omega)$ denote the set of 1-Lipschitz functions on $\Omega$.

Let $M(\Omega)$ be the set of finite signed Borel measures on $\Omega$, and $\Delta(\Omega)$ be the subset of probability measures. On the linear space $M(\Omega)$, we define the Kantorovich-Rubinstein norm: for each $\mu \in M(\Omega)$,

$$\|\mu\|_{KR} := |\mu(\Omega)| + \sup \left\{ \int_{\Omega} p(\omega) d\mu(\omega) : p \in L_1(\Omega), p(\omega_0) = 0 \right\},$$

where $\omega_0$ is an arbitrary fixed element of $\Omega$. Since $(\Omega, \rho)$ is a compact metric space, Theorem 6.9 and Remark 6.19 in Villani (2009) yield that $\|\cdot\|_{KR}$ metrizes the weak* topology on $\Delta(\Omega)$ and that $(\Delta(\Omega), \|\cdot\|_{KR})$ is a compact metric space. Let $\Delta(\Delta(\Omega))$ be the set of Borel probability measures on $\Delta(\Omega)$, endowed with the Kantorovich-Rubinstein distance. Then, $\Delta(\Delta(\Omega))$ is also a compact metric space.

2.2 Persuasion Problem

We now formally define the persuasion problem, as in Kamenica and Gentzkow (2011). The state space is $\Omega$, and there is a prior belief $\mu_0 \in \Delta(\Omega)$. An objective function $V : \Delta(\Omega) \to \mathbb{R}$ is bounded and upper semi-continuous. We will be imposing increasingly stronger assumptions on $V$ to derive increasingly stronger results throughout the paper.
The persuasion problem is to find a distribution of posterior beliefs $\tau \in \Delta(\Delta(\Omega))$ to

$$\maximize \int_{\Delta(\Omega)} V(\mu) d\tau(\mu) \quad \text{(P)}$$

subject to $\int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0$.

We will denote by $\mathcal{T}(\mu_0)$ the set of feasible distributions of posteriors, that is

$$\mathcal{T}(\mu_0) = \left\{ \tau \in \Delta(\Delta(\Omega)) : \int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0 \right\}.$$

We define the concave closure of $V$ to be the value of the persuasion problem:

$$\hat{V}(\mu_0) := \sup_{\tau \in \mathcal{T}(\mu_0)} \int_{\Delta(\Omega)} V(\mu) d\tau(\mu).$$

By Kamenica and Gentzkow (2011), the concave closure of $V$ at $\mu_0$ is the supremum of $z$ such that $(z, \mu_0)$ belongs to the convex hull of the graph of $V$ on $\Delta(\Omega)$.

The dual problem is to find a price function $p \in L(\Omega)$ to

$$\minimize \int_{\Omega} p(\omega) d\mu_0(\omega) \quad \text{(D)}$$

subject to $V(\mu) \leq \int_{\Omega} p(\omega) d\mu(\omega)$ for all $\mu \in \Delta(\Omega)$.

We will denote by $\mathcal{P}(V)$ the set of feasible price functions, that is,

$$\mathcal{P}(V) = \left\{ p \in L(\Omega) : V(\mu) \leq \int_{\Omega} p(\omega) d\mu(\omega) \text{ for all } \mu \in \Delta(\Omega) \right\}.$$

We define the concave envelope of $V$ at $\mu_0$ to be the value of the dual problem:

$$\overline{V}(\mu_0) := \inf_{p \in \mathcal{P}(V)} \int_{\Omega} p(\omega) d\mu_0(\omega).$$

As defined in Aliprantis and Border (2006), the concave envelope of $V$ at $\mu_0$ is the infimum of values taken at $\mu_0$ by all continuous affine functions on $M(\Omega)$ that bound $V$ from above.

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2In a previous draft of this paper, we have considered a more general dual problem by requiring only that $p$ is continuous (but not necessarily Lipschitz). While the previous approach allowed for strong duality to hold under slightly more permissive assumptions, we could not find any economic applications exploiting that additional generality. The current formulation, inspired by a comment from Doron Ravid, leads to a more elegant exposition.
on \( \Delta(\Omega) \). Our definition is equivalent: By Theorem 0 in Hanin (1992),\(^3\) the space dual to \((M(\Omega), \|\cdot\|_{KR})\) is the space \( L(\Omega) \), modulo the constant functions. Hence, any continuous linear function on \((M(\Omega), \|\cdot\|_{KR})\) can be represented as \( \int_{\Omega} p(\omega) d\mu(\omega) \) for some \( p \in L(\Omega) \).\(^4\)

We interpret the persuasion problem as a linear production problem of Gale (1960). The states are economic resources, and the probability measure \( \mu_0 \) is a producer’s endowment of resources. The set \( \Delta(\Omega) \) is the set of linear production processes available to the producer. A process \( \mu \in \Delta(\Omega) \) operated at unit level consumes the measure \( \mu \) of resources and generates income \( V(\mu) \). A production plan \( \tau \) describes the level at which each process \( \mu \) is operated. The primal problem is for the producer to find a production plan that exhausts the endowment \( \mu_0 \) and maximizes the total income.

To interpret the dual problem, imagine that there is a wholesaler who wants to buy out the producer. The wholesaler sets a unit price \( p(\omega) \) for each resource \( \omega \). The producer’s (opportunity) cost of operating a process \( \mu \) at unit level is thus \( \int_{\Omega} p(\omega) d\mu(\omega) \). A price function \( p \) is feasible for the wholesaler if the income generated by each process of the producer is not greater than the cost of operating the process, which makes the producer willing to sell all the resources. The dual problem is for the wholesaler to find feasible prices that minimize the total cost of buying up all the resources.\(^5\)

### 3 Duality

In this section, we establish weak and strong duality for the persuasion problem:

- **Weak duality** states that \( \hat{V}(\mu_0) \leq V(\mu_0) \), that is, the concave closure is bounded above by the concave envelope.

- **No duality gap** requires the equality \( \hat{V}(\mu_0) = V(\mu_0) \), that is, that the concave closure and the concave envelope coincide.

- **Primal and dual attainment** additionally require existence of solutions to the primal and dual problems, respectively. We use the term **strong duality** when both primal

\(^3\)Hanin (1992) credits the result to Kantorovich and Rubinstein (1958). The version of the result that we use is formulated in Exercise 8.10.143 in Bogachev (2007); see also Theorem 7.3 in Edwards (2011).

\(^4\)The distinction between affine and linear functions is immaterial for the persuasion problem because each continuous affine function \( \int_{\Omega} p(\omega) d\mu(\omega) + c \), with \( p \in L(\Omega) \) and \( c \in \mathbb{R} \), coincides with the continuous linear function \( \int_{\Omega} (p(\omega) + c) d\mu(\omega) \) on \( \Delta(\Omega) \).

\(^5\)A similar interpretation of states as resources in the context of persuasion appears in Galperti et al. (2021). Dworczak and Martini (2019) offer an interpretation with the producer replaced by a consumer, production plans by consumption bundles, and the wholesaler by a Walrasian auctioneer who sets prices in a “Persuasion economy” to clear the market.
and dual attainment (and hence also no duality gap) hold.\footnote{The exact use of these terms varies across authors. For example, Villani (2009) uses the term strong duality to refer to primal attainment and no duality gap. Our convention is consistent with the economics literature where strong duality typically includes existence of solutions to the dual problem (see, Daskalakis et al., 2017; Kleiner and Manelli, 2019 for recent examples).}

Weak duality serves as a verification tool. If we can find a feasible $\tau \in \mathcal{T}(\mu_0)$ and a feasible $p \in \mathcal{P}(V)$ such that $\int_{\Delta(\Omega)} V(\mu) d\tau(\mu) = \int_{\Omega} p(\omega) d\mu_0(\omega)$, then $\tau$ is optimal. Within our interpretation, weak duality states that the total income generated by the producer cannot exceed the total cost of the resources under feasible prices, which make the producer willing to sell the resources. Thus, if there exists a plan for the producer and feasible prices for the wholesaler that equalize the total income with the total cost, then this plan must be optimal for the producer, and the prices must be optimal for the wholesaler. However, in itself, weak duality does not guarantee that such solutions can be found.

No duality gap ensures that the bound imposed by weak duality is tight. Thus, a feasible $\tau \in \mathcal{T}(\mu_0)$ is optimal if and only if it achieves the value of the concave envelope $\hat{V}(\mu_0)$. However, the absence of a duality gap still does not guarantee that the optimality of $\tau$ can be verified by finding a feasible price function $p$.

Finally, primal and dual attainment ensure that the solutions to both the primal and the dual problems exist, and that the optimality of the primal solution can always be demonstrated by exhibiting a dual solution achieving the value of the concave envelope. Within our interpretation, strong duality means that there exists a feasible plan for the producer and feasible prices for the wholesaler such that the cost of each operated process is equal to the income it generates.

In the remainder of this section, we establish conditions under which weak duality, no duality gap, primal attainment, and dual attainment hold.

**Theorem 1** (Weak Duality). $\hat{V}(\mu_0) \leq \bar{V}(\mu_0)$.

*Proof.~* By definition of the Lebesgue integral, $\tau$ belongs to $\mathcal{T}(\mu_0)$ if and only if for any measurable and bounded $p : \Omega \to \mathbb{R}$,

$$\int_{\Delta(\Omega)} \int_{\Omega} p(\omega) d\mu(\omega) d\tau(\mu) = \int_{\Omega} p(\omega) d\mu_0(\omega).$$

Thus, for any $\tau \in \mathcal{T}(\mu_0)$ and any such $p$ that additionally satisfies $V(\mu) \leq \int_{\Omega} p(\omega) d\mu(\omega)$ for all $\mu \in \Delta(\Omega)$, we have

$$\int_{\Delta(\Omega)} V(\mu) d\tau(\mu) \leq \int_{\Delta(\Omega)} \int_{\Omega} p(\omega) d\mu(\omega) d\tau(\mu) = \int_{\Omega} p(\omega) d\mu_0(\omega).$$


Taking the supremum over $\mathcal{T}(\mu_0)$ on the left-hand side and the infimum over $\mathcal{P}(V)$ on the right-hand side (any $p \in \mathcal{P}(V)$ is measurable and bounded) yields the desired result. \hfill $\square$

**Theorem 2** (No duality gap and primal attainment). There is no duality gap,

$$\hat{V}(\mu_0) = \overline{V}(\mu_0),$$  \hfill (O)

and the value of the concave closure $\hat{V}(\mu_0)$ is attained by some feasible $\tau \in \mathcal{T}(\mu_0)$.

**Proof.** The proof is relegated to Appendix A.2. \hfill $\square$

The primal problem (P) corresponds to maximizing an upper semi-continuous function $V$ over the compact set of feasible distributions $\mathcal{T}(\mu_0)$, so existence of a solution follows from a standard argument. We demonstrate the absence of a duality gap (O) by applying the Fenchel-Moreau’s theorem (which itself is a consequence of an appropriate hyperplane separation theorem). In the persuasion context, no duality gap means that the concave closure and the concave envelope coincide.\footnote{When $\Omega$ is finite, equality (O) follows, for example, from Corollary 12.1.1 in Rockafellar (1970).} Theorem 2 thus implies that we can use these two notions interchangeably.

One consequence of duality in the persuasion setting is that we can provide a verification result for the persuasion problem and its dual. Within our interpretation, a feasible plan and supporting prices are optimal if and only if the cost of each operated process is equal to the income it generates.

**Corollary 1** (Complementary Slackness). Distribution $\tau \in \mathcal{T}(\mu_0)$ and price $p \in \mathcal{P}(V)$ are optimal solutions to (P) and (D), respectively, if and only if

$$V(\mu) = \int_{\Omega} p(\omega) d\mu(\omega), \quad \text{for all } \mu \in \text{supp}(\tau).$$  \hfill (C)

**Proof.** The proof is relegated to Appendix A.3. \hfill $\square$

In applications, Corollary 1 can be used to infer properties of solutions to the persuasion problem. However, for this approach to be applicable, we must ensure that a solution to the dual problem exists. Our final goal is to establish conditions under which dual attainment holds. Contrary to previous results, additional regularity conditions on $V$ are needed.

We say that $\overline{V}$ is superdifferentiable at $\mu_0$ if there exists a continuous linear function $H$ on $M(\Omega)$ (which we call a supporting hyperplane of $\overline{V}$ at $\mu_0$) such that $\overline{V}(\mu_0) = H(\mu_0)$ and $\overline{V}(\mu) \leq H(\mu)$ for all $\mu \in \Delta(\Omega)$. Note that the concave envelope $\overline{V}$ is a concave function. When $\Omega$ is finite, a concave function on $\Delta(\Omega)$ is also continuous on the interior of the domain,
Figure 3.1: An objective function $V$ and the optimal price function $p^*$ in the case of a binary state, $\Omega = \{0, 1\}$.

and hence it is superdifferentiable at all interior points (Theorems 7.12 and 7.24 in Aliprantis and Border, 2006). Interior points in case of finite $\Omega$ correspond to priors $\mu_0$ that have full support on $\Omega$. However, when $\Omega$ is infinite, the set of probability measures $\Delta(\Omega)$ has an empty (relative) interior—any $\mu_0 \in \Delta(\Omega)$ is a boundary point. As a result, the hyperplane separating $(\mu_0, V(\mu_0))$ from the graph of $V$ may be vertical, and hence the required linear function $H$ may fail to exist.8

Following Gale (1967), we say that $V$ has bounded steepness at $\mu_0$ if there exists a constant $L$ such that

$$\frac{V(\mu) - V(\mu_0)}{\|\mu - \mu_0\|_{KR}} \leq L, \text{ for all } \mu \in \Delta(\Omega).$$

Intuitively, bounded steepness says that the marginal increase in the value of the persuasion problem is bounded above for a small perturbation of the prior.

**Theorem 3** (Dual Attainment). The following statements are equivalent:

1. The problem (D) has an optimal solution.
2. $V$ is superdifferentiable at $\mu_0$.
3. $V$ has bounded steepness at $\mu_0$.

**Proof.** The proof is relegated to Appendix A.4. □

The proof relies on the results by Gale (1967) and the fact that continuous linear functions on $M(\Omega)$ can be identified with Lipschitz functions on $\Omega$. The optimal price function is then

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8For an analogy, consider a concave and continuous function $f(x) = \sqrt{x}$ on $[0, 1]$. This function is not superdifferentiable at the boundary point $x = 0$ because the supporting hyperplane would have to be vertical.
shown to be a supergradient of the concave envelope $V$ at the prior $\mu_0$. Geometrically, any price function $p$ defines a hyperplane in $\Delta(\Omega) \times \mathbb{R}$ that passes through each extreme point $(\delta_\omega, p(\omega))$. The price function $p$ is feasible for (D) if and only if this hyperplane lies above $V$ on $\Delta(\Omega)$. The dual problem is to find a hyperplane that lies above $V$ and whose value at the prior $\mu_0$ is minimized. Thus, the optimal hyperplane is a hyperplane that supports $V$ at $\mu_0$, and the optimal price $p^*(\omega)$ of each state $\omega$ is the value of the supporting hyperplane at $\delta_\omega$ (see Figure 3.1).

While Theorem 3 provides a necessary and sufficient condition for dual attainment, the condition is stated in terms of a non-primitive object, the concave envelope of $V$. Next, we present a useful sufficient condition on the primitive objective function $V$.

**Theorem 4** (Lipschitz Preservation). Let $V$ be Lipschitz on $\Delta(\Omega)$. Then $\hat{V}$ is also Lipschitz on $\Delta(\Omega)$. Consequently, $\hat{V}$ has bounded steepness at each $\mu_0 \in \Delta(\Omega)$.

**Proof.** The proof is relegated to Appendix A.1. □

**Corollary 2** (Strong duality). When $V$ is Lipschitz on $\Delta(\Omega)$, strong duality holds for the persuasion problem (P).

While the statement of Theorem 4 may seem intuitive, its proof is quite involved in the general (infinite-dimensional) case. Informally, we show that given two priors, $\mu_0$ and $\eta_0$, and an optimal distribution $\tau \in \mathcal{T}(\mu_0)$, we can find a perturbation $\eta_\mu$ of each posterior belief $\mu \in \text{supp}(\tau)$ such that the perturbed posteriors $\eta_\mu$ average out to $\eta_0$ under the distribution $\tau$. Moreover, the average distance between the posteriors $\mu$ and their perturbations $\eta_\mu$ is equal to the distance between $\mu_0$ and $\eta_0$. This implies that the value of the persuasion problem under the prior $\mu_0$ cannot exceed the value of the persuasion problem under the prior $\eta_0$ by more than $L \|\mu_0 - \eta_0\|_{KR}$ when $V$ is $L$-Lipschitz. Reversing the roles of $\mu_0$ and $\eta_0$ leads to the desired conclusion.

To the best of our knowledge, Theorems 3 and 4 provide the first general dual attainment result for Bayesian persuasion. At the same level of generality, Section 8 of Dworczak and Martini (2019) establishes weak duality by defining a price function on the space of beliefs $\Delta(\Omega)$ and requiring it to be “outer-convex” (a relaxation of convexity). Theorems 3 and 4 demonstrate that such a price function exists when $V$ is Lipschitz, and that prices can in fact be taken to be linear (the function $p$ on $\Omega$ induces a linear function on $\Delta(\Omega)$ satisfying all conditions postulated by Dworczak and Martini).

### 3.1 Illustration

We conclude the section with an illustration of our duality result, emphasizing the importance of dual attainment. We study optimality of two extreme information structures, full
disclosure, corresponding to a distribution $\tau_F \in \mathcal{T}(\mu_0)$ uniquely characterized by attaching probability one to the set of Dirac delta distributions on $\Omega$, and no disclosure, corresponding to a distribution $\tau_N \in \mathcal{T}(\mu_0)$ that attaches probability one to the prior $\mu_0$.

We first assume that $V$ is Lipschitz, so that, by Theorems 3 and 4, dual attainment holds. We show how duality implies the known sufficient conditions for optimality of $\tau_F$ and $\tau_N$, respectively, and proves their necessity. Next, we argue that these intuitive conditions are no longer valid when dual attainment fails.

Suppose that $\mu_0$ has full support on $\Omega$ and let $V$ be Lipschitz on $\Delta(\Omega)$. Then, full disclosure $\tau_F$ is optimal if and only if $V$ lies below a linear function that passes through each extreme point $(\delta_\omega, V(\delta_\omega))$:  

\[ V(\mu) \leq \int_{\Omega} V(\delta_\omega) d\mu(\omega) \text{ for all } \mu \in \Delta(\Omega). \]  

(F)

No disclosure $\tau_N$ is optimal if and only if $V$ is superdifferentiable at $\mu_0$.

To prove these two observations, note that Theorem 3 implies that the dual problem (D) has an optimal solution. Thus, by Corollary 1, a feasible distribution $\tau \in \mathcal{T}(\mu_0)$ is optimal if and only if the optimal price function $p \in \mathcal{P}(V)$ satisfies (C). The support of $\tau_F$ is the set of all Dirac measures $\delta_\omega$ on $\Omega$, so (C) simplifies to $p(\omega) = V(\delta_\omega)$ for all $\omega \in \Omega$. Thus, $\tau_F$ is optimal if and only if $V(\delta_\omega)$, treated as a function of $\omega$, belongs to $\mathcal{P}(V)$—this simplifies to (F). Similarly, the condition for optimality of $\tau_N$ follows from the observation that feasibility of $p$ along with (C) is equivalent to $p$ being the supergradient of $V$ at the prior.

Because sufficiency follows from weak duality (Theorem 1), the above conditions are sufficient even if we relax the assumptions on $V$ and $\mu_0$. However, we now show that when dual attainment fails, these conditions are no longer necessary.

Let $\mu_0$ be the Lebesgue measure on $\Omega = [0, 1]$ and let $V(\mu) = 1_{\{\mu=\delta_0/2+\delta_1/2\}}$. Since $\mu_0(\{0,1\}) = 0$, there does not exist $\tau \in \mathcal{T}(\mu_0)$ with $\tau(\delta_0/2 + \delta_1/2) > 0$, so each feasible distribution $\tau \in \mathcal{T}(\mu_0)$ is optimal. However, the above conditions for optimality of $\tau_F$ and $\tau_N$ both fail. In particular, (F) does not hold at $\mu = \delta_0/2 + \delta_1/2$. As for superdifferentiability of $V$ at the prior, suppose that $p \in \mathcal{P}(V)$ is its supergradient. Then, we would need $\int_0^1 p(\omega) d\omega = 0$, so that the supporting hyperplane defined by $p$ touches $V$ at the prior. But since $p$ is Lipschitz and non-negative, this implies that $p$ is identically 0; hence, the hyperplane defined by $p$ does not lie above the graph of $V$ at $\delta_0/2 + \delta_1/2$.

The above arguments indirectly show that the dual problem does not have an optimal solution. Indeed, the dual problem is to find a non-negative Lipschitz function $p$ satisfying

\[ \text{Dworczak and Martini (2019) call this condition outer-convexity and prove that (F) is necessary for full disclosure to be optimal for all priors.} \]

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\( p(0)/2 + p(1)/2 \geq 1 \) that minimizes \( \int_0^1 p(\omega) d\omega \). We know from Theorem 2, that the infimum is 0. Clearly, the infimum is not attained: It is approximated by a sequence of Lipschitz functions that take value 1 at \( \omega = 0 \) and \( \omega = 1 \), and converge to 0 on \((0, 1)\).

### 4 Moment persuasion

We now show how our approach specializes to the case of a persuasion problem in which the objective function depends only on certain moments of the posterior. In “moment persuasion,” we assume that, given some underlying state space \( \tilde{\Omega} \) and prior \( \tilde{\mu}_0 \),

\[
V(\mu) = v \left( \int_{\tilde{\Omega}} m(\tilde{\omega}) d\mu(\tilde{\omega}) \right), \quad \text{for all } \mu \in \Delta(\tilde{\Omega}),
\]

for some measurable \( m : \tilde{\Omega} \to \mathbb{R}^N \) and some real-valued function \( v \). It will be convenient to redefine the state space as \( \Omega = m(\text{supp}(\tilde{\mu}_0)) \) with the prior \( \mu_0 \) given by \( \mu_0(B) = \tilde{\mu}_0(m^{-1}(B)) \) for any measurable \( B \subset \Omega \), so that

\[
V(\mu) = v \left( \int_{\Omega} \omega d\mu(\omega) \right), \quad \text{for all } \mu \in \Delta(\Omega).
\]

We then define the space of “moments” \( X \) as the convex hull of \( \Omega \).

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We assume that \( X \) is a convex body (that is, a compact convex set with non-empty interior)\(^{11} \) and that \( v : X \to \mathbb{R} \) is Lipschitz with constant \( L \). That last assumption allows us to rely on dual attainment from Theorems 3 and 4, as shown by the following lemma.

**Lemma 1.** If \( v \) is Lipschitz, then \( V \) is also Lipschitz.

**Proof.** The proof is relegated to Appendix B.1.

In moment persuasion, a distribution \( \tau \) of posterior beliefs \( \mu \in \Delta(\Omega) \) influences the objective only through the induced distribution of moments. By Strassen’s theorem (for example, Theorem 7.A.1 in Shaked and Shanthikumar, 1994), a distribution \( \pi_X \in \Delta(X) \) of moments is feasible (that is, induced by some Bayes-plausible distribution of posterior beliefs) if and only if \( \mu_0 \) is a mean-preserving spread of \( \pi_X \). However, anticipating our results and following Kolotilin (2018), we will formulate the moment persuasion problem as

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\(^{10}\)By redefining the state space, we have converted a general case of moment persuasion to a problem in which the objective function only depends on a multi-dimensional vector of posterior means.

\(^{11}\)This is without loss of generality: As a convex set in \( \mathbb{R}^N \), \( X \) has a non-empty relative interior, so we can always embed \( X \) in a (possibly lower-dimensional) Euclidean space such that \( X \) has non-empty interior.
optimization over joint distributions of moments and states. Formally, we call a distribution \( \pi \in \Delta(X \times \Omega) \) feasible, denoted \( \pi \in \Pi(\mu_0) \), if

\[
\int_{X \times B} \mathrm{d}\pi(x,\omega) = \int_B \mathrm{d}\mu_0(\omega), \quad \text{for all measurable } B \subset \Omega,
\]

\[
\int_{B \times \Omega} (x - \omega) \mathrm{d}\pi(x,\omega) = 0, \quad \text{for all measurable } B \subset X,
\]

where the first equation is the Bayes-plausibility constraint, and the second equation is the martingale constraint.

We let \( \pi_X \) denote the marginal distribution of moments induced by \( \pi \). The primal problem (P) then simplifies to finding a joint distribution \( \pi \in \Delta(X \times \Omega) \) to

\[
\begin{aligned}
\text{maximize} & \quad \int_X v(x) \mathrm{d}\pi_X(x) \\
\text{subject to} & \quad \pi \in \Pi(\mu_0).
\end{aligned}
\]

(\( P_M \))

### 4.1 Prices for moments

The first major result of this section derives the implications of the general duality from Section 3 for the special case of moment persuasion.

**Theorem 5.** Fix an optimal solution \( p : \Omega \to \mathbb{R} \) to the dual problem (D). There exists an extension \( \bar{p} : X \to \mathbb{R} \) of \( p \) to \( X \) (\( p \) and \( \bar{p} \) coincide on \( \Omega \)) such that, for any optimal solution \( \pi \in \Pi(\mu_0) \) to (\( P_M \)),

1. \( \bar{p} \) is convex, Lipschitz, satisfies \( \bar{p} \geq v \), and

\[
\int_X v(x) \mathrm{d}\pi_X(x) = \int_\Omega \bar{p}(\omega) \mathrm{d}\mu_0(\omega);
\]

2. there exists a measurable function \( q : X \to \mathbb{R}^N \) such that \( \|q(x)\| \leq L \) for all \( x \in X \),

\[
\bar{p}(y) = \sup_{x \in X} \{v(x) + q(x) \cdot (y - x)\}, \quad \text{for all } y \in X,
\]

\[
\bar{p}(\omega) = v(x) + q(x) \cdot (\omega - x), \quad \text{for } \pi-\text{almost all } (x,\omega).
\]

Conversely, if there exists a feasible \( \pi \in \Pi(\mu_0) \) and a price function \( \bar{p} : X \to \mathbb{R} \) satisfying any one of conditions 1 or 2, then \( \pi \) is optimal for (\( P_M \)). (The last claim is true under a weaker assumption that \( v \) is measurable and bounded.)
Theorem 5 provides sufficient and necessary conditions for optimality of a candidate solution $\pi \in \Pi(\mu_0)$. It also highlights the additional structure that the price function has in moment persuasion. The main insight is that “prices for states” can be extended to “prices for moments,” and that prices for moments are convex. By providing the two conditions 1 and 2 that are jointly necessary but individually sufficient, the theorem connects our general duality result to existing duality approaches to moment persuasion (Kolotilin, 2018, Dworczak and Martini, 2019, Dizdar and Kovác, 2020).

Before discussing these points, we give an overview of the proof of Theorem 5. Because we have guaranteed dual attainment (by the assumption that $v$ is Lipschitz), there exists a solution $p$ to the dual problem (D), and there is no duality gap: Equality (O) simplifies to

$$\int_X v(x)d\pi_X(x) = \int_\Omega p(\omega)d\mu_0(\omega),$$

for any $\pi$ optimal for $(P_M)$. We can extend $p$ (prices for states) from $\Omega$ to $X$ (prices for moments) using the so-called “convex-roof” construction (Bucicovschi and Lebl, 2013):

$$\tilde{p}(x) := \inf \left\{ \int_\Omega p(\omega)d\mu(\omega) : \mu \in \Delta(\Omega), \int_\Omega \omega d\mu(\omega) = x \right\}, \text{ for all } x \in X. \quad (R)$$

It is easy to show that $\tilde{p}$ is convex, $\tilde{p} \geq v$, and hence $\tilde{p}$ satisfies the constraint in (D). Moreover, by definition, $\tilde{p}$ is point-wise smaller than $p$ on $\Omega$. If we could show that $\tilde{p}$ is Lipschitz, then $\tilde{p}$ restricted to $\Omega$ would be a solution to the dual (D), and condition 1 in Theorem 5 would hold.

However, $\tilde{p}$ does not even have to be continuous when $N$—the dimension of the space of moments—is three or higher (even though $p$ is Lipschitz). In Appendix C.3, we provide an example of moment persuasion in which there exists $p \in L(\Omega)$ that solves (D), but the convex roof of $p$ is discontinuous. We also construct an example with a non-Lipschitz $v$ in which there does not exist any convex continuous extension of optimal prices for states to prices for moments. These examples help explain why our assumptions on the objective $v$ are stronger than those imposed by Dworczak and Martini (2019) and Dizdar and Kovác (2020) in the one-dimensional case. The additional difficulties we face are a direct consequence of a multi-dimensional space of moments: We prove in Appendix C.3 that $\tilde{p}$ is Lipschitz when $\Omega$ contains the boundary of $X$—a condition that holds trivially in the one-dimensional case.

To circumvent these difficulties, we prove a lemma showing that the graph of $\tilde{p}$ can be

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12A careful reader might notice that this implies that some assumption of Berge’s Maximum Theorem must be violated. Indeed, it turns out that the feasibility correspondence $\Phi(x) = \{ \mu \in \Delta(\Omega) : \int_\Omega \omega d\mu(\omega) = x \}$ is not necessarily lower hemi-continuous in $\mathbb{R}^N$ for $N > 2$. However, because $\Phi$ is an upper hemi-continuous correspondence, $\tilde{p}$ is lower semi-continuous, by Lemma 17.30 in Aliprantis and Border (2006).
separated by a hyperplane (with a properly bounded gradient, as captured by the function \(q(x)\) from condition 2) from any point \((x, v(x))\) on the graph of the objective function \(v\). We can then define a new price function \(\bar{p} : X \to \mathbb{R}\) that is the supremum of all such hyperplanes. The resulting price function is a convex and Lipschitz extension of \(p\) that is “sandwiched” between \(p\) and \(v\). It follows that \(\bar{p}\) solves (D) (viewed as a function on \(\Omega\)) and that condition 1 of Theorem 5 holds. Additionally, using the function \(q(x)\), we can show that the complementary-slackness condition (C) takes a particularly simple form described in condition 2 of Theorem 5.

The opposite implication (that existence of prices satisfying either condition 1 or condition 2 of Theorem 5 implies that \(\pi\) is optimal) is straightforward and relies only on weak duality. In particular, we do not need the strong assumptions on \(v\).

Condition 1 in Theorem 5 shows that, in moment persuasion, all solutions to the dual problem (D) must be convex.\(^{13}\) To see that intuitively, note that in our interpretation of the dual problem (D) from Section 2, a measure \(\gamma \in \Delta(\Omega)\) of resources and one unit of resource \(x = \mathbb{E}_\gamma[\omega]\) are now equivalent for the producer. If prices failed to be convex, the producer could sell at effectively higher prices by engaging in such “mean-preserving” transformations of the resources. Thus, the wholesaler offers convex prices to begin with.

Theorem 5 recovers (under a stronger assumption) the known strong duality result for one-dimensional moment persuasion from Kolotilin (2018), Dworczak and Martini (2019), and Dizdar and Kováč (2020), and establishes (to the best of our knowledge, for the first time) strong duality for multi-dimensional moment persuasion. Appendix C.1 formally introduces the problem dual to \((P_M)\), and shows that the price function \(\bar{p}\) from Theorem 5 is a solution to that problem.

Finally, we explain why we stated Theorem 5 using two alternative constructions of the price function, corresponding to conditions 1 and 2. The price function from condition 1 is a direct analog of prices for moments in Dworczak and Martini (2019) who derive them as a multiplier on the mean-preserving spread constraint (represented in its integral form for the one-dimensional case). In contrast, the price function from condition 2, along with the function \(q\), are analogs of the dual variables from Kolotilin (2018) and Kolotilin et al. (2022) who derive them as multipliers on the two constraints defining the set \(\Pi(\mu_0)\) of joint distributions of moments and states. In particular, \(q\) is the multiplier on the martingale constraint. Thus, the two representations of prices for moments are a consequence of two alternative representations of feasible distributions for the primal problem. Theorem 5 not only extends both approaches to the multi-dimensional case, but also unifies them by showing

\(^{13}\)Since \(\Omega\) is not necessarily a convex set, we say that \(p\) is convex on \(\Omega\) if \(p(\omega) \leq \int_\Omega p(\omega')d\mu(\omega')\) for any \(\mu \in \Delta(\Omega)\) with \(\omega = \int_\Omega \omega' d\mu(\omega')\).
that they are special cases of the general duality from Section 3. We further comment on these relationships in Appendix C.2.

In the remainder of this section, we leverage Theorem 5 to derive structural properties of solutions to (P_M). Even though Theorem 5 guarantees existence of prices for moments, it does not provide a direct way to construct them. We show next that when \( v \) is continuously differentiable, we can take \( q(x) \) from condition 2 of Theorem 5 to be equal to the gradient of \( v \) at \( x \) on the support of any optimal \( \pi_X \).

### 4.2 Constructing solutions in the differentiable case

To derive tighter implications of duality for the properties of optimal solutions, we further strengthen our assumptions on the objective function. We assume that \( v \) is continuously differentiable on \( X \), and thus has a continuous gradient \( \nabla v \) on \( X \). We will show that, in this case, solving the problem (P_M) can be reduced to finding the support of the optimal distribution of moments.

For any closed set \( S \subset X \) (candidate support of the optimal distribution of moments), we define the function \( p_S \) on \( \Omega \) by

\[
p_S(\omega) := \max_{x \in S} \left\{ v(x) + \nabla v(x) \cdot (\omega - x) \right\}, \quad \text{for all } \omega \in \Omega. \tag{S}\]

In case \( \Omega \) is not convex, we extend \( p_S \) from \( \Omega \) to \( X \) using the convex-roof construction:

\[
p_S(x) := \inf \left\{ \int_{\Omega} p_S(\omega) d\mu(\omega) : \mu \in \Delta(\Omega), \int_{\Omega} \omega d\mu(\omega) = x \right\}, \quad \text{for all } x \in X \setminus \Omega.
\]

Finally, for any feasible \( \pi \in \Pi(\mu_0) \), consider the condition:

\[
\begin{align*}
p_S(x) &\geq v(x), \quad \text{for all } x \in X, \\
p_S(\omega) &= v(x) + \nabla v(x) \cdot (\omega - x), \quad \text{for all } (x, \omega) \in \text{supp}(\pi). \tag{M}
\end{align*}
\]

The following theorem connects condition (M) to optimality of \( \pi \).

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\(^{14}\) A continuously differentiable function is usually defined on an open set. As in Chapter 10 in Rudin (1976), we say that \( v \) is continuously differentiable on the compact set \( X \) if there is a continuously differentiable function \( \overline{v} \) defined on an open set \( \overline{X} \subset \mathbb{R}^N \) such that \( X \subset \overline{X} \) and \( \overline{v}(x) = v(x) \) for all \( x \in X \).

\(^{15}\) Note that because \( p_S \) is convex on \( \Omega \) by definition, it does not matter whether we use the convex roof for \( x \in X \setminus \Omega \) or for all \( x \in X \). The reader might be surprised that we rely on the convex roof construction after arguing that it sometimes fails to properly extend prices for states to the prices for moments. And indeed, the price function \( p_S \) we constructed does not necessarily satisfy all of the conditions of Theorem 5. Nevertheless, it turns out that \( p_S \) satisfies the conditions that are relevant for deriving properties of optimal solutions to (P_M) which is our ultimate goal.
Theorem 6. A joint distribution $\pi \in \Pi(\mu_0)$ is an optimal solution to $(P_M)$ if and only if condition (M) holds with $S = \text{supp}(\pi_X)$.

Proof. The proof is relegated to Appendix B.3. \hfill \Box

Theorem 6 gives rise to a “guess and verify” procedure that can be used to identify optimal solutions to $(P_M)$. The “guess” involves conjecturing the optimal support $S$ of moments. Fixing $S$, prices $p_S$ can be computed mechanically, and then condition (M) becomes necessary and sufficient for optimality of $\pi$ with support $S$. Moreover, once we find one solution, it is possible to characterize all solutions using a single set $S$ which can be viewed as a maximal possible support of an optimal solution.

Remark 1. Suppose that $\pi^* \in \Pi(\mu_0)$ is optimal for $(P_M)$, and let

$$S^* = \{x \in X : p_{\text{supp}(\pi_X)}(x) = v(x)\}.$$ 

Then, $\pi \in \Pi(\mu_0)$ is optimal for $(P_M)$ if and only if $\text{supp}(\pi_X) \subset S^*$ and condition (M) holds with $S = S^*$.\textsuperscript{16}

Proof. The proof is relegated to Appendix B.3. \hfill \Box

Remark 1 is particularly useful when we want to prove uniqueness of an optimal solution, or that the unique solution has some special structure. We turn to these issues next.

4.3 Structure of solutions

In this subsection, we focus on deriving the implications of Theorem 6 for the structure of optimal solutions to $(P_M)$. We provide a condition under which there is a unique optimal solution $\pi$ to $(P_M)$ that partitions the state space into convex sets, and pools the states in each element of the partition. This is a natural extension of the idea of monotone-partitional solutions from one-dimensional moment persuasion to the multi-dimensional case. We also generalize a result proven by Arieli et al. (2021) and Kleiner et al. (2021): In the single-dimensional case, there exists an optimal signal $\pi \in \Pi(\mu_0)$ with a bi-pooling structure; that is, any state $\omega \in \Omega$ can be appear together with at most two distinct posterior moments in $\text{supp}(\pi)$. We derive a multi-dimensional analog of this property.

\textsuperscript{16}It is easy to see that $p_S \geq v$ in this case, so only the second condition in (M) is relevant.
4.3.1 Optimality of convex-partitional signals

We first address the problem of when it is without loss of optimality for the Sender to restrict attention to convex-partitional signals. Because we do not assume that $\Omega$ is a convex set, we require the elements of the partition to be convex relative to $\Omega$. Specifically, we define the partition on the convex hull of $\Omega$ (that is, on $X$), and we require each element of the partition of $X$ to be convex.\footnote{To understand why we adopt this convention, consider the distribution $\pi$ induced by no disclosure. Intuitively, pooling all states should correspond to a convex-partitional signal. However, the support of this distribution over states conditional on the induced moment is equal to $\Omega$, and is hence not convex when $\Omega$ is not convex. We circumvent this by defining the partition on $X$; then, the unique element of that partition corresponding to no disclosure is $X$ itself, a convex set. And of course, this partition restricted to $\Omega$ still represents no disclosure.} Formally, we say that $\pi \in \Pi(\mu_0)$ is convex-partitional if there is a measurable map $\chi : X \to X$ such that, for all measurable sets $A \subset X$ and $B \subset \Omega$,

$$\pi(A, B) = \int_B 1\{\chi(\omega) \in A\}d\mu_0(\omega),$$

and, for all $x \in X$, the set $\chi^{-1}(x)$ is convex. Intuitively, $\chi$ represents a distribution that pools all states in $\chi^{-1}(x) \cap \Omega$ into the moment $x$. When $\mu_0$ is fixed, we will abuse notation by using $\chi$ itself to refer to the distribution $\pi$ that it defines.

**Theorem 7.** Suppose $\mu_0$ has a density with respect to the Lebesgue measure. Moreover, suppose there do not exist distinct $x, y \in X$ such that

$$\nabla v(x) = \nabla v(y),$$

$$v(x) - \nabla v(x) \cdot x = v(y) - \nabla v(y) \cdot y,$$

$$\lambda v(x) + (1 - \lambda)v(y) \geq v(\lambda x + (1 - \lambda)y), \quad \text{for all } \lambda \in [0, 1].$$

Then, there is a unique optimal solution to $(P_M)$, and that solution is convex-partitional.

**Proof.** The proof is relegated to Appendix B.4 and B.5.

Theorem 7 gives an easy-to-verify condition on the objective function $v$ under which the optimal distribution is unique and convex-partitional. The condition can be seen as an extension of the affine-closure property from Dworczak and Martini (2019) that guarantees optimality of a monotone partition in the one-dimensional case.\footnote{The condition in Theorem 7 is slightly stronger than the affine-closure property when $N = 1$; in Appendix B.5, we provide a condition that is more analogous to the affine-closure property and that implies optimality of a convex-partitional distribution when $\Omega$ is convex.} The assumption that $\mu_0$ is a continuous distribution allows us to circumvent the thorny issue of how to define a convex partition when there are atoms in the distribution of states (in this case, some of the...
atoms may need to be split among two or more elements of the partition). To the best of our knowledge, Theorem 7 provides the most permissive condition guaranteeing a convex-partitional signal for multi-dimensional moment persuasion. Prior to the current draft of our paper, Malamud and Schrimpf (2021) obtained a related condition.

In the remainder of this subsection, we give an overview of the proof of Theorem 7. The first part of the proof provides general insights about the structure of optimal solutions, and does not rely on any of the assumptions of Theorem 7. Thus, our goal in the overview is to present these additional results; they will be useful when we consider applications in the next section. The second part of the proof gives an explicit construction of the elements of the optimal convex partition from Theorem 7. To simplify exposition, we assume that Ω is convex (the proof in the appendix deals with the general case).

We begin by introducing some additional notation. Fix an optimal solution \( \pi^* \in \Pi(\mu_0) \) to \( (P_M) \), and define the set \( S^* \) as in Remark 1:

\[
S^* := \{ x \in X : p_{	ext{supp}(\pi^*_x)}(x) = v(x) \}.
\]

Recall that we can interpret \( S^* \) as the maximal possible support of an optimal solution. To simplify notation, let \( p^*(x) := p_{S^*}(x) \), for all \( x \in X \). Next, we define the set \( \Gamma \) that encodes the second property in condition \( (M) \):

\[
\Gamma := \{ (x, \omega) \in S^* \times \Omega : p^*(\omega) = v(x) + \nabla v(x) \cdot (\omega - x) \}.
\]

The set \( \Gamma \) is called a contact set in the linear programming literature. In light of Theorem 6 and Remark 1, a feasible \( \pi \in \Pi(\mu_0) \) is optimal if and only if \( \text{supp}(\pi) \subset \Gamma \). Finally, we define the \( x \)-section of \( \Gamma \),

\[
\Gamma_x := \{ \omega \in \Omega : (x, \omega) \in \Gamma \}.
\]

Intuitively, the set \( \Gamma_x \) contains states that can appear together with \( x \) in the support of an optimal solution; in other words, these (and only these) states can be pooled into the moment \( x \) under an optimal solution. The sets \( \Gamma_x \) can intersect in general. If \( \omega \in \Gamma_x \cap \Gamma_y \), then \( \omega \) could appear in the support of \( \pi \) both conditional on \( x \) and conditional on \( y \).

We show that \( \text{relint}(\Gamma_x) \cap \text{relint}(\Gamma_y) \neq \emptyset \implies \Gamma_x = \Gamma_y \), where \( \text{relint}(\cdot) \) stands for the relative interior of a set. Thus, the set \( \Gamma \) generates a partition of \( \Omega \) consisting of relatively open convex components \( \{ \text{relint}(\Gamma_x) : x \in S^* \} \) and the set of points on the boundaries of these components: \( X \setminus \bigcup_{x \in S^*} \text{relint}(\Gamma_x) \). If \( x \neq y \) implies that \( \text{relint}(\Gamma_x) \neq \text{relint}(\Gamma_y) \), then \( \pi \) has a very simple structure: For any \( x \in S^* \), states in \( \text{relint}(\Gamma_x) \) are pooled together into the posterior mean \( x \).
This is where the assumptions of Theorem 7 come in. When the condition on $v$ holds, it is indeed true that $x \neq y$ implies that $\text{relint}(\Gamma_x) \neq \text{relint}(\Gamma_y)$. When $\mu_0$ has a continuous distribution, we can ignore the measure-zero set of states on the boundaries of the convex elements of the partition. Thus, a convex-partitional signal is optimal. Moreover, the optimal $\chi: X \rightarrow X$ is uniquely determined, for $\mu_0$-almost all $\omega \in \Omega$, by

$$\chi(\omega) = \{x \in S^*: \omega \in \Gamma_x\} = \{x \in S^*: \nabla p^*(\omega) = \nabla v(x)\}.$$  

We illustrate this discussion with an application in the next section.

### 4.3.2 On the richness of optimal signals

In this subsection, we turn attention to the structure of solutions when the assumption of Theorem 7 fails. In the single-dimensional case, we know from the bi-pooling result of Arieli et al. (2021) and Kleiner et al. (2021) that even if no optimal signal is monotone-partitional, there still exist optimal signals with a relatively simple structure. We will show that the multi-dimensional analog of bi-pooling allows for signals that are much more complex. We note that our generalization is a direct consequence of duality, while Arieli et al. (2021) and Kleiner et al. (2021) rely on an extreme-point characterization of optimal signals.

To simplify exposition, we assume that $\Omega$ is convex, so that we can use the notation introduced in the preceding analysis. For a set $A \subset X$, let $\text{cl}(A)$ denote the closure of $A$, and $\text{ext}(A)$ denote the set of extreme points of the closed convex hull of $A$. Fixing a solution $\pi$ to $(P_M)$ and an optimal price function, let

$$S_x := \text{cl}(\text{supp}(\pi_X) \cap \text{relint}(\Gamma_x)).$$

Informally, conditional on $x$ being the realized posterior moment, $S_x$ is the set of all posterior moments that could have been generated by the optimal signal. For example, if the optimal signal is deterministic and convex-partitional, as in Theorem 7, then $S_x = \{x\}$. The bi-pooling result of Arieli et al. (2021) and Kleiner et al. (2021) states that in the single-dimensional moment persuasion case there exists an optimal solution such that $S_x$ has at most two elements. The following result extends that conclusion to the multi-dimensional case.

**Theorem 8.** Suppose $\mu_0$ has a density with respect to the Lebesgue measure. There exists an optimal solution $\pi \in \Pi(\mu_0)$ to $(P_M)$ such that, for $\pi_X$-almost all $x$, $S_x = \text{ext}(S_x)$.

**Proof.** The proof is relegated to Appendix B.6. \hfill $\Box$
The conclusion $S_x = \text{ext}(S_x)$ means that no posterior mean in $S_x$ can be expressed as a convex combination of other posterior means in $S_x$. Note that this generalizes the bi-pooling result of Arieli et al. (2021) and Kleiner et al. (2021) because in the single-dimensional case, for any set $S \subset \mathbb{R}$, $|\text{ext}(S)| \leq 2$. However, when the dimension $N$ of the state space is two or more, $\text{ext}(S_x)$ could be an infinite set, and hence the multi-dimensional analog of the class of bi-pooling signals is much richer. In Appendix B.6 we provide an example with $N = 2$ in which $S_x$ is infinite for any choice of optimal $\pi$, hence showing that the conclusion of Theorem 8 cannot be strengthened to predict a finite $S_x$.

The proof of Theorem 8 relies on the fact that $\text{supp}(\pi) \subset \Gamma$ is both necessary and sufficient for the optimality of $\pi \in \Pi(\mu_0)$. Therefore, we can modify an optimal $\pi$ as long as we maintain that condition. The modification we introduce is in the form of an auxiliary optimization problem in which we minimize the average norm of the posterior moments subject to maintaining the condition $\text{supp}(\pi) \subset \Gamma$. Intuitively, the auxiliary problem picks an optimal persuasion solution in which $S_x = \text{ext}(S_x)$ must be satisfied, as otherwise the value of the auxiliary minimization problem could be lowered by shifting probability mass towards some posterior mean $y \in S_x$ that can be expressed as a convex combination of other posterior means in $S_x$.

5 Application: Quadratic Objective

In this section, we show how our duality approach developed in the preceding section can be used to solve a class of persuasion problems in which $\mu_0$ has a density on $\Omega$ that is a compact convex set in $\mathbb{R}^2$ (so that $\Omega = X$), the objective function depends on a pair of moments $x = (x_1, x_2)$, and $v(x)$ is a quadratic form: $v(x) = x\Lambda x^T$. These assumptions are maintained throughout the section.

Variants of this model received considerable attention in the literature. The case $v(x) = x_1x_2$ is equivalent to the model of Rayo and Segal (2010), who analyzed it under the assumption that $\Omega$ is a finite set. Nikandrova and Pancs (2017) studied this problem under the assumption that $\Omega$ is a strictly convex curve. These two papers mostly focus on deriving necessary conditions for optimality. Tamura (2018) considers the case where $v$ is a general quadratic form in $\mathbb{R}^N$ but imposes strong symmetry assumptions on the prior distribution. Kramkov and Xu (2022) consider a problem (inspired by the insider trading model of Rochet and Vila, 1994) that turns out to be mathematically equivalent to a generalized version of our problem where the assumption $\Omega = X$ is not imposed—their analysis is limited in its

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economic predictions since their methods are designed to handle even fairly pathological distributions of the state. Our marginal contribution is to provide a tighter characterization of optimal solutions for the well-behaved case when $\Omega$ is a compact convex set (that is, when $\Omega = X$). Relative to Rayo and Segal (2010) and Nikandrova and Pancs (2017), we show that a set of necessary conditions taken from these two papers become jointly sufficient for optimality in our case. Prior to the current draft of this paper, Malamud and Schrimpf (2021) provided an alternative (less explicit) characterization of solutions under weaker assumptions.

We first argue that the case of a general quadratic form can easily be reduced to the special case $v(x) = x_1x_2$. Indeed, for any quadratic form, there exists a basis such that the quadratic form is diagonal: $v(x) = \lambda_1x_1^2 + \lambda_2x_2^2$. If $\lambda_1, \lambda_2 \geq 0$ (respectively, $\lambda_1, \lambda_2 \leq 0$), then full disclosure (respectively, no disclosure) is optimal, as follows from Section 3.1. If $\lambda_1$ and $\lambda_2$ have opposite signs, then there exists yet another basis such that $v(x) = x_1x_2$, which we assume henceforth.

It is known from Rayo and Segal (2010) that the posterior means induced by an optimal signal must belong to a monotone set. Using duality, we can establish a stronger claim. Formally, we will say that a set $S \subset X$ is

- **monotone** if $(x_1 - y_1)(x_2 - y_2) \geq 0$, for all $x, y \in S$;
- **maximal monotone** in $X$ if it is monotone, and for each $y \in X \setminus S$, there exists $x \in S$ such that $(x_1 - y_1)(x_2 - y_2) < 0$.
- **almost-maximal monotone** in $X$ if it is monotone, compact, and, for each $y \in X \setminus S$, there exists $x \in S$ such that $(x_1 - y_1)(x_2 - y_2) \leq 0$.

Intuitively, a monotone set $S$ in $\mathbb{R}^2$ has the property that if $x \in S$, then $S$ cannot intersect the interiors of either the north-west or the south-east quadrants centered at $x$. A monotone set is maximal in $X$ if it is not a proper subset of any monotone set in $X$. A maximal monotone set must be compact (when $X$ is compact, as assumed). An almost-maximal monotone set $S$ is a compact subset of a maximal monotone set $S'$ such that $S' \setminus S$ is a collection of open line segments that are either horizontal or vertical.

**Proposition 1.** If $\pi^* \in \Pi(\mu_0)$ is optimal, then the support of moments $\text{supp}(\pi_X^*)$ induced by $\pi^*$ is an almost-maximal monotone set in $X$.

**Proof.** Suppose that $\pi^* \in \Pi(\mu_0)$ is optimal. To simplify notation, let $p^* := p_{\text{supp}(\pi_X^*)}$ as defined by (S). By Remark 1, $p^* \geq v$ and $\text{supp}(\pi_X^*) \subset S^*$, where $S^* = \{x \in X : p^*(x) =$
$v(x)$; moreover, $p^* = p_{S^*}$, and hence, since $\Omega = X$ and $v(x) = x_1x_2$, we have, for all $x \in X$,

$$p^*(x) = \max_{y \in S^*} \{x_1y_2 + x_2y_1 - y_1y_2\}.$$

We claim that the set $S^*$ is monotone: Otherwise, we would have $x, y \in S^*$ such that $(x_1 - y_1)(x_2 - y_2) < 0$, but then

$$p^*(x) \geq x_1x_2 - (x_1 - y_1)(x_2 - y_2) > x_1x_2 = v(x),$$

contradicting that $x \in S^*$. Next, we claim that the set $S^*$ is maximal monotone in $X$. Otherwise, there would exist $x \in X \setminus S^*$ such that $(x_1 - y_1)(x_2 - y_2) \geq 0$ for all $y \in S^*$, and thus

$$p^*(x) = \max_{y \in S^*} \{x_1x_2 - (x_1 - y_1)(x_2 - y_2)\} \leq x_1x_2 = v(x).$$

But then, since $p^* \geq v$, we would have that $p^*(x) = v(x)$, contradicting that $x \notin S^*$.

Since supp$(\pi_X^*) \subset S^*$, and we have shown that $S^*$ is a monotone set, supp$(\pi_X^*)$ is also a monotone set. Finally, we claim that supp$(\pi_X^*)$ is almost-maximal monotone in $X$. Otherwise, there would exist $x \in X$ such that $(y_1 - x_1)(y_2 - x_2) > 0$ for all $y \in$ supp$(\pi_X^*)$, which implies that (since supp$(\pi_X^*)$ is compact)

$$p^*(x) = \max_{y \in \text{supp}(\pi_X^*)} \{x_1x_2 - (x_1 - y_1)(x_1 - y_2)\} < x_1x_2 = v(x),$$

contradicting that $p^* \geq v$. \qed

In light of Remark 1, the proof of Proposition 1 additionally implies that the optimal price function can always be derived from some candidate support $S$ of the distribution of moments that is a maximal monotone set. A natural class of maximal monotone sets in $X$ are graphs of continuous increasing functions.

The main result of this section describes necessary and sufficient conditions for the optimality of a solution $\pi^* \in \Pi(\mu_0)$ with supp$(\pi_X^*)$ equal to the graph Gr$(f)$ of a given well-behaved function $f$. By Theorem 7, the unique solution is convex-partitional; the optimal partition divides $\Omega$ into negatively-sloped line segments; a line segments that induces the posterior mean $(t, f(t))$ has slope $-f'(t)$. These observations are formalized in the following proposition.

**Proposition 2.** Let $f : [x_1, \bar{x}_1] \to \mathbb{R}$ be a twice continuously differentiable function, with $f'(t) > 0$ for all $t \in [x_1, \bar{x}_1]$, such that its graph Gr$(f)$ is a maximal monotone subset of $X$. An optimal $\pi^* \in \Pi(\mu_0)$ induces a support of moments supp$(\pi_X^*)$ equal to Gr$(f)$ if and only if
Ω can be partitioned (up to a measure zero set) into a collection of disjoint open line segments \( \{I_t\}_{t \in [\bar{x}_1, \bar{x}_1]} \) such that\(^{20}\)

1. \( \mathbb{E}[\omega | \omega \in I_t] = (t, f(t)) \), for almost all \( t \in [\bar{x}_1, \bar{x}_1] \);\(^{21}\)

2. \( I_t = \text{relint} \left( \{ \omega \in \Omega : t \in \arg \max_{s \in [\bar{x}_1, \bar{x}_1]} \{ \omega_1 f(s) + \omega_2 s - s f(s) \} \} \right) \), for all \( t \in [\bar{x}_1, \bar{x}_1] \).

Whenever the above conditions hold, the optimal signal is convex-partitional and pools the states within each \( I_t \); moreover, \( I_t \subseteq \{ \omega \in \Omega : \omega_2 = f(t) - f'(t)(\omega_1 - t) \} \), for all \( t \in [\bar{x}_1, \bar{x}_1] \).

**Proof.** We will prove that the existence of the required partition of \( \Omega \) is sufficient for optimality of the corresponding \( \pi^* \); we relegate the more technical proof of necessity to Appendix D.1.

Suppose that there exists a collection of line segments \( \{I_t\}_{t \in [\bar{x}_1, \bar{x}_1]} \) satisfying properties 1-2. We can define \( \pi^* \in \Pi(\mu_0) \) as the convex-partitional signal that pools states in each \( I_t \) (it is irrelevant how the signal is defined for \( \omega \in \Omega \) not belonging to any \( I_t \)). By the first property, the induced posterior-mean curve \( \text{supp}(\pi^*_X) \) is equal to \( \text{Gr}(f) \). Following Section 4.2, define the price function

\[
p_{\text{Gr}(f)}(x) = \max_{y \in \text{Gr}(f)} \{ v(y) + \nabla v(y) \cdot (x - y) \} = \max_{t \in [\bar{x}_1, \bar{x}_1]} \{ x_1 f(t) + x_2 t - tf(t) \}.
\]

We will verify that condition (M) holds; optimality of \( \pi^* \) will then follow from Theorem 6. First, we argue that \( p_{\text{Gr}(f)}(x) \geq v(x) \), for all \( x \in X \). It suffices to show that there exists a \( t \in [\bar{x}_1, \bar{x}_1] \) such that \( x_1 f(t) + x_2 t - tf(t) \geq x_1 x_2 \), or, equivalently, \( (t - x_1)(f(t) - x_2) \leq 0 \). The claim is obvious when \( x \in \text{Gr}(f) \), and follows from the fact that \( \text{Gr}(f) \) is maximal monotone in \( X \) when \( x \in X \setminus \text{Gr}(f) \). To complete the proof that (M) holds, note that, by the second property, for almost all \( \omega \in I_t \),

\[
p_{\text{Gr}(f)}(\omega) = v(x(t)) + \nabla v(x(t)) \cdot (\omega - x(t)) = \omega_1 f(t) + \omega_2 t - tf(t),
\]

This shows that the equality in (M) holds for all \( (x, \omega) \in \bigcup_{t \in [\bar{x}_1, \bar{x}_1]} (t, f(t)) \times I_t \); by continuity, the equality extends to the closure of this set, which is \( \text{supp}(\pi^*) \). (Note that we have not used the differentiability assumptions on \( f \) in that part of the proof.)

Finally, the inclusion \( I_t \subseteq \{ \omega \in \Omega : \omega_2 = f(t) - f'(t)(\omega_1 - t) \} \), for \( t \in (\bar{x}_1, \bar{x}_1) \), follows from the observation that, by the second property, the first-order condition \( (\omega_1 - t)f'(t) +

---

\(^{20}\)That is, \( \Omega \setminus \left( \bigcup_{t \in [\bar{x}_1, \bar{x}_1]} I_t \right) \) has zero (Lebesgue) measure.

\(^{21}\)Since \( I_t \) has zero measure under the prior, \( \mathbb{E}[\omega | \omega \in I_t] \) is formally defined almost everywhere via the conditional expectation of \( \omega \) conditional on a \( \sigma \)-algebra generated by \( \{I_t\}_{t \in [\bar{x}_1, \bar{x}_1]} \). We provide an explicit formula for the conditional expectation in Appendix D.1.2.
\[ \omega_2 - f(t) = 0 \] must hold for all \( \omega \in I_t \).\footnote{This observation shows that it would suffice to require \( \mathbb{E}[\omega_1 | \omega \in I_t] = t \) in the first property in Proposition 1. Indeed, \( (\omega_1 - t)f'(t) + \omega_2 - f(t) = 0 \) for all \( t \in (\underline{x}_1, \bar{x}_1) \) and \( \omega \in I_t \) implies that \( (\mathbb{E}[\omega_1 | \omega \in I_t] - t)f'(t) + \mathbb{E}[\omega_2 | \omega \in I_t] - f(t) = 0 \), from which the second required equality \( \mathbb{E}[\omega_2 | \omega \in I_t] = f(t) \) follows.}

For \( t \in \{x_1, x_1\} \), the proof of the inclusion is more complicated, and thus relegated to Appendix D.1.

Proposition 2 provides a tight characterization of optimal signals under the additional regularity requirement that the induced posterior mean curve is sufficiently regular (a graph of a twice differentiable function). If an optimal signal \( \pi^* \) induces \( \text{supp}(\pi^*_X) = \text{Gr}(f) \), then it must have a simple convex-partitional structure in which only states belonging to negatively-sloped line segments \( I_t \) are pooled together. Moreover, the slopes of these line segments are uniquely pinned down by \( f \). Our proof in Appendix D.1 additionally reveals that the closures of these line segments can only intersect at the endpoints. The endpoints can be found by solving the optimization problem in the second property in Proposition 2 (we give a more explicit characterization in the proof). This direction of Proposition 2 can be used to verify the optimality of a conjectured posterior mean curve—below, we provide an illustration by deriving a necessary condition for posterior means to lie on a line.

**Corollary 3.** If for some constants \( a > 0 \) and \( b \in \mathbb{R} \), the posterior means induced by the optimal signal belong to a line segment \( \{(t, at + b) : t \in \mathbb{R} \} \cap \Omega \), then the optimal signal is to reveal the realization of \( a\omega_1 + \omega_2 \).

Indeed, Proposition 2 implies that in this case there must exist a partition of \( \Omega \) into line segments \( I_t \) that are parallel to each other, with slope \(-a\). Since the closures of these line segments do not intersect, \( \text{cl}(I_t) \) is the set of all states in \( \Omega \) belonging to the line with slope \(-a\) that crosses the posterior mean curve at \( \omega_1 = t \). Thus, the optimal signal in this case—corresponding to pooling the states in each \( I_t \)—is precisely the signal that reveals the realization of \( a\omega_1 + \omega_2 \). In other words, a necessary condition for optimality of an affine posterior mean curve is that the optimal signal reveals a particular weighted average of the two states.

The other direction of Proposition 2 provides a way to construct the optimal signal. Suppose that we partition \( \Omega \) (up to a measure-zero set) into negatively-sloped open line segments in such a way that pooling the states within these line segments induces a posterior mean curve that is a graph of some continuous function \( f \). Then, this signal is optimal as long as the second property holds. Moreover, if \( f \) is differentiable and the closures of these line segments are disjoint, then it suffices to verify that the slope of the line segment inducing posterior mean \( (t, f(t)) \) is \(-f'(t)\). Below, we provide an illustration by showing that the necessary condition identified by Corollary 3 can be sufficient.
Corollary 4. If for some constants $a > 0$ and $b \in \mathbb{R}$, $E[\omega | a\omega_1 + \omega_2 = 2at + b] = (t, at + b)$, for almost all $t$, then disclosing the realization of $a\omega_1 + \omega_2$ is an optimal signal.

Indeed, let $\pi \in \Pi(\mu_0)$ be a signal corresponding to disclosure of the realization of $a\omega_1 + \omega_2$. That signal admits a representation in terms of a partition of $\Omega$ into parallel open line segments $\{I_t\}_{t \in [\underline{x}_1, \bar{x}_1]}$, where $I_t = \text{relint} (\{\omega \in \Omega : a\omega_1 + \omega_2 = 2at + b\})$, and the range $[\underline{x}_1, \bar{x}_1]$ is defined by the property that $(t, at + b) \in \Omega$. The assumption $E[\omega | a\omega_1 + \omega_2 = 2at + b] = (t, at + b)$ then implies that the induced posterior mean curve is a line segment with slope $a$ that is a monotone maximal set in $\Omega$. Finally, the second property in Proposition 2 holds since

$$\text{relint} (\{\omega \in \Omega : t \in \arg \max_{s \in [\underline{x}_1, \bar{x}_1]} \{\omega_1 f(s) + \omega_2 s - sf(s)\}\}) = \text{relint} (\{\omega \in \Omega : a\omega_1 + \omega_2 = 2at + b\})$$

which was precisely our definition of $I_t$. Thus, Proposition 2 shows that $\pi$ is optimal.

Suppose that the distribution of $\omega$ is symmetric around the line $\omega_1 = \omega_2$. An immediate consequence of Corollary 4 (obtained by setting $a = 1$ and $b = 0$) is that disclosing $\omega_1 + \omega_2$ is then the optimal signal.

Finally, we can provide a partial converse to Corollary 4 under a regularity condition.

Corollary 5. Fix $a > 0$ and suppose that $f(s) := E[\omega | a\omega_1 + \omega_2 = s]$ is twice continuously differentiable with $f' > 0$. If disclosing the realization of $a\omega_1 + \omega_2$ is an optimal signal, then there must exist $b \in \mathbb{R}$ such that $E[\omega | a\omega_1 + \omega_2 = 2at + b] = (t, at + b)$, for almost all $t$.

Indeed, if disclosing $a\omega_1 + \omega_2$ is optimal, then the open line segments $I_t$ partitioning $\Omega$ (whose existence is guaranteed by Proposition 2 under the regularity assumption) must be parallel and have slope $-a$. But then, we must have that $\omega_2 = f(t) - f'(t)(\omega_1 - t)$ if and only if $\omega_2 = 2at + b - a\omega_1$, for some $b$, which is only possible when $f(t) = at + b$.

References


Appendix

A Proofs for Section 3

We will prove the results in Section 3 in a different order than they appear in Section 3. We start with a lemma.

Lemma 2. Let \( V : \Delta(\Omega) \to \mathbb{R} \cup \{-\infty\} \) be bounded from above and upper semi-continuous. Then (P) has an optimal solution.

Proof. Because the map \( \tau \to \int_{\Delta(\Omega)} \mu d\tau(\mu) \) is continuous, the feasible set \( \mathcal{T}(\mu_0) \) is compact, being a closed subset of the compact set \( \Delta(\Delta(\Omega)) \). Moreover, \( \mathcal{T}(\mu_0) \) is non-empty, as it contains the Dirac measure \( \delta_{\mu_0} \) at \( \mu_0 \), which corresponds to no disclosure. Since \( V \) is bounded from above and upper semi-continuous, the function \( \tau \to \int V(\mu)d\tau(\mu) \) is also upper semi-continuous and thus attains its maximum on the compact set \( \mathcal{T}(\mu_0) \), by Weierstrass’s theorem. This standard argument shows that an optimal solution \( \tau^* \) to the problem (P) exists.

Next, we prove Theorem 4 (we will need it in the proof of Theorem 2).
A.1 Proof of Theorem 4

We start with a key lemma.

**Lemma 3.** For any $\mu_0 \in \Delta(\Omega)$ and any finitely-supported $\eta_0 \in \Delta(\Omega)$, there exists $\tau \in T(\mu_0)$ that attains the concave closure $\hat{V}(\mu_0)$, and $\{\eta_\mu\}_{\mu \in \text{supp}(\tau)} \subset \Delta(\Omega)$ such that

$$
\int_{\Delta(\Omega)} \eta_\mu d\tau(\mu) = \eta_0 \quad \text{and} \quad \int_{\Delta(\Omega)} \|\mu - \eta_\mu\|_{KR} d\tau(\mu) = \|\mu_0 - \eta_0\|_{KR}.
$$

(A.1)

Before proving Lemma 3, we show that it implies Theorem 4. Since $V$ is assumed to be Lipschitz, there exists $L \in \mathbb{R}$, such that, for all $\mu_0, \eta_0 \in \Delta(\Omega)$,

$$
|V(\mu_0) - V(\eta_0)| \leq L \|\mu_0 - \eta_0\|_{KR}.
$$

If $\eta_0$ is finitely supported, then by Lemma 3, we have that

$$
\hat{V}(\mu_0) - \hat{V}(\eta_0) \leq \int_{\Delta(\Omega)} V(\mu) d\tau(\mu) - \int_{\Delta(\Omega)} V(\eta_\mu) d\tau(\mu)
\leq \int_{\Delta(\Omega)} L \|\mu - \eta_\mu\|_{KR} d\tau(\mu) = L \|\mu_0 - \eta_0\|_{KR}.
$$

Because $\Omega$ is a compact metric space, by Theorem 6.18 in Villani (2009), for each $\zeta_0 \in \Delta(\Omega)$ and $\varepsilon > 0$, there exists a finitely-supported $\eta_0 \in \Delta(\Omega)$ such that $\|\zeta_0 - \eta_0\|_{KR} \leq \varepsilon$; but then

$$
|\hat{V}(\mu_0) - \hat{V}(\zeta_0)| \leq |\hat{V}(\mu_0) - \hat{V}(\eta_0)| + |\hat{V}(\zeta_0) - \hat{V}(\eta_0)|
\leq L \|\mu_0 - \eta_0\|_{KR} + L \|\zeta_0 - \eta_0\|_{KR}
= L \|\mu_0 - \zeta_0 + \zeta_0 - \eta_0\|_{KR} + L \|\zeta_0 - \eta_0\|_{KR}
\leq L \|\mu_0 - \zeta_0\|_{KR} + 2L \|\zeta_0 - \eta_0\|_{KR}
\leq L \|\mu_0 - \zeta_0\|_{KR} + 2L \varepsilon,
$$

where the second inequality holds because we can replace $\mu_0$ with $\zeta_0$ in preceding arguments. Because $\varepsilon$ was arbitrary, we must have that for any $\mu_0, \zeta_0 \in \Delta(\Omega)$,

$$
|\hat{V}(\mu_0) - \hat{V}(\zeta_0)| \leq L \|\mu_0 - \zeta_0\|_{KR}.
$$

By reversing the roles of $\mu_0$ and $\zeta_0$, we conclude that $\hat{V}$ is Lipschitz (with constant $L$). Thus, it remains to prove Lemma 3.

*Proof of Lemma 3:* First, since $V$ is Lipschitz, it is upper semi-continuous, and hence, by Lemma 2, for any $\mu_0 \in \Delta(\Omega)$, there exists $\tau \in T(\mu_0)$ that attains the concave closure of
\( V \) at \( \mu_0 \),

\[
\hat{V}(\mu_0) = \int_{\Delta(\Omega)} V(\mu) d\tau(\mu).
\]

Fix \( \eta_0 \in \Delta(\Omega) \) with \( \mu_0 \neq \eta_0 \) and \( |\text{supp}(\eta_0)| < \infty \). The idea behind our proof below is to “perturb” each posterior belief \( \mu \in \text{supp}(\tau) \) in such a way that perturbed posteriors average out to \( \eta_0 \), and the average distance between each posterior \( \mu \) and its perturbation is exactly the same as the distance between the “priors” \( \mu_0 \) and \( \eta_0 \). A naive argument would be to perturb each posterior \( \mu \) by the same magnitude and in the same direction \( \eta_0 - \mu_0 \); such an argument would work if \( \Omega \) were finite and each \( \mu \) were interior. However, in general, there is no way to guarantee that such perturbed posteriors are actually probability measures. Thus, our actual construction is more complicated; the idea is to decompose the difference \( \eta_0 - \mu_0 \) into its positive- and negative-measure part to better control the perturbation.

Denote \( \nu_0 = \eta_0 - \mu_0 \). Consider the Hahn decomposition of \( \nu_0 \) into two nonnegative measures \( \nu_0^+ \) and \( \nu_0^- \) which are singular to each other (see Chapter 3.1 in Bogachev, 2007). Specifically, denote

\[
\Omega^+ = \{ \omega \in \text{supp}(\eta_0) : \eta_0(\omega) \geq \mu_0(\omega) \}, \\
\Omega^- = \{ \omega \in \text{supp}(\eta_0) : \eta_0(\omega) < \mu_0(\omega) \}, \\
\Omega^c = \Omega \setminus (\Omega^+ \cup \Omega^-).
\]

Then \( \nu_0 = \nu_0^+ - \nu_0^- \) such that \( \nu_0^+, \nu_0^- \in M^+(\Omega) \), \( \nu_0^+(\Omega) = \nu_0^+(\Omega^+) \), and \( \nu_0^-(\Omega) = \nu_0^- (\Omega \setminus \Omega^+) \). Note that, by our assumption that \( \eta_0 \) has finite support, the sets \( \Omega^+ \) and \( \Omega^- \) are finite, and the measure \( \nu_0^+ \) also has finite support.

Since we have to keep track of the distance between the measures \( \mu_0 \) and \( \eta_0 \) when defining the “perturbed” posteriors, it will be convenient to represent the norm of \( \nu_0 \) when defining an optimal transport problem. As follows from Theorems 3.1, 4.4, 4.5, and Corollary 7.4 in Edwards (2011),

\[
\|\nu_0\|_{KR} = \max_{p \in L^1(\Omega)} \int_\Omega p(\omega)d\nu_0(\omega) = \min_{\pi \in \Pi(\nu_0^+, \nu_0^-)} \int_{\Omega \times \Omega} \rho(\omega^+, \omega^-)d\pi(\omega^+, \omega^-) \quad (A.2)
\]

where \( \Pi(\nu_0^+, \nu_0^-) \) is the set of non-negative measures \( \pi \in M^+(\Omega \times \Omega) \) such that, for each measurable set \( B \subset \Omega \), we have \( \pi(B \times \Omega) = \nu_0^+(B) \) and \( \pi(\Omega \times B) = \nu_0^-(B) \). That is, the Kantorovich-Rubinstein norm of the (signed) measure \( \nu_0 \) with \( \nu_0(\Omega) = 0 \) is the value of the optimal transport problem of transferring the positive part of \( \nu_0 \) to the negative part of \( \nu_0 \), where the cost function is given by the metric on the state space. Note, in particular, that the supremum and infimum are shown to be attained, hence they are replaced by the
maximum and minimum. Moreover, by Theorem 8.1 in Edwards (2011), \( p \in L_1(\Omega) \) and \( \pi \in \Pi(\nu^+_0, \nu^-_0) \) are optimal solutions if and only if

\[
p(\omega^+) - p(\omega^-) = \rho(\omega^+, \omega^-) \quad \text{for all } (\omega^+, \omega^-) \in \text{supp}(\pi).
\]

Henceforth, we fix some optimal \( p \) and \( \pi \) (the function \( p \) will be used in the last step of the proof).

Since \( \Omega \) is a compact metric space, the disintegration theorem implies that there exists a measurable map \( \omega^- \mapsto \pi(\cdot|\omega^-) \), from \( \Omega \) into \( \Delta(\Omega^+) \), uniquely determined for \( \nu^-_0 \)-almost all \( \omega^- \in \Omega \), such that for each \( \omega^+ \in \Omega^+ \) and each measurable set \( B \subset \Omega \),

\[
\pi(\omega^+, B) = \int_B \pi(\omega^+|\omega^-) d\nu^-_0(\omega^-).
\]

Intuitively, \( \pi(\omega^+|\omega^-) \) is the conditional probability of the state \( \omega^+ \) conditional on each realization \( \omega^- \), under the joint distribution \( \pi \).

For each \( \mu \in \text{supp}(\tau) \), we (uniquely) define \( \nu^-_\mu, \nu^+_\mu \in M^+(\Omega) \) so that \( \nu^-_\mu(\Omega) = \nu^-_\mu(\Omega \setminus \Omega^+) \), \( \nu^+_\mu(\Omega) = \nu^+_\mu(\Omega^+) \), and

\[
\nu^-_\mu(\omega^-) = \frac{\mu_0(\omega^-) - \eta_0(\omega^-)}{\mu_0(\omega^-)} \mu(\omega^-), \quad \text{for } \omega^- \in \Omega^-, \\
\nu^+_\mu(\omega^+) = \sum_{\omega^- \in \Omega^-} \pi(\omega^+|\omega^-) \frac{\mu_0(\omega^-) - \eta_0(\omega^-)}{\mu_0(\omega^-)} \mu(\omega^-) + \int_{\Omega^c} \pi(\omega^+|\omega^-) d\mu(\omega^-), \quad \text{for } \omega^+ \in \Omega^+.
\]

Note that these measures are well defined because, by definition, \( \Omega^- \) can only contain states \( \omega^- \) that have strictly positive probability under \( \mu_0 \) (in particular, \( \Omega^- \) is empty when \( \mu_0 \) is an atomless distribution).

Finally, let \( \nu_\mu = \nu^+_\mu - \nu^-_\mu \) and \( \eta_\mu = \mu + \nu_\mu \). Intuitively, \( \nu_\mu \) is exactly the “perturbation” applied to posterior \( \mu \).

The proof of Lemma 3 (and hence Theorem 4) follows from the following two lemmas. The first lemma makes sure that the perturbed posteriors are well-defined probability measures that average out to the “prior” \( \eta_0 \).

**Lemma 4.** For each \( \mu \in \text{supp}(\tau) \), we have \( \eta_\mu \in \Delta(\Omega) \). Moreover, \( \int_{\Delta(\Omega)} \eta_\mu d\tau(\mu) = \eta_0 \).

**Proof.** By definition, \( \nu^+_\mu \) is concentrated on \( \Omega^+ \) and \( \nu^-_\mu \) is concentrated on \( \Omega \setminus \Omega^+ \), so \( \nu_\mu = \nu^+_\mu - \nu^-_\mu \) is the Hahn decomposition. First, we make sure that \( \eta_\mu \) is a non-negative measure.
For each $\omega^- \in \Omega^-$,
\[
\eta_\mu(\omega^-) = \mu(\omega^-) - \frac{\mu_0(\omega^-) - \eta_0(\omega^-)}{\mu_0(\omega^-)}\mu(\omega^-) = \frac{\eta_0(\omega^-)}{\mu_0(\omega^-)}\mu(\omega^-) \geq 0;
\]
for each measurable set $B \subset \Omega^c$,
\[
\eta_\mu(B) = \mu(B) - \mu(B) = 0;
\]
and for each $\omega^+ \in \Omega^+$,
\[
\eta_\mu(\omega^+) = \mu(\omega^+) + \nu_\mu^+(\omega^+) \geq 0.
\]
Second, we make sure that the measure of the whole space under $\eta_\mu$ is 1:
\[
\eta_\mu(\Omega) = \sum_{\omega^- \in \Omega^-} \frac{\eta_0(\omega^-)}{\mu_0(\omega^-)}\mu(\omega^-) + \sum_{\omega^+ \in \Omega^+} \pi(\omega^+|\omega^-)\frac{\mu_0(\omega^-) - \eta_0(\omega^-)}{\mu_0(\omega^-)}\mu(\omega^-) + \int_{\Omega^c} \pi(\omega^+|\omega^-)d\mu(\omega^-)
\]
\[
= \sum_{\omega^- \in \Omega^-} \frac{\eta_0(\omega^-)}{\mu_0(\omega^-)}\mu(\omega^-) + \sum_{\omega^+ \in \Omega^+} \mu(\omega^+) + \sum_{\omega^- \in \Omega^-} \frac{\mu_0(\omega^-) - \eta_0(\omega^-)}{\mu_0(\omega^-)}\mu(\omega^-) + \int_{\Omega^c} d\mu(\omega^-)
\]
\[
= \mu(\Omega^+) + \mu(\Omega^-) + \mu(\Omega^c) = 1.
\]
Finally, we verify that $\eta_\mu$ average out to $\eta_0$ under $\tau$. Taking into account $\int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0$, we have, for each $\omega^- \in \Omega^-$,
\[
\int_{\Delta(\Omega)} \eta_\mu(\omega^-)d\tau(\mu) = \int_{\Delta(\Omega)} \frac{\eta_0(\omega^-)}{\mu_0(\omega^-)}\mu(\omega^-)d\tau(\mu) = \eta_0(\omega^-),
\]
and, for each $\omega^+ \in \Omega^+$,
\[
\int_{\Delta(\Omega)} \eta_\mu(\omega^+)d\tau(\mu)
\]
\[
= \int_{\Delta(\Omega)} \left(\mu(\omega^+) + \sum_{\omega^- \in \Omega^-} \pi(\omega^+|\omega^-)\frac{\mu_0(\omega^-) - \eta_0(\omega^-)}{\mu_0(\omega^-)}\mu(\omega^-) + \int_{\Omega^c} \pi(\omega^+|\omega^-)d\mu(\omega^-)\right) d\tau(\mu)
\]
\[
= \mu_0(\omega^+) + \sum_{\omega^- \in \Omega^-} \pi(\omega^+|\omega^-)(\mu_0(\omega^-) - \eta_0(\omega^-)) + \int_{\Omega^c} \pi(\omega^+|\omega^-)d\mu_0(\omega^-)
\]
\[
= \mu_0(\omega^+) + \int_{\Omega^c \Omega^+} \pi(\omega^+|\omega^-)d\nu_0(\omega^-) = \mu_0(\omega^+) + \pi(\omega^+, \Omega) = \mu_0(\omega^+) + \nu_\mu^+(\omega^+) = \eta_0(\omega^+),
\]
showing that $\int_{\Delta(\Omega)} \eta_\mu d\tau(\mu) = \eta_0$.  

\[\square\]
In the last step of the proof, we ensure that the average distance between \( \mu \) and its “perturbed” version \( \eta \) is equal to the distance between the “priors” \( \mu_0 \) and \( \eta_0 \).

**Lemma 5.** We have \( \int_{\Delta(\Omega)} \|\nu\|_{KR} d\tau(\mu) = \|\nu_0\|_{KR} \).

**Proof.** For any \( \mu \in \text{supp}(\tau) \), let \( \pi_\mu \in M^+(\Omega \times \Omega) \) be such that \( \text{supp}(\pi_\mu(\cdot, \Omega)) \subset \Omega^+ \), and for each \( \omega^+ \in \Omega^+ \) and each measurable set \( B \subset \Omega \),

\[
\pi_\mu(\omega^+, B) = \sum_{\omega^- \in \Omega^+ \cap B} \pi(\omega^+ | \omega^-) \frac{\mu_0(\omega^-) - \eta_0(\omega^-)}{\mu(\omega^-)} \mu(\omega^-) + \int_{\Omega^+ \cap B} \pi(\omega^+ | \omega^-) d\mu(\omega^-).
\]

Notice that \( \pi_\mu \in \Pi(\nu^+, \nu^-) \), that is, \( \pi_\mu \) is a plan transporting \( \nu^+ \) to \( \nu^- \). Indeed, for each \( \omega^+ \in \Omega^+ \) and each measurable set \( B \subset \Omega \), we clearly have \( \pi(\omega^+, \Omega) = \nu^+(\omega^+) \) and \( \pi(\Omega, B) = \nu^-(B) \). Moreover,

\[
\pi_\mu \in \arg \min_{\bar{\pi} \in \Pi(\nu^+, \nu^-)} \int_{\Omega^+ \times \Omega} \rho(\omega^+, \omega^-) d\bar{\pi}(\omega^+, \omega^-)
\]

\[
p \in \arg \max_{q \in L_1(\Omega)} \int_{\Omega} q(\omega) d\nu(\omega),
\]

because \( \text{supp}(\pi_\mu) \subset \text{supp}(\pi) \) by definition of \( \pi_\mu \), and thus

\[
p(\omega^+) - p(\omega^-) = \rho(\omega^+, \omega^-) \quad \text{for all } (\omega^+, \omega^-) \in \text{supp}(\pi_\mu)
\]

which—as observed above—is equivalent to optimality of \( \pi_\mu \) and \( p \). It follows that (recall equation (A.2))

\[
\|\nu_\mu\|_{KR} = \int_{\Omega^+ \times \Omega} \rho(\omega^+, \omega^-) d\pi_\mu(\omega^+, \omega^-).
\]

Therefore,

\[
\int_{\Delta(\Omega)} \|\nu\|_{KR} d\tau(\mu) = \int_{\Delta(\Omega)} \int_{\Omega^+ \times \Omega} \rho(\omega^+, \omega^-) d\pi_\mu(\omega^+, \omega^-) d\tau(\mu) = \int_{\Omega^+ \times \Omega} \rho(\omega^+, \omega^-) d\pi(\omega^+, \omega^-) = \|\nu_0\|_{KR},
\]

where the second equality holds because \( \int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0 \) implies \( \int_{\Delta(\Omega)} \pi_\mu d\tau(\mu) = \pi \).

**A.2 Proof of Theorem 2**

Existence of an optimal solution to the primal problem follows from Lemma 2.

To prove the rest of the theorem, we introduce some basic tools from convex analysis,
used in the proof of the next lemma.\textsuperscript{23} Let $E$ be a normed vector space and $E^*$ its topological dual space, that is, the space of all continuous linear functions on $E$. Let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be an extended-valued function that is not identically $\{+\infty\}$. The Legendre transform of $\varphi$ is the function $\varphi^* : E^* \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi^*(z^*) = \sup_{z \in E} \{\langle z^*, z \rangle - \varphi(z)\} \quad \text{for all } z^* \in E^*,$$

where $\langle \cdot, \cdot \rangle$ is the duality product between $E$ and $E^*$. It is easy to verify that $\varphi^*$ is convex, lower semi-continuous, and not identically $\{+\infty\}$. Moreover, $\varphi$ and $\varphi^*$ satisfy the so-called Young’s inequality,

$$\langle z^*, z \rangle \leq \varphi(z) + \varphi^*(z^*) \quad \text{for all } z \in E \text{ and } z^* \in E^*.$$

Next, define the function $\varphi^{**} : E \to \mathbb{R} \cup \{+\infty\}$ as the Legendre transform of $\varphi^*$, restricted from $E^{**}$ to $E$,

$$\varphi^{**}(z) = \sup_{z^* \in E^*} \{\langle z^*, z \rangle - \varphi^*(z^*)\} \quad \text{for all } z \in E.$$

Clearly, $\varphi^{**}$ is a convex and lower semi-continuous function satisfying $\varphi^{**}(z) \leq \varphi(z)$ for all $z \in E$. Fenchel-Moreau’s Theorem states that if $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuous, and not identically $\{+\infty\}$, then $\varphi^{**} = \varphi$. We remark that the Fenchel-Moreau’s theorem is a consequence of an appropriate hyperplane separation theorem.\textsuperscript{24}

We prove the theorem in two steps. First, we show the conclusion for Lipschitz objective functions. Here, we rely on the (already proven) Theorem 4. Second, we use an approximation argument to extend the conclusion to all bounded and upper semi-continuous objectives.

**Lemma 6.** Let $V \in L(\Delta(\Omega))$. Then (O) holds.

**Proof.** Let $E = (M(\Omega), \|\cdot\|_{KR})$; then, as argued in the main text, $E^* = L(\Omega)$. Define the function $\varphi$ on $M(\Omega)$ as

$$\varphi(\eta) = \begin{cases} -\sup_{\tau \in \mathcal{T}(\eta)} \int_{\Delta(\Omega)} V(\mu) d\tau(\mu), & \eta \in \Delta(\Omega), \\ +\infty, & \eta \notin \Delta(\Omega). \end{cases}$$

First, we note that $\varphi$ is convex. Indeed, let $\eta_1, \eta_2 \in M(\Omega)$ and $\lambda \in (0, 1)$. If $\eta_1, \eta_2 \in \Delta(\Omega)$,

\textsuperscript{23}See Chapter 1.4 in Brezis (2011) for further details.

\textsuperscript{24}Indeed, an earlier version of our paper contained a proof of strong duality that directly relied on a hyperplane separation theorem.
then, by Lemma 2, there exist $\tau_1 \in \mathcal{T}(\eta_1)$ and $\tau_2 \in \mathcal{T}(\eta_2)$ such that

$$
\varphi(\eta_1) = -\int_{\Delta(\Omega)} V(\mu) d\tau_1(\mu) \in \mathbb{R} \quad \text{and} \quad \varphi(\eta_2) = -\int_{\Delta(\Omega)} V(\mu) d\tau_2(\mu) \in \mathbb{R}.
$$

By definition of $\mathcal{T}$,

$$
\lambda \tau_1 + (1 - \lambda) \tau_2 \in \mathcal{T}(\lambda \eta_1 + (1 - \lambda) \eta_2)
$$

and hence, by definition of $\varphi$,

$$
\varphi(\lambda \eta_1 + (1 - \lambda) \eta_2) \leq -\int_{\Delta(\Omega)} V(\mu) d(\lambda \tau_1 + (1 - \lambda) \tau_2)
$$

$$
= \lambda \varphi(\eta_1) + (1 - \lambda) \varphi(\eta_2).
$$

If $\eta_1 \notin \Delta(\Omega)$ or $\eta_2 \notin \Delta(\Omega)$, then, trivially,

$$
\varphi(\lambda \eta_1 + (1 - \lambda) \eta_2) \leq \lambda \varphi(\eta_1) + (1 - \lambda) \varphi(\eta_2) = +\infty.
$$

Second, we note that $\varphi : M(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous, because $\varphi$ is Lipschitz on the compact set $\Delta(\Omega)$, by Theorem 4.

Let us compute the Legendre transform of $\varphi$. For each $q \in L(\Omega),$

$$
\varphi^*(q) = \sup_{\eta \in M(\Omega)} \left\{ \int_{\Omega} q(\omega) d\eta(\omega) - \varphi(\eta) \right\}
$$

$$
= \sup_{\eta \in \Delta(\Omega), \tau \in \mathcal{T}(\eta)} \left\{ \int_{\Omega} q(\omega) d\eta(\omega) + \int_{\Delta(\Omega)} V(\mu) d\tau(\mu) \right\}
$$

$$
= \sup_{\eta \in \Delta(\Omega), \tau \in \mathcal{T}(\eta)} \left\{ \int_{\Delta(\Omega)} \left( \int_{\Omega} q(\omega) d\mu(\omega) + V(\mu) \right) d\tau(\mu) \right\}
$$

$$
= \sup_{\eta \in \Delta(\Omega)} \left\{ \int_{\Omega} q(\omega) d\eta(\omega) + V(\eta) \right\},
$$

where the last equality follows from the fact that by treating $\tilde{V}(\mu) := \int_{\Omega} q(\omega) d\mu(\omega) + V(\mu)$ as an objective function, we obtain a persuasion problem in which we choose both a prior $\eta$ and a distribution $\tau$ of posteriors, which averages out to the prior, so it is optimal to choose a prior $\eta \in \operatorname{arg\,max}_{\mu \in \Delta(\Omega)} V(\mu)$ and a degenerate distribution $\tau = \delta_{\eta}.^{25}$

---

^{25}This observation is also made in the proof of Theorem 2 in Dworczak (2020).
Let us, finally, compute $\varphi^{**}(\mu_0)$.

\[
\varphi^{**}(\mu_0) = \sup_{q \in L(\Omega)} \left\{ \int_{\Omega} q(\omega) d\mu_0(\omega) - \varphi(q) \right\}
= \sup_{q \in L(\Omega)} \left\{ \int_{\Omega} q(\omega) d\mu_0(\omega) \right\} - \sup_{\eta \in \Delta(\Omega)} \left\{ \int_{\Omega} \eta(\omega) (q(\omega) + V(\eta)) \right\}
= -\inf_{p \in L(\Omega)} \left\{ \int_{\Omega} p(\omega) d\mu_0(\omega) + \sup_{\eta \in \Delta(\Omega)} \left\{ V(\eta) - \int_{\Omega} p(\omega) d\eta(\omega) \right\} \right\}
= -\inf_{p \in P(V)} \left\{ \int_{\Omega} p(\omega) d\mu_0(\omega) : \sup_{\eta \in \Delta(\Omega)} \left\{ V(\eta) - \int_{\Omega} p(\omega) d\eta(\omega) \right\} = 0 \right\},
\]

where the third equality follows from substituting $q = -p$, and the fourth follows because, for any fixed $\eta$, adding a constant to $p$ does not change the value of the outer infimum—it is thus without loss of generality to normalize $p$ by insisting that the inner supremum is equal to 0 (note that the inner supremum is attained and finite at each $p \in L(\Omega)$). Fenchel-Moreau’s theorem implies that $\varphi = \varphi^{**}$, so (O) follows from $\varphi(\mu_0) = \varphi^{**}(\mu_0)$.

**Lemma 7.** Let $V$ be bounded and upper semi-continuous. Then (O) holds.

**Proof.** This follows from a standard approximation argument, as, for example, in the proof of Theorem 1.3 in Villani (2003). By Baire’s theorem (see, for example, Box 1.5 in Santambrogio, 2015), there exists a non-increasing sequence of Lipschitz functions $V_k \in L(\Delta(\Omega))$ converging pointwise to $V$. That is, $V_k(\mu) \geq V_{k+1}(\mu)$ for all $\mu \in \Delta(\Omega)$ and $k \in \mathbb{N}$, and $\lim_{k \to \infty} V_k(\mu) = V(\mu)$ for all $\mu \in \Delta(\Omega)$. Let $\tau_k^*$ denote an optimal solution to (P) with the objective function $V_k$. For each $k \in \mathbb{N}$, we have

\[
\int_{\Delta(\Omega)} V(\mu) d\tau^*(\mu) \leq \inf_{p \in P(V)} \int_{\Omega} p(\omega) d\mu_0(\omega) \leq \inf_{p \in P(V_k)} \int_{\Omega} p(\omega) d\mu_0(\omega) = \int_{\Delta(\Omega)} V_k(\mu) d\tau_k^*(\mu),
\]

where the first inequality holds by Theorem 1, the second inequality holds by $P(V_k) \subset P(V)$ for $V_k \geq V$, and the equality holds by Lemma 6 for Lipschitz $V_k$. To establish (O) for upper semi-continuous $V$, it is thus sufficient to show that

\[
\lim_{k \to \infty} \int_{\Delta(\Omega)} V_k(\mu) d\tau_k^*(\mu) \leq \int_{\Delta(\Omega)} V(\mu) d\tau^*(\mu).
\]

Thanks to compactness of $\mathcal{T}(\mu_0)$, up to extraction of a subsequence, we can suppose that $\tau_k^*$
converges weakly to some \( \tau \in \mathcal{T}(\mu_0) \). Then for each \( j \in \mathbb{N} \), we have

\[
\lim_{k \to \infty} \int_{\Delta(\Omega)} V_k(\mu) d\tau_k^*(\mu) \leq \lim_{k \to \infty} \int_{\Delta(\Omega)} V_j(\mu) d\tau_k^*(\mu) = \int_{\Delta(\Omega)} V_j(\mu) d\tau(\mu),
\]

where the first inequality holds because \( V_k \leq V_j \) for \( k \geq j \), and the equality holds because \( V_j \) is (Lipschitz) continuous and \( \tau_k^* \to \tau \). Then letting \( j \) go to infinity and invoking the monotone convergence theorem,

\[
\lim_{j \to \infty} \int_{\Delta(\Omega)} V_j(\mu) d\tau(\mu) = \int_{\Delta(\Omega)} V(\mu) d\tau(\mu),
\]

we obtain

\[
\lim_{k \to \infty} \int_{\Delta(\Omega)} V_k(\mu) d\tau_k^*(\mu) \leq \int_{\Delta(\Omega)} V(\mu) d\tau(\mu) \leq \int_{\Delta(\Omega)} V(\mu) d\tau^*(\mu),
\]

where the last inequality holds because \( \tau^* \) is an optimal solution to (P). This establishes (O) for upper semi-continuous \( V \). As a by-product, it also shows the optimality of \( \bar{\tau} \).

\[\text{A.3 Proof of Corollary 1}\]

By Theorem 2, \( \tau \in \mathcal{T}(\mu_0) \) and \( p \in \mathcal{P}(V) \) are optimal solutions to (P) and (D) if and only if

\[
\int_{\Delta(\Omega)} V(\mu) d\tau(\mu) = \int_{\Omega} p(\omega) d\mu_0(\omega) \iff \int_{\Delta(\Omega)} \left( V(\mu) - \int_{\Omega} p(\omega) d\mu(\omega) \right) d\tau(\mu) = 0.
\]

Since the term in parenthesis is non-positive for \( p \in \mathcal{P}(V) \), it follows that \( \tau(\Gamma) = 1 \) where

\[
\Gamma = \left\{ \mu \in \Delta(\Omega) : V(\mu) = \int_{\Omega} p(\omega) d\mu(\omega) \right\} = \left\{ \mu \in \Delta(\Omega) : V(\mu) \geq \int_{\Omega} p(\omega) d\mu(\omega) \right\}.
\]

The set \( \Gamma \) is closed because \( V(\mu) \) is upper semi-continuous in \( \mu \) and \( \int_{\Omega} p(\omega) d\mu(\omega) \) is continuous in \( \mu \), given that each \( p \in \mathcal{P}(V) \) is continuous. Thus, \( \text{supp}(\tau) \subset \Gamma \) and (C) follows, since \( \text{supp}(\tau) \) is defined as the smallest closed set on which \( \tau \) is concentrated.

\[\text{A.4 Proof of Theorem 3}\]

By Duality Theorem in Gale (1967), \( V \) is superdifferentiable at \( \mu_0 \) if and only if \( V \) has bounded steepness at \( \mu_0 \). Thus, Theorem 3 follows from the following lemma.

\[\text{26In the persuasion literature, a similar argument appears in the proof of Theorem 1 in Dizdar and Kováč (2020) for the special case of linear persuasion.}\]
Lemma 8. There exists an optimal solution \( p \in \mathcal{P}(V) \) to (D) if and only if \( V \) is superdifferentiable at \( \mu_0 \).

Proof. If \( \hat{V} \) is superdifferentiable at \( \mu_0 \), then by using once more the fact that \((M(\Omega), \|\cdot\|_{KR})^* = L(\Omega)\), we conclude that there exists \( p \in L(\Omega) \) such that
\[
\hat{V}(\mu_0) = \int_{\Omega} p(\omega) d\mu_0(\omega) \quad \text{and} \quad \hat{V}(\mu) \leq \int_{\Omega} p(\omega) d\mu(\omega), \quad \text{for all } \mu \in \Delta(\Omega).
\]
Thus,
\[
V(\mu) \leq \hat{V}(\mu) \leq \int_{\Omega} p(\omega) d\mu(\omega), \quad \text{for all } \mu \in \Delta(\Omega),
\]
so \( p \in \mathcal{P}(V) \) is an optimal solution to (D), by Theorem 1.

Conversely, if \( p \in \mathcal{P}(V) \) is optimal, then by Theorem 1 we have \( p \in L(\Omega) \),
\[
\hat{V}(\mu_0) = \int_{\Omega} p(\omega) d\mu_0(\omega), \quad \text{and} \quad V(\mu) \leq \int_{\Omega} p(\omega) d\mu(\omega), \quad \text{for all } \mu \in \Delta(\Omega),
\]
so
\[
\hat{V}(\mu) \leq \int_{\Omega} p(\omega) d\mu(\omega), \quad \text{for all } \mu \in \Delta(\Omega),
\]
by definition of \( \hat{V} \) as the concave envelope. Thus, \( \hat{V} \) has a supergradient \( p \) at \( \mu_0 \). This of course means that \( \hat{V} \) is superdifferentiable at \( \mu_0 \) (simply define \( H(\mu) = \int_{\Omega} p(\omega) d\mu(\omega) \) which is a continuous linear function on \( M(\Omega) \) since \( p \in L(\Omega) \)). \( \square \)

B Proofs for Section 4

B.1 Proof of Lemma 1

Suppose that \( v \) is \( L \)-Lipschitz on \( X \subset \mathbb{R}^N \). Since all norms are equivalent in an \( N \)-dimensional Euclidean space, without loss of generality, we endow \( \mathbb{R}^N \) with the Euclidean norm,
\[
\|x\| = \sqrt{\sum_{i=1}^{N} x_i^2}, \quad \text{for all } x \in \mathbb{R}^N.
\]
For any $\mu, \eta \in \Delta(\Omega)$, with $\mu \neq \eta$,
\[
\frac{|V(\mu) - V(\eta)|}{\|\mu - \eta\|_{KR}} = \frac{|v(E_\mu[\omega]) - v(E_\eta[\omega])|}{\|\mu - \eta\|_{KR}} = \frac{|v(E_\mu[\omega]) - v(E_\eta[\omega])|}{\|E_\mu[\omega] - E_\eta[\omega]\|} \frac{\|E_\mu[\omega] - E_\eta[\omega]\|}{\|\mu - \eta\|_{KR}} \leq L \frac{\|E_\mu[\omega] - E_\eta[\omega]\|}{\|\mu - \eta\|_{KR}}.
\]
Because the function $f(\omega) = \omega_i$ is 1-Lipschitz,
\[
|E_\mu[\omega_i] - E_\eta[\omega_i]| = \left| \int_\Omega \omega_i d(\mu - \eta)(\omega) \right| \leq \|\mu - \eta\|_{KR},
\]
and thus
\[
\|E_\mu[\omega] - E_\eta[\omega]\| = \sqrt{\sum_{i=1}^N (E_\mu[\omega_i] - E_\eta[\omega_i])^2} \leq \sqrt{N} \|\mu - \eta\|_{KR},
\]
showing that $V$ is $L\sqrt{N}$-Lipschitz.

### B.2 Proof of Theorem 5

By Lemma 1, we know that $V : \Delta(\Omega) \rightarrow \mathbb{R}$ is Lipschitz, since $v$ is Lipschitz. It follows from Theorems 2, 3, and 4 that there exists a solution $p \in L(\Omega)$ to the dual problem (D); moreover, since $(P_M)$ is a special case of the general problem $(P)$, $\pi \in \Pi(\mu_0)$ is then optimal for $(P_M)$ if and only if
\[
\int_X v(x) d\pi(x) = \int_\Omega p(\omega) d\mu_0(\omega).
\]
Let $\hat{p}$ be the convex roof extension of $p$ from $\Omega$ to $X$, as in the main text. By construction, $\hat{p} \leq p$ on $\Omega$. Moreover, the infimum in the definition of $\hat{p}$ is attained because $p$ is (Lipschitz) continuous on $\Omega$ and the set of feasible distributions is compact. Hence, for any $x \in X$, we can write $\hat{p}(x) = \int_\Omega p(\omega) d\mu_x(\omega)$ for some $\mu_x \in \Delta(\Omega)$ with $\int_\Omega \omega d\mu_x(\omega) = x$. By definition of $\hat{p}$, for any $x, y \in X$, $\lambda \in (0, 1)$, we have
\[
\lambda \hat{p}(x) + (1 - \lambda) \hat{p}(y) = \int_\Omega p(\omega) d(\lambda \mu_x + (1 - \lambda) \mu_y)(\omega) \geq \hat{p}(\lambda x + (1 - \lambda) y),
\]
showing that $\hat{p}$ is convex. Moreover, by feasibility of $p$, for any $x \in X$,
\[
\hat{p}(x) = \int_\Omega p(\omega) d\mu_x(\omega) \geq V(\mu_x) = v(x).
\]
Next, we prove a key lemma.
Lemma 9. Let \( v \) be \( L \)-Lipschitz and \( \bar{p} \geq v \). There exists a measurable function \( q : X \to \mathbb{R}^N \) such that \( \|q(x)\| \leq L \) for all \( x \in X \), and

\[
\bar{p}(y) \geq v(x) + q(x) \cdot (y - x), \quad \text{for all } y, x \in X.
\]

Proof. Define

\[
F(x) := \{ r \in \mathbb{R}^N : \bar{p}(y) \geq v(x) + r \cdot (y - x), \quad \text{for all } y \in X \},
\]

and let

\[
q(x) := \arg \min_{r \in F(x)} \|r\|, \quad \text{for all } x \in X.
\]

Note that \( F(x) \) is closed-valued and convex-valued. Thus, if \( F(x) \) is non-empty, then \( q(x) \) exists and is unique because \( q(x) \) is the projection of 0 onto the non-empty closed convex set \( F(x) \). If we can additionally prove that \( \|q(x)\| \leq L \) for all \( x \in X \), then \( q \) will be measurable by the measurable maximum theorem (Theorem 18.19 in Aliprantis and Border, 2006). To see that, note that the definition of \( q \) will not change if we additionally require that \( \|r\| \leq L \), so that the correspondence \( x \mapsto F(x) \cap \{ r \in \mathbb{R}^N : \|r\| \leq L \} \) is compact-valued and upper hemi-continuous (given that \( \bar{p} \) is lower semi-continuous and \( v \) is continuous), and thus measurable, by Theorem 18.20 in Aliprantis and Border (2006).

We deal with some easy cases first. If \( 0 \in F(x) \), then \( q(x) = 0 \) and \( 0 = \|q(x)\| \leq L \). Next, if \( 0 \notin F(x) \) but \( \bar{p}(x) = v(x) \), then we have, for any \( y \in X \),

\[
\bar{p}(y) - \bar{p}(x) \geq v(y) - v(x) \geq -L\|y - x\|,
\]

because \( \bar{p} \geq v \) and \( v \) is \( L \)-Lipschitz. By the Duality Theorem in Gale (1967), \( q(x) \) is well defined and

\[
\|q(x)\| = -\inf_{y \in X} \frac{\bar{p}(y) - \bar{p}(x)}{\|y - x\|} \leq L.
\]

Thus, for the rest of the proof, we fix an arbitrary \( x \in X \) such that \( 0 \notin F(x) \) and \( \bar{p}(x) > v(x) \).

We first show that \( F(x) \) is non-empty. Because, \( \bar{p}(x) > v(x) \), the point \((x,v(x))\) does not belong to the epigraph of \( \bar{p} \), defined as \( \text{epi } \bar{p} := \{(y,t) \in X \times \mathbb{R} : t \geq \bar{p}(y)\} \). Note that \( \text{epi } \bar{p} \) is closed and convex, because \( \bar{p} \) is lower semi-continuous (see footnote 12) and convex. By the separation theorem (for example, Corollary 11.4.1 in Rockafellar, 1970), there exists \((\alpha,\beta) \in \mathbb{R}^N \times \mathbb{R} \) such that, for all \( y \in X \) and \( t \geq \bar{p}(y) \),

\[
\alpha \cdot y + \beta t > \alpha \cdot x + \beta v(x).
\]
Clearly, $\beta \geq 0$; otherwise the inequality would be violated for sufficiently large $t$. Moreover, $\beta \neq 0$; otherwise the inequality would be violated for $(y, t) = (x, \tilde{\rho}(x))$. Thus, evaluating the inequality for $t = \tilde{\rho}(y)$, for all $y \in X$, proves that $-\alpha/\beta$ belongs to $F(x)$. Thus, $F(x)$ is indeed non-empty (and hence $q(x)$ is well-defined).

We now show that $\|q(x)\| \leq L$. Define the set

$$Y := \{y \in X : \tilde{\rho}(y) = v(x) + q(x) \cdot (y - x)\}$$

Note that $Y$ is non-empty: If there does not exist $y \in X$ such that $\tilde{\rho}(y) = v(x) + q(x) \cdot (y - x)$, then the constraint in the definition of $F(x)$ is slack, so it is possible to reduce $\|r\|$, contradicting that $q(x)$ is a minimizer (this step uses the fact that $\tilde{\rho}$ is lower semi-continuous). Since $\tilde{\rho}$ is convex, the set $Y$ is convex. Since $\tilde{\rho}(x) > v(x)$, the set $Y$ cannot contain $x$. Also, let

$$E := \{e \in \mathbb{R}^N : e \cdot q(x) < 0\}.$$

We will prove that there exists $y^* \in Y$ such that $e \cdot (y^* - x) \geq 0$ for all $e \in E$. Suppose that such $y^*$ does not exist. Since any such $y^*$ must satisfy $y^* - x = -tq(x)$ for some $t \geq 0$, we conclude that the compact convex set $Y - x := \{y - x : y \in Y\}$ and the closed convex cone $\{-tq(x) : t \geq 0\}$ must be disjoint. By the separation theorem (for example, Corollary 11.4.1 in Rockafellar, 1970), there exists $e \in \mathbb{R}^N$ such that

$$\max_{y \in Y} e \cdot (y - x) < \inf_{t \geq 0} e \cdot (-tq(x)).$$

Notice that we must have $e \cdot q(x) \leq 0$, as otherwise the right-hand side is $-\infty$ and the inequality cannot hold. In fact, there exists $e \in \mathbb{R}^N$ such that $e \cdot q(x) < 0$, because we can always replace $e$ with $e - \varepsilon q(x)$ for a sufficiently small $\varepsilon > 0$ without violating the above inequality, given that $Y$ is compact. Since there is $e \in E$ such that $e \cdot (y - x) < 0$ for all $y \in Y$, there is $\delta > 0$ such that for all $z$ in the $\delta$-neighborhood of $Y$, we have $e \cdot (z - x) < 0$, and thus for all $\varepsilon > 0$,

$$v(x) + (q(x) + \varepsilon e) \cdot (z - x) < v(x) + q(x) \cdot (z - x).$$

Since $\tilde{\rho}(z) > v(x) + q(x) \cdot (z - x)$ for $z \notin Y$, and $\tilde{\rho}$ is convex and lower semi-continuous, there exists $\gamma > 0$ such that for all $z \in X$ outside the $\delta$-neighborhood of $Y$, we have

$$\tilde{\rho}(z) > v(x) + q(x) \cdot (z - x) + \gamma.$$
Consequently, there exists a sufficiently small $\varepsilon > 0$ such that, for all $z \in X$,

$$\hat{p}(z) > v(x) + (q(x) + \varepsilon e) \cdot (z - x).$$

This is a contradiction with the definition of $q(x)$. Indeed, the above inequality shows that $q(x) + \varepsilon e \in F(x)$ and, by the fact that $e \in E$ and $q(x) \neq 0$, we have $\|q(x) + \varepsilon e\| < \|q(x)\|$ for sufficiently small $\varepsilon > 0$.

We have thus proven that there exists $y^* \in Y$ such that $e \cdot (y^* - x) \geq 0$ for all $e \in E$. Since $e \cdot (y^* - x) \geq 0$ for all $e \in E$ and $Y$ does not contain $x$, it follows that there exists $t > 0$ such that

$$x - y^* = tq.$$

Thus,

$$q(x) \cdot (x - y^*) = \|q(x)\| \|x - y^*\|.$$

And since $y^* \in Y$, we have that

$$v(x) - \hat{p}(y^*) = q(x) \cdot (x - y^*).$$

Putting these two equalities together, we conclude that

$$\|q(x)\| \|x - y^*\| = v(x) - \hat{p}(y^*) \leq v(x) - v(y^*) \leq L \|x - y^*\|,$$

showing that $\|q(x)\| \leq L$. \qed

Fixing $q(x)$ from Lemma 9, we define

$$\bar{p}(y) := \sup_{x \in X} \{v(x) + q(x) \cdot (y - x)\}, \quad \text{for all } y \in X.$$

Note that $\bar{p}$ is convex as a pointwise supremum of affine functions. It lies everywhere above $v$, by definition. Finally, we show that $\bar{p}$ is $L$-Lipschitz. Take any $y, z \in X$. Let $x_n$ be a sequence of points in $X$ that generate the supremum in the definition of $\bar{p}(y)$. Because $X$ is compact and $q$ is bounded, we can assume that $x_n$ and $q(x_n)$ converge. Then, we have that

$$\bar{p}(y) - \bar{p}(z) = \lim_{n \to \infty} \{v(x_n) + q(x_n) \cdot (y - x_n)\} - \bar{p}(z)$$

$$\leq \lim_{n \to \infty} \{v(x_n) + q(x_n) \cdot (y - x_n) - v(x_n) - q(x_n) \cdot (z - x_n)\}$$

$$= \lim_{n \to \infty} \{q(x_n)\} \cdot (y - z) \leq L \|y - z\|. $$
Because \( y \) and \( z \) were arbitrary, this proves that \( \bar{p} \) is \( L \)-Lipschitz.

Finally, notice that \( \bar{p} \leq \bar{p} \), by Lemma 9. Therefore, on \( \Omega \), we have that

\[
\bar{p} \leq \bar{p} \leq p.
\]

Since \( \bar{p} \) is Lipschitz, \( \bar{p} \geq v \) and \( \bar{p} \) is convex, it follows that \( \bar{p} \) (restricted to \( \Omega \)) is feasible for the dual \( (D) \); indeed, for any \( \mu \in \Delta(\Omega) \),

\[
\int_\Omega \bar{p}(\omega)d\mu(\omega) \geq \bar{p} \left( \int_\Omega \omega d\mu(\omega) \right) \geq v \left( \int_\Omega \omega d\mu(\omega) \right) = V(\mu).
\]

But since \( p \) solves the dual problem \( (D) \), we must have that \( p = \bar{p} \) almost surely on \( \Omega \). Since both these function are (Lipschitz) continuous, we can conclude that \( p \) and \( \bar{p} \) coincide on \( \Omega \). In particular, we have shown that \( \bar{p} \) is convex and solves \( (D) \) when restricted to \( \Omega \).

Next, we prove that if \( \pi \in \Pi(\mu_0) \) is optimal for \( (P_M) \), then conditions 1 and 2 hold. We have already shown that \( \bar{p} \) is convex, Lipschitz, and satisfies \( \bar{p} \geq v \). To finish the proof that condition 1 holds, note that

\[
\int_X v(x)d\pi_X(x) = \int_\Omega p(\omega)d\mu_0(\omega) = \int_\Omega \bar{p}(\omega)d\mu_0(\omega),
\]

where the first equality is due to the absence of a duality gap (Theorem 2) and the second is by the fact that \( p = \bar{p} \) on \( \Omega \). We can also prove that condition 2 holds: \( \bar{p} \) satisfies the required equality by definition given that \( q \) is from Lemma 9; moreover,

\[
\int_{X \times \Omega} (v(x) + q(x) \cdot (\omega - x))d\pi(x, \omega) = \int_X v(x)d\pi_X(x) = \int_\Omega \bar{p}(\omega)d\mu_0(\omega) = \int_{X \times \Omega} \bar{p}(\omega)d\pi(x, \omega),
\]

where the first and last equality follow from the feasibility of \( \pi \), and the second equality was established above. Because, by definition, \( \bar{p}(\omega) \geq v(x) + q(x) \cdot (\omega - x) \) for all \( (x, \omega) \), we must have that for \( \pi \)-almost all \( (x, \omega) \),

\[
v(x) + q(x) \cdot (\omega - x) = \bar{p}(\omega).
\]

It remains to show that any one of conditions 1 or 2 imply optimality of \( \pi \in \Pi(\mu_0) \). Note that we will not use the assumption that \( v \) is Lipschitz in that part of the proof.

Assume that condition 1 holds. Note that under these assumptions, \( \bar{p} \) is feasible for the dual \( (D) \) when viewed as a function on \( \Omega \) (in particular, as shown previously, convexity and \( \bar{p} \geq v \) imply that \( \int_\Omega \bar{p}(\omega)d\mu(\omega) \geq V(\mu) \), for all \( \mu \in \Delta(\Omega) \)). But then the fact that \( \pi_X \) achieves no duality gap means that \( \pi \) must be optimal.
Assume that condition 2 holds. Note that under these assumptions, we have shown previously (using only the definition of \( \bar{p} \) and the property that \( q \) is measurable with \( \|q(x)\| \leq L \) for all \( x \in X \)) that \( \bar{p} \) is feasible for the dual (D) on \( \Omega \). Moreover, by the last equation of condition 2,
\[
\int_\Omega \bar{p}(\omega)\,d\mu_0(\omega) = \int_{X \times \Omega} (v(x) + q(x) \cdot (\omega - x))\,d\pi(x,\omega) = \int_X v(x)\,d\pi_X(x),
\]
showing that \( \bar{p} \) and \( \pi_X \) achieve no duality gap, and hence \( \pi \) is optimal.

**B.3 Proof of Theorem 6 and Remark 1**

Since \( v \) is continuously differentiable on the compact set \( X \), it is \( L \)-Lipschitz on \( X \) where
\[
L := \max_{x \in X} \|\nabla v(x)\| < \infty,
\]
so all previous results apply. We now prove the two implications of the equivalence separately.

**If.** Fix \( \pi \in \Pi(\mu_0) \), and let \( S = \text{supp}(\pi_X) \). The function \( p_S \) is convex (see footnote 15). Moreover, by condition (M), \( p_S \geq v \). Thus, there exists a function \( q \) as in Lemma 9. Then, for any feasible \( \bar{\pi} \in \Pi(\mu_0) \), we have
\[
\int_{X \times \Omega} v(x)\,d\bar{\pi}(x,\omega) = \int_{X \times \Omega} (v(x) + q(x) \cdot (\omega - x))\,d\bar{\pi}(x,\omega)
\leq \int_{X \times \Omega} p_S(\omega)\,d\bar{\pi}(x,\omega)
= \int_\Omega p_S(\omega)\,d\mu_0(\omega)
= \int_{X \times \Omega} p_S(\omega)\,d\pi(x,\omega)
= \int_{X \times \Omega} (v(x) + \nabla v(x) \cdot (\omega - x))\,d\pi(x,\omega)
= \int_{X \times \Omega} v(x)\,d\pi(x,\omega),
\]
showing that \( \pi \) is optimal. The inequality follows from Lemma 9. The second to last equality holds by condition (M). The remaining equalities follow from the feasibility of \( \bar{\pi} \) and \( \pi \).

**Only if.** Fix an optimal distribution \( \pi \in \Pi(\mu_0) \). By Theorem 5, there exists an optimal solution \( p \) to (D) and it is convex on \( \Omega \). Define the convex roof extension \( \hat{p} \) of \( p \) from \( \Omega \) to \( X \), as in formula (R). For each \( x \in X \), the infimum in the definition of \( \hat{p}(x) \) is attained at
some $\mu_x \in \Delta(\Omega)$. By feasibility of $p$, for any $x \in X$,
\[
\tilde{p}(x) = \int_{\Omega} p(\omega) d\mu_x(\omega) \geq V(\mu_x) = v(x).
\]

Consequently,
\[
\int_X v(x) d\pi_X(x) \leq \int_X \tilde{p}(x) d\pi_X(x) \leq \int_\Omega \tilde{p}(\omega) d\mu_0(\omega) = \int_\Omega p(\omega) d\mu_0(\omega),
\]
where the first inequality holds because $\tilde{p} \geq v$, the second inequality holds because $\tilde{p}$ is convex and $\mu_0$ is a mean-preserving spread of $\pi_X$, and the equality holds because $\tilde{p}$ coincides with $p$ on $\Omega$, given that $p$ is convex on $\Omega$. Hence condition 1 in Theorem 5 implies that all inequalities hold with equality,
\[
\int_X v(x) d\pi_X(x) = \int_X \tilde{p}(x) d\pi_X(x) = \int_\Omega \tilde{p}(\omega) d\mu_0(\omega).
\]

Thus, $\pi_X(\tilde{S}) = 1$, where $\tilde{S} = \{x \in X : v(x) = \tilde{p}(x)\}$. Since $X$ is closed, $v$ is continuous, $\tilde{p}$ is lower semi-continuous (see footnote 12), and the set $\tilde{S}$ can be equivalently written as $\tilde{S} = \{x \in X : v(x) \geq \tilde{p}(x)\}$, it follows that the set $\tilde{S}$ is closed. Thus, $\text{supp}(\pi_X) \subset \tilde{S}$.

Taking into account that $v$ is continuously differentiable and $\tilde{p}$ is convex and satisfies $\tilde{p} \geq v$, we obtain that $\tilde{p}$ has a subgradient $\nabla v(x)$ at each $x \in \tilde{S}$, so, for all $y \in X$,
\[
\tilde{p}(y) \geq \tilde{p}(x) + \nabla v(x) \cdot (y - x) = v(x) + \nabla v(x) \cdot (y - x).
\]

Indeed, for $x \in \tilde{S}$, $y \in X$, and $\varepsilon > 0$, we have
\[
\tilde{p}(y) - \tilde{p}(x) \geq \frac{1}{\varepsilon} (\tilde{p}(x + \varepsilon(y - x)) - \tilde{p}(x)) \geq \frac{1}{\varepsilon} (v(x + \varepsilon(y - x)) - v(x)),
\]
where the first inequality is by convexity of $\tilde{p}$, and the second inequality is by $\tilde{p} \geq v$ and $\tilde{p}(x) = v(x)$. Taking $\varepsilon \downarrow 0$ yields that $\nabla v(x)$ is a subgradient of $\tilde{p}$ at $x \in \tilde{S}$.

Thus, since $\pi \in \Pi(\mu_0)$ and $p = \tilde{p}$ on $\Omega$, we have
\[
\int_{\Omega} p(\omega) d\mu_0(\omega) = \int_{X \times \Omega} p(\omega) d\pi(x, \omega)
\geq \int_{X \times \Omega} (v(x) + \nabla v(x) \cdot (\omega - x)) d\pi(x, \omega)
= \int_{X \times \Omega} v(x) d\pi(x, \omega).
\]
As shown above, the inequality holds with equality, so \( \pi(\bar{\Gamma}) = 1 \), where

\[
\bar{\Gamma} = \{(x, \omega) \in \bar{S} \times \Omega : \bar{p}(\omega) = v(x) + \nabla v(x) \cdot (\omega - x)\}
\]

Note that the set \( \bar{\Gamma} \) is closed, given that \( \bar{S} \) and \( \Omega \) are closed and \( \nabla v \) and \( \bar{p} \) are continuous on \( X \) and \( \Omega \), respectively. Thus, \( \text{supp}(\pi) \subset \bar{\Gamma} \). But then we have that, for all \( \omega \in \Omega \),

\[
p_{\text{supp}(\pi_X)}(\omega) = \max_{x \in \text{supp}(\pi_X)} \{v(x) + \nabla v(x) \cdot (\omega - x)\} = \bar{p}(\omega),
\]

where the first equality is by the definition of \( p_S \), and the second equality is by \( \text{supp}(\pi) \subset \bar{\Gamma} \). This shows that \( p_{\text{supp}(\pi_X)}(\omega) = \bar{p}(\omega) = p(\omega) \) for \( \omega \in \Omega \), and hence also that \( p_{\text{supp}(\pi_X)}(x) = \bar{p}(x) \) for \( x \in X \). Thus, we have shown that \( p_{\text{supp}(\pi_X)} \) satisfies condition (M), which finishes the proof of the theorem.

Finally, we explain why the above proof also implies Remark 1. First, note that in the “only if” part of the proof we established \( p_{\text{supp}(\pi_X)} \equiv \bar{p} \) for an arbitrary optimal \( \pi \). It follows that \( S^*, \) as defined in Remark 1, is equal to \( \bar{S} \) in the proof (note that \( \bar{S} \) does not depend on which optimal solution \( \pi \) we consider). Thus, we also have that \( p_{S^*} \equiv \bar{p} \).

Fix a feasible \( \pi \in \Pi(\mu_0) \). Suppose that \( \pi \) is optimal for \((P_M)\). Then, the “only if” part of the above proof shows that \( \text{supp}(\pi_X) \subset \bar{S} \) and \( \text{supp}(\pi) \subset \bar{\Gamma} \). As argued in the previous paragraph, we can replace \( \bar{S} \) with \( S^* \) and \( \bar{p} \) with \( p_{S^*} \) and hence condition (M) holds with \( S = S^* \). Conversely, if \( \text{supp}(\pi_X) \subset S^* \) and condition (M) holds with \( S = S^* \), then the “if” part of the proof shows that \( \pi \) is optimal for \((P_M)\).

### B.4 Notation and preliminaries for the remaining proofs

In this appendix, we introduce new notation and collect some implications of Theorem 6 that will be used to prove subsequent results in Section 4. The notation allows us to accommodate the general case when \( \Omega \) is not necessarily convex (so that \( \Omega \neq X \)). In the special case when \( \Omega = X \), this new notation coincides with the notation used in the discussion following Theorem 7 in the main text (for which we assumed that \( \Omega = X \)).

As explained in Section 4.3, it will be convenient to consider solutions \( \pi \in \Pi(\mu_0) \) on the extended space \( X \times X \) even though \( \text{supp}(\pi) \subseteq X \times \Omega \). To make our notation more intuitive, we will use the symbols \( x, y, z \in X \) to refer to moments, and \( \omega \in X \) to refer to the “extended states.”

For a closed set \( S \subset X \), let \( p_S : X \to \mathbb{R} \) be defined as in Section 4.2. Let \( S^* \) be defined
as in Remark 1. Specifically, $S^*$ is the closed subset of $X$ such that

$$S^* = \{ x \in X : p_{S^*}(x) = v(x) \},$$

and condition (M) holds with $S = S^*$ (for any optimal solution $\pi$). Define the function $p^*: X \to \mathbb{R}$,

$$p^*(\omega) := \max_{x \in S^*} \{ v(x) + \nabla v(x) \cdot (\omega - x) \}, \quad \text{for all } \omega \in X.$$

Note that this definition agrees with the one introduced in Section 4.3 when $\Omega = X$ because $p^*$ and $p_{S^*}$ coincide on $\Omega$; however, $p^*$ and $p_{S^*}$ may differ on $X \setminus \Omega$.

Define the contact set $\Gamma \subset X \times X$,

$$\Gamma := \{ (x, \omega) \in S^* \times X : p^*(\omega) = v(x) + \nabla v(x) \cdot (\omega - x) \},$$

and its $x$-section,

$$\Gamma_x := \{ \omega \in X : (x, \omega) \in \Gamma \}, \quad \text{for all } x \in S^*.$$

Define the correspondence $\mathcal{X} : X \rightrightarrows X$ by

$$\mathcal{X}(\omega) := \arg \max_{x \in S^*} \{ v(x) + \nabla v(x) \cdot (\omega - x) \}, \quad \text{for all } \omega \in X,$$

and fix any measurable selection $\chi : X \to X$ from $\mathcal{X}$, which exists by the measurable maximum theorem (Theorem 18.19 in Aliprantis and Border, 2006).

**Lemma 10.**

1. The function $p^*$ is convex and Lipschitz on $X$. Moreover, $p^*$ is differentiable at any $\omega \in \text{int}(X)$ if and only if the set $\{ \nabla v(x) : x \in \mathcal{X}(\omega) \}$ is a singleton, and in that case $\nabla p(\omega) = \nabla v(x)$ for all $x \in \mathcal{X}(\omega)$.

2. The set $\Gamma \subseteq X \times X$ is closed. Its projection along the first coordinate is $S^*$, and its projection along the second coordinate is $X$. For each $x \in S^*$, $\Gamma_x$ is a compact convex set such that $x \in \Gamma_x$ and

$$\Gamma_x = \arg \min_{\omega \in X} \{ p^*(\omega) - \nabla v(x) \cdot \omega \}.$$
Proof. 1. Clearly, \( p^* \) is convex on \( X \) as a pointwise maximum of affine functions. Moreover, it is Lipschitz on \( X \) because, for any \( \omega, \omega' \in X \),

\[
p^*(\omega) - p^*(\omega') \leq v(\chi(\omega)) + \nabla v(\chi(\omega)) \cdot (\omega - \chi(\omega)) - v(\chi(\omega)) - \nabla v(\chi(\omega)) \cdot (\omega' - \chi(\omega)) = \nabla v(\chi(\omega)) \cdot (\omega - \omega') \leq L\|\omega - \omega'\|,
\]

with \( L \) defined (as in Appendix B.3) as the maximal value of the norm of the gradient of \( v \) on \( X \).

The remainder of part 1 is a consequence of the envelope theorem. For \( N = 1 \), this follows immediately from Corollary 4 in Milgrom and Segal (2002). Below, we extend their analysis to the general case \( N \geq 1 \).

Suppose, by contradiction, that \( p^* \) is differentiable at \( \omega \in \text{int} \ X \) but there exist \( x, y \in \mathcal{X}(\omega) \) such that \( \nabla v(x) \neq \nabla v(y) \). Denote \( u := \nabla v(x) - \nabla v(y) \), so that \( \nabla v(x) \cdot u > \nabla v(y) \cdot u \).

Since \( \omega \in \text{int} \ X \), we have \( \omega \pm hu \in X \) for small enough \( h > 0 \). Moreover, by the definitions of \( p^* \) and \( \mathcal{X} \),

\[
\frac{p^*(\omega + hu) - p^*(\omega)}{h} \geq \nabla v(x) \cdot u \quad \text{and} \quad \frac{p^*(\omega - hu) - p^*(\omega)}{h} \geq -\nabla v(y) \cdot u,
\]

and thus

\[
-\lim_{h \downarrow 0} \frac{p^*(\omega - hu) - p^*(\omega)}{h} \leq \nabla v(y) \cdot u < \nabla v(x) \cdot u \leq \lim_{h \downarrow 0} \frac{p^*(\omega + hu) - p^*(\omega)}{h},
\]

showing that \( p^* \) is not differentiable at \( \omega \).

Conversely, suppose that \( \omega \in \text{int} \ X \) and \( \{\nabla v(x) : x \in \mathcal{X}(y)\} \) is a singleton. Fix any \( u \in \mathbb{R}^N \) and small enough \( h'' > h' > 0 \), so that \( \omega + h'u \) and \( \omega + h''u \) are both in \( X \). By the definition of \( p^* \),

\[
\nabla v(\chi(\omega + h'u)) \cdot u \leq \frac{p^*(\omega + h''u) - p^*(\omega + h'u)}{h'' - h'} \leq \nabla v(\chi(\omega + h''u)) \cdot u.
\]

Taking the limit superior in this inequality as \( h' \downarrow 0 \) yields

\[
\limsup_{h' \downarrow 0} \nabla v(\chi(\omega + h'u)) \cdot u \leq \frac{p^*(\omega + h''u) - p^*(y)}{h''} \leq \nabla v(\chi(\omega + h''u)) \cdot u.
\]
Taking the limit inferior in the resulting inequality as $h'' \downarrow 0$ yields

\[
\limsup_{h'' \downarrow 0} \nabla v(\chi(\omega + h'u)) \cdot u \leq \liminf_{h'' \downarrow 0} \frac{p^*(\omega + h''u) - p^*(\omega)}{h''} \leq \liminf_{h'' \downarrow 0} \nabla v(\chi(\omega + h''u)) \cdot u.
\]

Since the limit superior is never smaller than the limit inferior, we conclude that the two limits coincide, and hence

\[
\lim_{h'' \downarrow 0} \frac{p^*(\omega + hu) - p^*(\omega)}{h} = \lim_{h'' \downarrow 0} \nabla v(\chi(\omega + hu)) \cdot u.
\]

Since the correspondence $\mathcal{X}: X \Rightarrow X$ is upper hemicontinuous, a version of the Berge’s Maximum Theorem (see Lemma 17.30 in Aliprantis and Border, 2006) yields

\[
\lim_{h'' \downarrow 0} p^*(\omega + hu) - p^*(\omega) = \lim_{h'' \downarrow 0} \nabla v(\chi(\omega + hu)) \cdot u \leq \max_{x \in \mathcal{X}(\omega)} \nabla v(x) \cdot u.
\]

Since $\{\nabla v(x) : x \in \mathcal{X}(\omega)\}$ is a singleton, we have $\max_{x \in \mathcal{X}(\omega)} \nabla v(x) \cdot u = \nabla v(x) \cdot u$ for all $x \in \mathcal{X}(\omega)$. Finally, taking into account that, by definition of $p^*$, for any $x \in \mathcal{X}(\omega)$ and any small enough $h > 0$, we have

\[
\nabla v(x) \cdot u \leq \frac{p^*(\omega + hu) - p^*(\omega)}{h},
\]

it follows that

\[
\lim_{h'' \downarrow 0} \frac{p^*(\omega + hu) - p^*(\omega)}{h} = \nabla v(x) \cdot u, \quad \text{for all } x \in \mathcal{X}(\omega),
\]

showing that $p^*$ is differentiable at $y$ and $\nabla p^*(\omega) = \nabla v(x)$ for all $x \in \mathcal{X}(\omega)$.

2. The set $\Gamma$ is closed, because the function $p^*(\omega) - v(x) - \nabla v(x) \cdot (\omega - x)$ is continuous in $(x, \omega)$ on $X \times X$. The projection of $\Gamma$ along the second coordinate is $X$, because $(\chi(\omega), \omega) \in \Gamma$ for each $\omega \in X$. The projection of $\Gamma$ along the first coordinate is $S^*$ by the definition of $S^*$ and the fact that $\Gamma_x$ is non-empty, for any $x \in S^*$, which is shown in the next paragraph.

Fix any $x \in S^*$. We have

\[
\Gamma_x = \{\omega \in X : p^*(\omega) = v(x) + \nabla v(x) \cdot (\omega - x)\}
\]

\[
= \{\omega \in X : p^*(\omega) \leq v(x) + \nabla v(x) \cdot (\omega - x)\},
\]

where the first equality is by the definition of $\Gamma$ and $\Gamma_x$, and the second equality is by the
definition of $p^*$, which, in particular, implies that

$$p^*(\omega) \geq v(x) + \nabla v(x) \cdot (\omega - x), \quad \text{for all } \omega \in X.$$  

Thus, the set $\Gamma_x$ is compact and convex, as it is a sublevel set of the (Lipschitz) continuous and convex function $p^*(\omega) - v(x) - \nabla v(x) \cdot (\omega - x)$ (viewed as a function of $\omega$). Moreover, we have $x \in \Gamma_x$, because

$$v(x) = p_{S^*}(x) \geq p^*(x) \geq v(x),$$

where the equality is by $x \in S^*$, the first inequality is by definition of $p_{S^*}$, and the last inequality is by the definition of $p^*$ and the fact that $x \in S^*$. Since $p^*(x) = v(x)$, we have

$$p^*(\omega) \geq p^*(x) + \nabla v(x) \cdot (\omega - x), \quad \text{for all } \omega \in X,$$

and thus

$$\Gamma_x = \arg \min_{\omega \in X} \{ p^*(\omega) - \nabla v(x) \cdot \omega \}.$$  

We have thus shown that $\Gamma_x$ is the projection along the first coordinate of the face of the epigraph of $p^*$ exposed by the direction $(-1, \nabla v(x))$. Then, implication (a) is immediate, whereas implications (b) and (c) follow from Corollary 18.1.2 and Theorem 18.1 in Rockafellar (1970). For completeness, we provide short self-contained proofs of (b) and (c). To show (c), let $\omega \in \text{relint } \Gamma_x \cap \Gamma_y$. Since $\Gamma_x$ is convex, for any $\omega' \in \Gamma_x$ with $\omega' \neq \omega$, there exists $\omega'' \in \Gamma_x$ and $\lambda \in (0, 1)$ such that $\omega = \lambda \omega' + (1 - \lambda) \omega''$. Next, by the definition of $p^*$, we have

$$p^*(\omega') \geq v(y) + \nabla v(y) \cdot (\omega' - y) \quad \text{and} \quad p^*(\omega'') \geq v(y) + \nabla v(y) \cdot (\omega'' - y).$$

Both inequalities must hold with equality, as otherwise we would have

$$p^*(\omega) \geq \lambda p^*(\omega') + (1 - \lambda) p^*(\omega'') > v(y) + \nabla v(y) \cdot (\omega - y),$$

contradicting that $\omega \in \Gamma_y$. Since $\omega'$ is arbitrary, we get $\Gamma_x \subset \Gamma_y$, proving (c). To prove (b), notice that if $\text{relint } \Gamma_x \cap \text{relint } \Gamma_y \neq \emptyset$, then $\text{relint } \Gamma_x \cap \Gamma_y \neq \emptyset$ and $\text{relint } \Gamma_y \cap \Gamma_x \neq \emptyset$, implying that $\Gamma_x \subset \Gamma_y$ and $\Gamma_y \subset \Gamma_x$, and thus $\Gamma_x = \Gamma_y$.  

### B.5 Proof of Theorem 7

The proof builds on and adopts the notation from Appendix B.4.

Let $\bar{X}$ be the set of interior points of $X$ where $p^*$ is differentiable. The set of boundary
points of the convex set $X$ is Lebesgue-negligible, by Theorem 1 in Lang (1986). The set of interior points of $X$ where $p^*$ is not differentiable is Lebesgue-negligible by Rademacher’s theorem (Theorem 10.8 in Villani, 2009). Thus, taking into account that $\mu_0$ has a density on $X$, we conclude that $\mu_0(\tilde{X}) = 1$.

Fix $\omega \in \tilde{X}$. We claim that $\mathcal{X}(\omega)$ is a singleton. Suppose, by contradiction, that there exist distinct $x, y \in \mathcal{X}(\omega)$. Since $\omega \in \text{int } X$ and $p^*$ is differentiable at $\omega$, part 1 of Lemma 10 yields

$$\nabla p^*(\omega) = \nabla v(x) = \nabla v(y).$$

In turn, part 2 of Lemma 10 yields $x \in \Gamma_x, y \in \Gamma_y$, and $\Gamma_x = \Gamma_y$, and thus, given that $p^*$ is affine on $\Gamma_x$ by the definition of $\Gamma_x$, we have $p^*(y) = p^*(x) + \nabla p^*(\omega) \cdot (y - x)$ or equivalently

$$v(x) - \nabla v(x) \cdot x = v(y) - \nabla v(y) \cdot y.$$

Next, for all $\lambda \in [0, 1]$, we have $p_{S^*}(\lambda x + (1 - \lambda)y) = \lambda v(x) + (1 - \lambda)v(y)$ as follows from

$$\lambda v(x) + (1 - \lambda)v(y) = \lambda p^*(x) + (1 - \lambda)p^*(y)
= p^*(\lambda x + (1 - \lambda)y)
\leq p_{S^*}(\lambda x + (1 - \lambda)y)
\leq \lambda p_{S^*}(x) + (1 - \lambda)p_{S^*}(y)
= \lambda v(x) + (1 - \lambda)v(y),$$

where the first equality is by $x \in \Gamma_x$ and $y \in \Gamma_y$, the second equality is by affinity of $p^*$ on the convex set $\Gamma_x = \Gamma_y$, the first and second inequalities are by the definition of $p_{S^*}$, and the last equality is by $p_{S^*} = v$ on $S^*$. Taking into account that $p_{S^*} \geq v$ on $X$, we get

$$\lambda v(x) + (1 - \lambda)v(y) \geq v(\lambda x + (1 - \lambda)y), \text{ for all } \lambda \in [0, 1],$$

contradicting the assumptions of the theorem. Thus, $\mathcal{X}(\omega)$ is a singleton $\{\chi(\omega)\}$ for each $\omega \in \tilde{X}$ where $\chi(\omega)$ is determined by

$$\{\chi(\omega)\} = \{x \in S^* : \omega \in \Gamma_x\} = \{x \in S^* : \nabla p^*(\omega) = \nabla v(x)\}.$$

The first equality is by the definition of $\mathcal{X}$, and the second is by part 1 of Lemma 10.

Finally, for any optimal $\pi \in \Pi(\mu_0)$, we have

$$1 = \pi(\Gamma) = \pi \left( \bigcup_{\omega \in \tilde{X}} \{\chi(\omega)\} \times \{\omega\} \right),$$

50
where the first equality is by Remark 1, and the second equality is by $\Gamma = \bigcup_{\omega \in X} \mathcal{X}(\omega) \times \{\omega\}$, $\mathcal{X}(\omega) = \{\chi(\omega)\}$ for $\omega \in \tilde{X}$, and $\mu_0(\tilde{X}) = 1$. Since $\chi(\omega)$ is determined by $p^*$ for $\mu_0$-almost all $\omega \in X$, and $p^*$ is independent of $\pi$, we conclude that $\pi$ is uniquely determined by

$$
\pi(A, B) = \int_B 1\{\chi(\omega) \in A\} d\mu_0(\omega), \quad \text{for all measurable } A \subset X \text{ and } B \subset X.
$$

B.6 Proof of Theorem 8

By the same argument as in the proof of Theorem 7, $\Omega$ is partitioned (up to a measure zero set) into a collection of disjoint (relatively) open sets $\Xi = \{\text{relint}(\Gamma_x)\}_{x \in S^*}$ such that $(y, \omega) \in \Gamma$ if and only if $\text{relint}(\Gamma_x) = \text{relint}(\Gamma_y)$.

Consider an auxiliary problem of finding a joint distribution $\pi \in \Pi(\mu_0)$ to maximize

$$
\int_{X \times \Omega} w(x, \omega) d\pi(x, \omega)
$$

where

$$
w(x, \omega) = \begin{cases} 
-\|x\|^2, & (x, \omega) \in \Gamma, \\
-\infty, & (x, \omega) \in (X \times X) \setminus \Gamma.
\end{cases}
$$

Note that $\int_{X \times \Omega} w(x, \omega) d\pi(x, \omega)$ is finite for $\pi \in \Pi(\mu_0)$ if and only if $\text{supp}(\pi) \subset \Gamma$ (i.e., $\pi$ is optimal for the primary problem). Thus, since $w$ is upper semicontinuous and bounded from above, by Lemma 2, there exists an optimal solution $\pi \in \Pi(\mu_0)$ to the auxiliary problem, which is also optimal for the primal problem $(P_M)$.

Consider a $\sigma$–algebra generated by sets in $\Xi$. Let $\phi$ be the marginal distribution over $\Xi$ induced by $\pi$. By the disintegration theorem, there exists a measurable map $\xi \mapsto \zeta(\cdot | \xi)$ from $\Xi$ to $\Delta(X \times \Omega)$ such that for every “test function” $h \in C(X \times \Omega)$, we have

$$
\int_{X \times \Omega} h(x, \omega) d\pi(x, \omega) = \int_{\Xi} \int_{X \times \Omega} h(x, \omega) d\zeta(x, \omega | \xi) d\phi(\xi).
$$

Let $\zeta_X(\cdot | \xi)$ and $\zeta_{\Omega}(\cdot | \xi)$, for $\xi \in \Xi$, denote the marginal distributions of $x$ and $\omega$ induced by $\pi$. Then, there exists a set $\tilde{X} \subset \text{supp}(\pi_X)$ with $\pi_X(\tilde{X}) = 1$ such that, for all $x \in \tilde{X}$, we have

$$
\text{supp}(\zeta_{\Omega}(\cdot | \text{relint}(\Gamma_x))) \subset \text{cl}(\text{relint}(\Gamma_x)),
$$

$$
\text{supp}(\zeta_X(\cdot | \text{relint}(\Gamma_x))) = \text{cl}(\text{supp}(\pi_X) \cap \text{relint}(\Gamma_x)),
$$

$$
\int_{\tilde{X} \times \Omega} (\omega - x) d\zeta(x, \omega | \text{relint}(\Gamma_x)) = 0, \quad \text{for all measurable } \tilde{X} \subset X,
$$

$$
\int_{X \times \Omega} w(x, \omega) d\zeta(x, \omega | \text{relint}(\Gamma_x)) \geq \int_{X \times \Omega} w(x, \omega) d\tilde{\pi}(x, \omega), \quad \text{for all } \tilde{\pi} \in \Pi(\zeta(\cdot | \text{relint}(\Gamma_x))).
$$

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Suppose, by contradiction, that there exist distinct \( x^0, x^1, \ldots, x^n \) in \( S_x \) such that \( x^0 = \lambda^1 x^1 + \cdots + \lambda^n x^n \) where \( \lambda^1, \ldots, \lambda^n > 0 \) and \( \lambda^1 + \cdots + \lambda^n = 1 \). Since \( x^1, \ldots, x^n \in S_x \), for all \( i = 1, \ldots, n \), and \( \delta > 0 \), we have \( \zeta_x(B_\delta(x^i)|\text{relint}(\Gamma_x)) > 0 \). There exists a sufficiently small \( \delta \) such that for some \( \lambda^1_\delta, \ldots, \lambda^n_\delta > 0 \) with \( \lambda^1_\delta + \cdots + \lambda^n_\delta = 1 \), we have \( x^0 = \lambda^1_\delta x^1_\delta + \cdots + \lambda^n_\delta x^n_\delta \) where \( x^i_\delta = \mathbb{E}_{\zeta_x(\cdot | \text{relint}(\Gamma_x))}[x|B_\delta(x^i)] \). Moreover, there exists \( \varepsilon > 0 \) such that \( \varepsilon \lambda^i_\delta < \zeta_x(B_\delta(x^i)|\text{relint}(\Gamma_x)) \) for all \( i \). Finally, there exists \( \tilde{\pi} \in \Pi(\zeta_\Omega(\cdot | \text{relint}(\Gamma_x))) \) such that

\[
\tilde{\pi}_x = \zeta_x(\cdot | \text{relint}(\Gamma_x)) + \varepsilon \delta x^0 - \varepsilon \sum_i \lambda^i_\delta \zeta_x(\cdot | \text{relint}(\Gamma_x) \cap B_\delta(x^i)).
\]

But then, by Jensen’s inequality, we have

\[
\int_{X \times \Omega} w(x, \omega) d\tilde{\pi}(x, \omega) - \int_{X \times \Omega} w(x, \omega) d\zeta(x, \omega | \text{relint}(\Gamma_x)) = \varepsilon \left( \sum_i \lambda^i_\delta \int_{X} x^2 d\zeta_x(x | \text{relint}(\Gamma_x) \cap B_\delta(x^i)) - (x^0)^2 \right) 
\geq \varepsilon \left( \sum_i \lambda^i_\delta (x^i_\delta)^2 - (x^0)^2 \right) > 0,
\]

yielding a contradiction.

**B.6.1 Example**

Suppose that \( \mu_0 \) is uniformly distributed on the circle \( \Omega = X = \{ (\omega_1, \omega_2) \in \mathbb{R}^2 : \|\omega\| \leq 4\pi \} \), and suppose that the objective function is \( v(x) = -\cos(\|x\|) \). It is easy to verify that the optimal solution to (D) is \( p(x) = 1 \) for all \( x \) and that \( \Gamma_x = \text{int}(\Omega) \) for all \( x \in S^* = \{ x \in \mathbb{R}^2 : \|x\| \in \{\pi, 3\pi\} \} \). There exists an optimal signal that satisfies \( S_x = \text{ext}(S_x) = \{ x \in \mathbb{R}^2 : \|x\| = \pi \} \), in line with Theorem 8 (and there also exist other optimal signals with \( S_x \neq \text{ext}(S_x) \)). However, there does not exist an optimal signal such that \( S_x \) is a finite set.

**B.7 An alternative statement of Theorem 7**

In this appendix, we present an alternative version of Theorem 7 under the assumption \( \Omega = X \), which allows us to relax the condition imposed on the objective function \( v \).

**Theorem 9.** Suppose \( \mu_0 \) has a density on \( \Omega = X \) with respect to the Lebesgue measure.
Moreover, suppose there do not exist distinct \( x, y \in X \) and \( \varepsilon > 0 \) such that

\[
\nabla v(x) = \nabla v(z),
\]

\[
\lambda x + (1 - \lambda)z \in X, \quad \text{for all } \lambda \in [-\varepsilon, 1 + \varepsilon],
\]

\[
\lambda v(x) + (1 - \lambda)v(z) \geq v(\lambda x + (1 - \lambda)z), \quad \text{for all } \lambda \in [-\varepsilon, 1 + \varepsilon].
\]

Then there is a unique optimal solution, and that solution is convex-partitional.

C Additional results for Section 4

In this appendix, we complement the analysis of Section 4 by formulating the dual problem for moment persuasion. We also introduce an alternative formulation of the dual problem, and show that the price function from Theorem 5 solves both of these problems. This in turn allows us to sharpen the connection between our results and existing duality methods. Finally, we include examples illustrating the difficulties associated with using the convex roof construction in the multi-dimensional case, as well as results showing that the convex roof solves the dual problem (and its alternative) under additional assumptions.

C.1 Two duality formulations for moment persuasion

The problem dual to \((P_M)\) is to find functions \( p : X \to \mathbb{R} \) and \( q : X \to \mathbb{R}^N \) to

\[
\minimize \int_{\Omega} p(\omega) d\mu_0(\omega)
\]

subject to \( p(y) \geq v(x) + q(x) \cdot (y - x) \) for all \( x, y \in X \),

\( p \) is Lipschitz on \( X \), \( q \) is measurable and bounded on \( X \).

\((D_M)\)

This duality formulation is a consequence of the fact that in our primal problem we represent feasible solutions as joint distributions of moments and states (similarly to Kolotilin, 2018 and Kolotilin et al., 2022). The dual variable \( p \) is a multiplier on the Bayes-plausibility constraint, while the dual variable \( q \) is a multiplier on the martingale constraint.

When, instead, feasibility for the primal problem is described in terms of marginal distributions of moments using a mean-preserving spread constraint (as in Dworczak and Martini, 2019 and Dizdar and Kováč, 2020), we can write the dual problem as finding a function
We now show that these problems can both be treated as duals to \((P_M)\) in the sense that their values provide the relevant upper bound on the value of \((P_M)\) that is tight and attained by the price function identified in Theorem 5.

**Proposition 3.**

1. **Weak duality:** If \(v\) is measurable and bounded, then for any \(\pi\) feasible for \((P_M)\) and any \(p\) feasible for either \((D_M)\) or \((D'_M)\), \(\int_X v(x) d\pi(x) \leq \int_\Omega p(\omega) d\mu_0(\omega)\).

2. **No duality gap and primal attainment:** If \(v\) is bounded and upper semi-continuous, then there exists an optimal solution to \((P_M)\), and the problems \((P_M), (D_M), (D'_M)\) all have the same value.

3. **Dual attainment:** If \(v\) is Lipschitz, then the price function \(\bar{p}\) from Theorem 5 solves \((D'_M)\), and together with the function \(q\) from condition 2 of Theorem 5 solves \((D_M)\).

**Proof.** *Weak duality.* Suppose that \(\pi\) is feasible for \((P_M)\). If \((p, q)\) is feasible for \((D_M)\), then

\[
\int_X v(x) d\pi(x) = \int_{X \times \Omega} (v(x) + q(x) \cdot (\omega - x)) d\pi(x, \omega) \leq \int_{X \times \Omega} p(\omega) d\pi(x, \omega) = \int_\Omega p(\omega) d\mu_0(\omega).
\]

If instead \(p\) is feasible for \((D'_M)\), then

\[
\int_X v(x) d\pi(x) \leq \int_X p(x) d\pi(x) \leq \int_\Omega p(\omega) d\mu_0(\omega).
\]

**No duality gap and primal attainment.** When \(v\) is bounded and upper semi-continuous on \(X\), the corresponding \(V\) is also bounded and upper semi-continuous on \(\Delta(\Omega)\), and hence, by Lemma 2, the problem \((P_M)\) has an optimal solution \(\pi^* \in \Pi(\mu_0)\).

Thus, weak duality above implies that \(\max \, (P_M) \leq \inf \, (D_M)\). Moreover, if \(p\) is feasible for \((D'_M)\), then, by Corollary 13.3.3 in Rockafellar (1970), \(p\) has a bounded subgradient (which we denote \(q\)), so that, for all \(x, y \in X\),

\[
p(y) \geq p(x) + q(x) \cdot (y - x) \geq v(x) + q(x) \cdot (y - x),
\]
showing that \((p, q)\) is feasible for \((D_M)\) and hence \(\max (P_M) \leq \inf (D_M) \leq \inf (D'_M)\).

Thus, it suffices to show that \(\max (P_M) = \inf (D'_M)\). The proof is essentially the same as the proof of Lemma 7. Let \(\mathcal{P}_M(v)\) denote the sets of functions \(p : X \to \mathbb{R}\) feasible for \((D'_M)\). By Baire’s theorem, there exists a non-increasing sequence of Lipschitz functions \(v_k\) converging pointwise to \(v\). Let \(\pi^*_k\) denote an optimal solution to \((P_M)\) with the objective function \(v_k\). For each \(k \in \mathbb{N}\), we have

\[
\int_{X \times \Omega} v(x) d\pi^*_k(x, \omega) \leq \inf_{p \in \mathcal{P}_M(v_k)} \int_{X} p(x) d\mu_0(x) \leq \min_{p \in \mathcal{P}_M(v_k)} \int_{\Omega} p(\omega) d\mu_0(\omega) = \int_{X \times \Omega} v_k(x) d\pi^*_k(x, \omega),
\]

where the first inequality holds by \(\max (P_M) \leq \inf (D'_M)\), the second inequality holds by \(\mathcal{P}_M(v_k) \subset \mathcal{P}_M(v)\) for \(v_k \geq v\), and the equality holds by Theorem 5. It is thus sufficient to show that

\[
\lim_{k \to \infty} \int_{X \times \Omega} v_k(x) d\pi^*_k(x, \omega) \leq \int_{X \times \Omega} v(x) d\pi^*(x, \omega).
\]

Thanks to compactness of \(\Pi(\mu_0)\), up to extraction of a subsequence, we can suppose that \(\pi^*_k\) converges weakly to some \(\pi \in \Pi(\mu_0)\). Then for each \(j \in \mathbb{N}\), we have

\[
\lim_{k \to \infty} \int_{X \times \Omega} v_k(x) d\pi^*_k(x, \omega) \leq \lim_{k \to \infty} \int_{X \times \Omega} v_j(x) d\pi^*_k(x, \omega) = \int_{X \times \Omega} v_j(x) d\pi(x, \omega),
\]

where the first inequality holds because \(v_k \leq v_j\) for \(k \geq j\), and the equality holds because \(v_j\) is (Lipschitz) continuous and \(\pi^*_k \to \pi\). Then letting \(j\) go to infinity and invoking the monotone convergence theorem,

\[
\lim_{j \to \infty} \int_{X \times \Omega} v_j(x) d\pi(x, \omega) = \int_{X \times \Omega} v(x) d\pi(x, \omega),
\]

we obtain

\[
\lim_{k \to \infty} \int_{X \times \Omega} v_k(x) d\pi^*_k(x, \omega) \leq \int_{X \times \Omega} v(x) d\pi(x, \omega) \leq \int_{X \times \Omega} v(x) d\pi^*(x, \omega),
\]

where the last inequality holds because \(\pi^*\) is an optimal solution to \((P_M)\). This establishes that \(\max (P_M) = \inf (D_M) = \inf (D'_M)\).

**Dual attainment.** When \(v\) is Lipschitz, Theorem 5 guarantees existence of \(\bar{p}\) and \(q\) with all required properties, and such that for any \(\pi\) optimal for \((P_M)\),

\[
\int_{X} v(x) d\pi_X(x) = \int_{\Omega} \bar{p}(\omega) d\mu_0(\omega).
\]

It follows that \(\bar{p}\) solves \((D'_M)\) and \((\bar{p}, q)\) solve \((D_M)\). \(\square\)
Proposition 3 formalizes our claim from Section 4 that the two conditions in Theorem 5 correspond to two alternative formulations of the problem dual to \((P_M)\). At the same time, the proposition shows that these two problems have the same solution, at least under the conditions of Theorem 5. This observation allows us to describe the exact connection between our general duality result and existing duality approaches to moment persuasions.

C.2 Relationship to existing duality methods

The linear persuasion problem in which the objective function depends only on the expected one-dimensional state has received special attention (see, for example, Gentzkow and Kamenica, 2016, Kolotilin et al., 2017, Kolotilin, 2018, Dworczak and Martini, 2019, and Dizdar and Kováč, 2020). Linear persuasion is a special case of moment persuasion when \(\omega \in \mathbb{R}\) and \(m(\omega) = \omega\). Theorem 3 is mathematically more general than the existing strong duality results in the sense that it applies on a larger domain of problems; in fact, bounded steepness of the concave closure is shown to be necessary for dual attainment so it must imply all existing sufficient conditions. However, verifying bounded steepness of the concave closure may be difficult in applications. Previous papers identified easier-to-verify regularity conditions on the primitives under which strong duality holds. Our Theorem 4 identifies Lipschitz continuity of \(V\) as a simple sufficient condition; while this condition is stronger than the one identified for linear persuasion (by Dworczak and Martini, 2019, and especially Dizdar and Kováč, 2020), it has the advantage of being fully universal—it applies to any persuasion problem.

When the objective function is Lipschitz, Theorem 5 generalizes Theorems 1 and 2 in Dworczak and Martini (2019): By a simple transformation, condition 1 of Theorem 5 establishes existence of a convex and (Lipschitz) continuous function \(p^*\) and a cumulative distribution function \(G^*\) of moments such that

\[
\begin{align*}
\text{supp}(G^*) \subseteq \{x \in X : p^*(x) = v(x)\}, \\
\int_{\Omega} p^*(x)dF_0(x) = \int_{\Omega} p^*(x)dG^*(x), \\
G^* \preceq_{\text{ex}} F_0, \text{ and } p^* \geq v.
\end{align*}
\]

Moreover, the theorem resolves (positively) the conjecture of Dworczak and Martini that if the objective function \(V\) is measurable with respect to a moment \(m(\omega)\), then so is the corresponding price function.

It is worth noting that we impose stronger regularity conditions on the price function compared to Dworczak and Martini. In our dual formulation \((D'_M)\), we assume that prices \(p\)
are Lipschitz continuous, while Dworczak and Martini only assume continuity. The general trade-off is that stronger regularity conditions on the dual variable make it more difficult to prove that the dual problem has a solution in the assumed class, but—conditional on proving existence—impose tighter structure on the solutions to the primal problem. We impose a stronger condition on $p$ because Lipschitz continuity is directly implied by Theorem 5. Dizdar and Kováč (2020) prove, under weaker assumptions on the objective function, that the prices that solve the dual problem of Dworczak and Martini are in fact Lipschitz.\textsuperscript{27} Thus, it seems that in most economically relevant cases imposing Lipschitz continuity of prices in the dual is without significant loss of generality.\textsuperscript{28}

Kolotilin (2018) and Galperti et al. (2021) use an alternative approach to the persuasion problem. Instead of working with an objective function $V : \Delta(\Omega) \rightarrow \mathbb{R}$, they consider a sender and a receiver whose utility functions are $w : A \times \Omega \rightarrow \mathbb{R}$ and $u : A \times \Omega \rightarrow \mathbb{R}$ where $A$ is the space of the receiver’s actions. The sender chooses a joint distribution $\pi \in \Delta(A \times \Omega)$ of the state $\omega$ and the recommended action $a$. On top of the Bayes plausibility constraint, $\pi$ must satisfy the obedience constraint, which requires each recommended action to be incentive-compatible for the receiver given the beliefs it induces.

Generally, it is possible to reformulate the alternative problem as our problem, and vice versa. To illustrate this point suppose that $\Omega$ is a finite set. Consider an alternative problem in which the sender’s and receiver’s utility functions are $w(a, \omega)$ and $u(a, \omega)$. Kamenica and Gentzkow (2011) show that this problem is equivalent to our problem in which the objective function is $V(\mu) = \mathbb{E}_\mu[w(a^*(\mu), \omega)]$ with $a^*(\mu) \in \arg \max_{a \in A} \mathbb{E}_\mu[u(a, \omega)]$. Conversely, consider our problem in which the objective function is $V$. This problem is equivalent to the alternative problem in which the action space is $A = \Delta(\Omega)$, and the sender’s and receiver’s utility functions are $w(a, \omega) = V(a)$ and $u(a, \omega) = 2a(\omega) - \sum_{\omega' \in \Omega} a^2(\omega')$. Indeed, given a posterior $\mu$, the receiver takes an action $a^*(\mu) = \mu$, which maximizes his expected utility $\sum_{\omega \in \Omega}(2a(\omega)\mu(\omega) - a^2(\omega))$, and thus the objective function is $V(\mu)$.

By setting $A = X$ in the model of Kolotilin (2018), we can draw a tight connection between the two duality approaches. The dual problem in Kolotilin (2018) is to find a

\textsuperscript{27}Dizdar and Kováč show that the dual problem in linear persuasion has an optimal solution by demonstrating that feasible solutions can be restricted to a compact set of uniformly Lipschitz functions. Our proof strategy is different: We construct the optimal solution (a price function on the space of moments) from the supergradient of the concave closure of $V$.

\textsuperscript{28}That being said, it is easy to come up with examples where the dual problem $(D_M')$ has a solution in the class of continuous functions but not in the class of Lipschitz functions. For instance, when $\mu_0$ is fully supported on $\Omega = [0, 1]$ and $w(x) = -\sqrt{x}$, $p(x) = -\sqrt{x}$ is continuous and achieves the lower bound in $(D_M')$, but a Lipschitz solution does not exist.
continuous function $p : \Omega \to \mathbb{R}$ and a bounded measurable function $q : A \to \mathbb{R}$ to

$$
\minimize \int_{\Omega} p(\omega) d\mu_0(\omega)
$$

subject to $p(\omega) + q(a)(a - \omega) \geq v(a)$ for all $(a, \omega) \in A \times \Omega$,

where $p$ and $q$ are multipliers for the Bayes plausibility and obedience constraints. Thus, the problem $(D_A)$ corresponds to our dual problem $(D_M)$, and condition 2 of Theorem 5 establishes that this problem is solved by the price function $\bar{p}$ derived from our general duality results from Section 3.

C.3 Comments on the convex-roof construction

In this appendix, we further investigate the properties of the convex-roof construction that underlies the proof of Theorem 5. Our goal is twofold: On one hand, we are interested in regularity conditions under which the convex roof is (Lipschitz) continuous, guaranteeing that it can be used as the price function $\bar{p}$ satisfying conditions 1 and 2 of Theorem 5 (and hence as a solution to the dual problems $(D_M')$ and $(D_M)$). On the other hand, we show (by means of examples) that the convex roof can behave in surprisingly pathological ways when the space of moments is multi-dimensional, explaining why we need stronger assumptions to extend existing duality methods to the multi-dimensional case.

The main result in this appendix shows that if the support of the prior contains the boundary of its convex hull, then the convex roof preserves the Lipschitz constant of the objective function, and hence the convex roof could be used as a solution to problems $(D_M)$ and $(D_M')$.

**Proposition 4.** Let $v$ be $L$-Lipschitz on $X$, and let $\Omega$ contain the boundary of $X$. Then $\bar{p}$ is $L$-Lipschitz on $X$.

**Proof.** By the proof of Theorem 5, there exists a price function $\bar{p} : X \to \mathbb{R}$ that is convex and $L$-Lipschitz. Moreover, for each $z \in X$, we have $\bar{p}(z) \geq \tilde{p}(z)$ and, for each $y \in \Omega$, there exists a sequence $x_n \in X$ converging to some $x \in X$ such that $q(x_n)$ converges to some $r(y) \in \mathbb{R}^N$, with $\|r(y)\| \leq L$, and

$$
\tilde{p}(y) = \bar{p}(y) = \lim_{n \to \infty} \left\{ v(x_n) + q(x_n) \cdot (y - x_n) \right\}.
$$

Thus, for each $z \in X$ and each $y \in \text{bd} X \subset \Omega$, we have

$$
\bar{p}(z) - \bar{p}(y) \geq \lim_{n \to \infty} \left\{ v(x_n) + q(x_n) \cdot (z - x_n) - v(x_n) - q(x_n) \cdot (y - x_n) \right\} = r(y) \cdot (z - y),
$$

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showing that \( r(y) \), with \( \| r(y) \| \leq L \), is a subgradient of \( \hat{p} \) at \( y \in \text{bd} \, X \).

By Theorem 7.12 in Aliprantis and Border (2006), at each \( z \in \text{int} \, X \), the convex roof \( \hat{p} \) has a subgradient \( r(z) \in \mathbb{R}^N \). We claim that \( \| r(z) \| \leq L \). Suppose that \( r(z) \neq 0 \), as otherwise the claim is trivial. Since \( z \in \text{int} \, X \) and \( \| r(z) \| > 0 \), there exists \( t > 0 \) such that \( y := z + tr(z) \in \text{bd} \, X \subset \Omega \). Hence,

\[
L \| y - z \| \geq \hat{p}(y) - \hat{p}(z) = \hat{p}(y) - \hat{p}(z) \\
\geq \hat{p}(z) + r(z) \cdot (y - z) - \hat{p}(z) = r(z) \cdot (y - z) \\
= \| r(z) \| \| y - z \|,
\]

showing that \( \| r(z) \| \leq L \).

Thus, for each \( z, y \in X \), we have

\[
\hat{p}(z) - \hat{p}(y) \leq r(y) \cdot (z - y) \leq \| r(y) \| \| z - y \| \leq L \| y - z \|,
\]

showing that \( \hat{p} \) is \( L \)-Lipschitz on \( X \).

Next, we construct an example showing that the assumptions of Proposition 4 are not redundant: \( \hat{p} \) does not always preserve the Lipschitz constant of \( v \) even when \( N = 2 \) and \( \Omega \) is finite.

**Example 1.** Let \( \Omega = \{(-l, 0), (0, 1), (l, 0)\} \) with \( l > 1 \) and \( v(x) = |x_1| \) for \( x \in X \), which is 1-Lipschitz. We can apply Corollary 1 to show that full disclosure is optimal and thus \( p \) that coincides with \( v \) on \( \Omega \) solves (D). Indeed, condition (C) holds, and, by Jensen’s inequality,

\[
V(\mu) = \left| \int_\Omega \omega_1 d\mu(\omega) \right| \leq \int_\Omega |\omega_1| d\mu(\omega) = \int_\Omega p(\omega) d\mu(\omega) \text{ for all } \mu \in \Delta(\Omega).
\]

It is easy to see that \( \hat{p} \) is given by \( \hat{p}(x) = l(1 - x_2) \) for all \( x \in X \), so the Lipschitz constant of \( \hat{p} \) is \( l > 1 \). Of course, by Theorem 5, there exists a different convex extension \( \bar{p} \) of \( p \) from \( \Omega \) to \( X \) (for example, consider \( \bar{p} = v \) on \( X \)) that is convex, 1-Lipschitz, and satisfies \( \bar{p} \geq v \).

The next example demonstrates the additional difficulties that arise when the dimension of the space of moments is three (or higher). In this case, even when the objective function is continuously differentiable, and the set of extreme points of \( X \) is compact, the convex roof may be discontinuous.
Example 2. The example is adapted from Example 5.1 in Bucicovschi and Lebl (2013). Let

\[ K = \{(x_1, x_2, x_3) : x_1 = -1, x_2^2 + x_3^2 = 1\} \cup \{(x_1, x_2, x_3) : x_1 = 1, x_2^2 + x_3^2 = 1\}, \]

and \( \omega^* = (0, 0, 1) \). Define \( \Omega = K \cup \{\omega^*\} \), and note that its convex hull \( X \) is a cylinder:

\[ X = \{(x_1, x_2, x_3) : -1 \leq x_1 \leq 1, x_2^2 + x_3^2 \leq 1\}. \]

Define the objective function as \( v(x) = x_1^2 \) for \( x \in X \), which is Lipschitz. We can again apply Corollary 1 to show that \( p \) that coincides with \( v \) on \( \Omega \) solves (D).

We will now show that the convex roof \( \tilde{p} \) is discontinuous at \( \omega^* \). On any line segment \( \{(x_1, x_2, x_3) : -1 \leq x_1 \leq 1, x_2 = y, x_3 = z\} \) with \( y \neq 0 \) and \( y^2 + z^2 = 1 \), the convex roof \( \tilde{p} \) must be identically 1. This shows that \( \tilde{p} \) is discontinuous at \( \omega^* = (0, 0, 1) \), because \( \tilde{p}(\omega^*) = 0 \) yet \( \tilde{p}(\omega^n) = 1 \) for the sequence \( \omega^n = (0, 1/n, \sqrt{1-1/n^2}) \) that converges to \( \omega^* \), as \( n \to \infty \).

By Theorem 5, there exists a convex, Lipschitz extension \( \bar{p} \) (for example, \( \bar{p} = v \)).

Finally, we construct an instance of moment persuasion (with a discontinuous objective function) in which there exists an optimal convex and Lipschitz price function on \( \Omega \) solving the original dual (D), but the price function cannot be extended to a convex and continuous function on \( X \). This example, unlike the previous ones, goes beyond indicating a problem with the convex-roof construction; it shows that—beyond the case of a Lipschitz \( v \)— requiring the price function to be (Lipschitz) continuous on \( X \) in the multi-dimensional moment persuasion problem may be too demanding.

Example 3. The example is adapted from Example 5.4 in Bucicovschi and Lebl (2013). Let

\[ K = \{(x_1, x_2, x_3) : -1 \leq x_1 \leq -x_3, x_2^2 + x_3^2 = 1\} \cup \{(x_1, x_2, x_3) : x_1 = 1, x_2^2 + x_3^2 = 1\}, \]

and \( \omega^* = (0, 0, 1) \). Define \( \Omega = K \cup \{\omega^*\} \), and note that its convex hull \( X \) is the same cylinder:

\[ X = \{(x_1, x_2, x_3) : -1 \leq x_1 \leq 1, x_2^2 + x_3^2 \leq 1\}. \]

Define the objective function

\[ v(x) = \begin{cases} 
1, & x \in K, \\
0, & x = \omega^*, \\
-1, & x \notin K \cup \{\omega^*\}.
\end{cases} \]

Because the sets \( K \) and \( \{\omega^*\} \) are closed and disjoint, the function \( v \) is upper semi-continuous.
We claim that full disclosure is optimal in this instance of moment persuasion. We can again apply Corollary 1 by defining \( p = v \) on \( \Omega \). Then, \( p \) is trivially Lipschitz, and condition (C) holds, so all we have to check is that for all \( x \in X \), and \( \mu \in \Delta(\Omega) \) such that \( \int_{\Omega} \omega d\mu(\omega) = x \), \( \int_{\Omega} p(\omega)d\mu(\omega) \geq v(x) \). When \( x \notin K \), this is trivial because \( p \geq 0 \). When \( x \in K \), the conclusion is trivial for all \( \mu \) with support in \( K \). So the only case we have to check is when \( x \in K \) but \( \text{supp}(\mu) \) contains the point \( \omega^* \). We will prove that this case cannot arise. Indeed, since \( \omega^* \) is an isolated point of \( \Omega \), it would have to be that \( \mu(\omega^*) > 0 \) and

\[
x = \mu(\omega^*)\omega^* + \int_K \omega d\mu(\omega).
\]

But \( x \in K \) implies that, for almost all \( \omega \in \text{supp}(\mu) \), \( \omega_2 = 0 \) and \( \omega_3 = 1 \) (as otherwise \( x_2^2 + x_3^2 < 1 \)). But the only points in \( K \) with that property are \((-1, 0, 1)\) and \((1, 0, 1)\). This is a contradiction with \( \mu(\omega^*) > 0 \), because \( \mu(\omega^*) > 0 \) implies that \( x_1 \in (-1, 1) \).

We will now show that there does not exist a convex and continuous extension of \( p \) to \( X \). On any line segment \( \{(x_1, x_2, x_3) : -1 \leq x_1 \leq 1, x_2 = y, x_3 = z\} \) with \( y \neq 0 \) and \( y^2 + z^2 = 1 \), the function \( p \) takes the value 1 for \( x_1 \in [-1, -\varepsilon] \cup \{1\} \). Hence, any convex extension \( \bar{p} \) of \( p \) must be identically equal to 1 on such a line segment. This, however, means that such \( \bar{p} \) must be discontinuous at \( \omega^* = (0, 0, 1) \). Indeed, \( \bar{p}(\omega^*) = 0 \), but \( \bar{p}(\omega^n) = 1 \) for the sequence \( \omega^n = (0, 1/n, \sqrt{1 - 1/n^2}) \) that converges to \( \omega^* \).

\[\blacksquare\]

### D Proofs for Section 5

#### D.1 Proof of Proposition 2

In this appendix, we prove the necessity part of Proposition 2, and show that \( I_t \subseteq \{\omega \in \Omega : \omega_2 = f(t) - f'(t)(\omega_1 - t)\} \), for all \( t \in [\underline{x}_1, \overline{x}_1] \), and \( I_t \) is a proper line segment, for all \( t \in (\underline{x}_1, \overline{x}_1) \). Fix an optimal \( \pi^* \in \Pi(\mu_0) \). Since \( \mu_0 \) has a density and \( \nabla v(x) = (x_2, x_1) \neq (y_2, y_1) = \nabla v(y) \) for \( x \neq y \), Theorem 7 implies that \( \pi^* \) is the unique optimal signal, and that it is convex-partitional. Suppose that \( \text{supp}(\pi^*_{\chi}) \) is the graph of the function \( f \), as described in the proposition.

By the definition of \( \Gamma_x \) from Section 4.3, for each \( t \in [\underline{x}_1, \overline{x}_1] \),

\[
\Gamma(t, f(t)) = \{\omega \in \Omega : t \in \text{arg max}_{s \in [\underline{x}_1, \overline{x}_1]} \{\omega_1 f(s) + \omega_2 s - sf(s)\}\}.
\]

First, consider \( t \in (\underline{x}_1, \overline{x}_1) \). The necessary first order condition yields \( \omega_2 = f(y) - \).
\( f'(y)(\omega_1 - y) \) for all \( \omega \in \Gamma_{(t,f(t))} \). Define, for all \( t \in [\underline{x}_1, \bar{x}_1] \),

\[
\ell_t := \min_{\omega \in \mathcal{X}} \omega_1 - t,
\]

subject to \( \omega_2 = f(t) - f'(t)(\omega_1 - t) \),

\[
\omega_2 + \frac{(t - \omega_1)(f(t) - \omega_2)}{s - \omega_1} \leq f(s), \quad \text{for all } s \in (\omega_1, \bar{x}_1],
\]

and

\[
\bar{l}_t := \max_{\omega \in \mathcal{X}} \omega_1 - t,
\]

subject to \( \omega_2 = f(t) - f'(t)(\omega_1 - t) \),

\[
\omega_2 + \frac{(t - \omega_1)(f(t) - \omega_2)}{s - \omega_1} \geq f(s), \quad \text{for all } s \in [\underline{x}_1, \omega_1).
\]

Notice that \( (t + \ell_t, f(t) - f'(t)\ell_t) \) and \( (t + \bar{l}_t, f(t) - f'(t)\bar{l}_t) \) are the points in \( \Gamma_{(t,f(t))} \) with the lowest and highest first coordinate. To see this, consider \( \omega \in \Gamma_{(t,f(t))} \) with \( t > \omega_1 \) (and thus \( f(t) - \omega_2 = -f'(t)(t - \omega_1) < 0 \)) and notice that, for \( s \leq \omega_1 \), we have \( f(s) \leq f(t) < \omega_2 \), and thus

\[
(t - \omega_1)(f(t) - \omega_2) < 0 \leq (s - \omega_1)(f(s) - \omega_2).
\]

Consequently, \( \omega \in \Omega \) with \( \omega_1 < t \) belongs to \( \Gamma_{(t,f(t))} \) if and only if

\[
\omega_2 = f(t) - f'(t)(\omega_1 - t),
\]

\[
\omega_2 + \frac{(t - \omega_1)(f(t) - \omega_2)}{s - \omega_1} \leq f(s), \quad \text{for all } s \in (\omega_1, \bar{x}_1].
\]

Since \( (t, f(t)) \in \Gamma_{(t,f(t))} \), it follows that \( (t + \ell_t, f(t) - f'(t)\ell_t) \) is indeed the point in \( \Gamma_{(t,f(t))} \) with the lowest first coordinate. An analogous argument shows that \( (t + \bar{l}_t, f(t) - f'(t)\bar{l}_t) \) is the point in \( \Gamma_{(t,f(t))} \) with the highest first coordinate. Finally, since, by Lemma 10, \( \Gamma_{(t,f(t))} \) is convex, it follows that

\[
\Gamma_{(t,f(t))} = \bar{I}_t := \{ \omega \in \Omega : \omega_1 = x_1 + l, \omega_2 = f(x_1) - f'(x_1)l, \ l \in \mathbb{I}(x_1, \bar{l}(x_1)) \}.
\]

It turns out that the above property also holds for \( x \in \text{supp}(\pi^*_X) \) with \( x_1 \in \{\underline{x}_1, \bar{x}_1\} \). However, the proof of that fact is significantly more complicated, and hence we relegate its proof till the end.

**Lemma 11.** \( \Gamma_{(t,f(t))} = \bar{I}_t \) for \( t \in \{\underline{x}_1, \bar{x}_1\} \).

**Proof.** See Appendix D.1.1.
By Lemma 11, we can conclude that $\Gamma_{(t,f(t))} = T_t$ for each $t \in [\bar{x}_1, \bar{x}_1]$. Since the projection of the contact set $\Gamma$ along the second coordinate is $X = \Omega$, it follows that $\Omega = \bigcup_{t \in [\bar{x}_1, \bar{x}_1]} T_t$. Define $I_t = \text{relint}(T_t)$, for $t \in [\bar{x}_1, \bar{x}_1]$.29 By part 2(b) in Lemma 10, for $t \neq s$, the open line segments $I_t$ and $I_s$ do not intersect. In fact, part 2(c) in Lemma 10 yields a stronger conclusion that, for $t \neq s$, the closed line segments $T_t$ and $T_s$ can intersect only at a common endpoint. By part 1 in Lemma 10, the price function $p_{\text{Gr}(f)}(\omega)$, as defined in Section 4.2, is differentiable at each point in $I_t$, but is not differentiable at an endpoint of $I_t$ if this endpoint lies in the interior of $\Omega$. Thus, as in the proof of Theorem 7, invoking Theorem 1 in Lang (1986) and Theorem 10.8 in Villani (2009), we conclude that $\Omega \setminus \bigcup_{t \in [\bar{x}_1, \bar{x}_1]} I_t$ has zero (Lebesgue) measure. In sum, we have established that there exists a collection $\{I_t\}_{t \in [\bar{x}_1, \bar{x}_1]}$ of open disjoint line segments that partition $\Omega$, up to a measure-zero set.

The first property then follows directly from the above characterization of the optimal signal $\pi^*$ and the assumption that $\text{supp}(\pi^*_X) = \text{Gr}(f)$. The second property follows from the definition of $T_t$. Moreover, the inclusion $I_t \subseteq \{\omega \in \Omega : \omega_2 = f(t) - f'(t)(\omega_1 - t)\}$, for $t \in [\bar{x}_1, \bar{x}_1]$, follows directly from the fact that $\Gamma_{(t,f(t))} = T_t$ for each $t \in [\bar{x}_1, \bar{x}_1]$. This finishes the proof of the proposition.

### D.1.1 Proof of Lemma 11

We start by proving yet another lemma.

**Lemma 12.** There exists $\varepsilon > 0$ such that

$$\omega_2 + \frac{(x_1 - \omega_1)(f(x_1) - \omega_2)}{y_1 - \omega_1} < f(y_1),$$

for all $x_1 \in [\bar{x}_1, \bar{x}_1]$, $\omega_1 \in [x_1 - \varepsilon, x_1]$, $y_1 \in (\omega_1, x_1) \cup (x_1, \bar{x}_1)$, and $\omega_2 = f(x_1) - f'(x_1)(\omega_1 - x_1)$.

**Proof.** Since $f'$ and $f''$ are continuous and $f' > 0$ on the compact set $[\bar{x}_1, \bar{x}_1]$, there exists $\varepsilon > 0$ such that $2h'(\hat{x}_1) + \varepsilon f''(\hat{x}_1) > 0$ for all $\hat{x}_1, \tilde{x}_1 \in [\bar{x}_1, \bar{x}_1]$. It is easy to see that

$$\omega_2 + \frac{(x_1 - \omega_1)(f(x_1) - \omega_2)}{y_1 - \omega_1} \leq f(x_1) + f'(x_1)\varepsilon - \frac{f'(x_1)\varepsilon^2}{y_1 - x_1 + \varepsilon},$$

for all $x_1 \in [\bar{x}_1, \bar{x}_1]$, $\omega_1 \in [x_1 - \varepsilon, x_1]$, and $y_1 \in (\omega_1, x_1) \cup (x_1, \bar{x}_1)$. Thus, it suffices to show that

$$f(x_1) + f'(x_1)\varepsilon - \frac{f'(x_1)\varepsilon^2}{y_1 - x_1 + \varepsilon} < f(y_1),$$

for all $x_1 \in [\bar{x}_1, \bar{x}_1]$ and $y_1 \in (x_1 - \varepsilon, x_1) \cup (x_1, \bar{x}_1)$.

---

29Note that $I_t$ is a point when $T_t$ is degenerate, since a point is a relatively open set.
Denote $f' = \min_{\bar{x}_1 \in [\underline{x}_1, \overline{x}_1]} f'(\bar{x}_1)$ and $f'' = \min_{\bar{x}_1 \in [\underline{x}_1, \overline{x}_1]} f''(\bar{x}_1)$. If $f'' \geq 0$, the inequality holds because $f(y_1)$ is convex in $y_1$ with derivative $f'(x_1)$ at $x_1$, while

$$f(x_1) + f'(x_1)\varepsilon - \frac{f'(x_1)\varepsilon^2}{y_1 - x_1 + \varepsilon}$$

is strictly concave in $y_1$ with derivative $f'(x_1)$. So assume that $f'' < 0$.

Since $f''(\bar{x}_1) \geq f''$ and $f'(\bar{x}_1) \geq f'$ for all $\bar{x}_1 \in [\underline{x}_1, \overline{x}_1]$, we have $f(y_1) \geq f(y_1)$ for all $y_1 \in [\underline{x}_1, \overline{x}_1]$ where

$$f(y_1) = \begin{cases} f(x_1) + f'(x_1)(y_1 - x_1) + \frac{f''(\varepsilon - (y_1 - x_1))}{2}(y_1 - x_1)^2, & y_1 - x_1 \leq \frac{f'(x_1) - f'}{-f''}, \\
 f(x_1) + f'(x_1)\frac{f'(x_1) - f'}{-f''} + \frac{f''}{2} \left(\frac{f'(x_1) - f'}{-f''}\right)^2 + f'(y_1 - x_1 - \frac{f'(x_1) - f'}{-f''}), & y_1 - x_1 > \frac{f'(x_1) - f'}{-f''}. \end{cases}$$

So it suffices to show that

$$f(x_1) + f'(x_1)\varepsilon - \frac{f'(x_1)\varepsilon^2}{y_1 - x_1 + \varepsilon} < f(y_1). \quad (D.1)$$

By direct calculation, for $y_1 - x_1 \in (-\varepsilon, 0)$, inequality $(D.1)$ is equivalent to $2f'(x_1) + f''(\varepsilon - (y_1 - x_1)) > 0$, which holds because $y_1 < x_1$ and $2f'(\hat{x}_1) + \varepsilon f''(\hat{x}_1) > 0$ for all $\hat{x}_1, \hat{x}_1 \in [\underline{x}_1, \overline{x}_1]$. For $y_1 - x_1 \in (0, (f'(x_1) - f')/(-f''))$, inequality $(D.1)$ is equivalent to $f'(x_1) + f' + f''\varepsilon > 0$, which holds because $2f'(\hat{x}_1) + \varepsilon f''(\hat{x}_1) > 0$ for all $\hat{x}_1, \hat{x}_1 \in [\underline{x}_1, \overline{x}_1]$. Finally, for $y_1 - x_1 > (f'(x_1) - f')/(-f'')$, inequality $(D.1)$ is equivalent to

$$\frac{(f'(x_1) - f')^2}{2(-f'')} - (y_1 - x_1 + \varepsilon) + f'(y_1 - x_1)(y_1 - x_1 + \varepsilon) - f'(x_1)(y_1 - x_1)\varepsilon > 0,$$

where the left hand side is quadratic in $y_1$ with derivative at $y_1 = x_1 + (f'(x_1) - f')/(-f'')$ being positive (and thus the left hand side is increasing in $y_1$) if and only if $3f' + f'(x_1) + 2f''\varepsilon > 0$, which holds because $2h'(\hat{x}_1) + \varepsilon f''(\hat{x}_1) > 0$ for all $\hat{x}_1, \hat{x}_1 \in [\underline{x}_1, \overline{x}_1]$. Thus, inequality $(D.1)$ holds for $y_1 - x_1 > (f'(x_1) - f')/(-f'')$, because it holds for $y_1 - x_1 = (f'(x_1) - f')/(-f'')$, as shown above.

We are now ready to prove Lemma 11. We will focus on the case $t = \underline{x}_1$ since the other case is fully analogous. The necessary Kuhn-Tucker condition yields $\omega_2 \leq f(\underline{x}_1) - f'(\underline{x}_1)(\omega_1 - \underline{x}_1)$ for all $\omega \in \Gamma(\underline{x}_1, f(\underline{x}_1))$. We claim that $\omega_2 \geq f(\underline{x}_1) - f'(\underline{x}_1)(\omega_1 - \underline{x}_1)$ for all $\omega \in X$, and thus $\omega_2 = f(\underline{x}_1) - f'(\underline{x}_1)(\omega_1 - \underline{x}_1)$ for all $\omega \in \Gamma(\underline{x}_1, f(\underline{x}_1))$, so $\Gamma(\underline{x}_1, f(\underline{x}_1)) = \Gamma(\underline{x}_1, f(\underline{x}_1)) = \Gamma(\underline{x}_1, f(\underline{x}_1))$, by the same argument as previously. Towards a contradiction, suppose that there exists $z \in X$ such that $z < f(\underline{x}_1) - f'(\underline{x}_1)(z_1 - \underline{x}_1)$ and $z_1 < x_1$ (the case $z_1 > x_1$ is analogous and omitted). Since $X$
is convex and the graph of $f$ is a maximal monotone set in $X$, it follows that $z_2 \geq f(x_1)$ and that there exists $\varepsilon > 0$ such that, for all $\omega_1 \in (x_1 - \varepsilon, x_1)$, points $(\omega_1, f(x_1) - f'(x_1)(\omega_1 - x_1))$ and $(\omega_1, f(x_1) - \iota(\omega_1 - x_1))$ with $\iota = (z_2 - f(x_1))/(x_1 - z_1) \in [0, f'(x_1))$ belong to $X$. It is easy to see that, for all $\omega_1 < x_1$ and $y_1 > x_1$, we have

$$\omega_2 - \iota(\omega_1 - x_1) - \iota\frac{(x_1 - \omega_1)^2}{y_1 - \omega_1} < \omega_2 - f'(x_1)(\omega_1 - x_1) - f'(x_1)\frac{(x_1 - \omega_1)^2}{y_1 - \omega_1}.$$ 

Thus, by Lemma 12, for sufficiently small $\varepsilon > 0$, points $(\omega_1, f(x_1) - f'(x_1)(\omega_1 - x_1))$ and $(\omega_1, f(x_1) - \iota(\omega_1 - x_1))$ belong to $\Gamma_x$. But then $\Gamma_x$ has a nonempty interior, and all points in the interior belong only to $\Gamma_x$, by Lemma 10. Consequently, since $\mu_0$ has a strictly positive density on $X$,

$$\int_{\Gamma_x} (\omega_2 - f(x_1) - f'(x_1)(\omega_1 - x_1))d\mu_0(\omega) = \int_{\text{int}(\Gamma_x)} (\omega_2 - f(x_1) - f'(x_1)(\omega_1 - x_1))d\mu_0(\omega) < 0,$$

as the boundary of the convex set $\Gamma_x$ has zero Lebesgue measure, by Theorem 1 in Lang (1986), and the integrand is strictly negative on the interior of $\Gamma_x$, as implied by the Kuhn-Tucker condition. This shows that any $\pi$ supported on $\Gamma$ cannot be in $\Pi(\mu_0)$, as it violates the second constraint in the definition of $\Pi(\mu_0)$. A contradiction.

### D.1.2 An explicit formula for property 1 in Proposition 2

Let $g$ denote the density of the prior distribution $\mu_0$ on $\Omega$.

**Lemma 13.** Property 1 in Proposition 2 holds if and only if for almost all $t \in [\underline{x}_1, \overline{x}_1]$,

$$\int_{\underline{x}_1}^{\overline{x}_1} l(2f'(t) - f''(t)l)g(t + l, f(t) - f'(t)l)dl = 0.$$ 

**Proof.** Define $\widetilde{\Omega} = \bigcup_{t \in [\underline{x}_1, \overline{x}_1]} I_t$ and recall that $\mu_0(\widetilde{\Omega}) = 1$. By footnote 22, $E[\omega | \omega \in I_t] = (t, f(t))$ is equivalent to $E[\omega | \omega \in I_t] = t$. Let $G$ be the distribution function of the posterior mean of $\omega_1$ induced by $\pi^*$, so that, for all $t \in [\underline{x}_1, \overline{x}_1]$, we have

$$G(t) = \int_{\omega_1 \in [\underline{x}_1, \overline{x}_1]} g(\omega_1, \omega_2)d\omega_1d\omega_2.$$ 

By the definition of the conditional expectation, property 1 in Proposition 2 holds if and
only if, for all \( t \in [\underline{x}_1, \overline{x}_1] \), we have
\[
\int_{\underline{x}_1}^{t} \, s \, dG(s) = \int_{\bigcup_{s \in [\underline{x}_1, t]} I_s} \omega_1 g(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2.
\]

Consider a change of variables on \( \tilde{\Omega} \) given by the following transformation: \((\omega_1, \omega_2) = (t + l, f(t) - f'(t)l)\) where \( t \in [\underline{x}_1, \overline{x}_1] \) and \( l \in (l, \overline{l}) \). This transformation is diffeomorphism, as (1) it is one-to-one and onto \( \tilde{\Omega} \), because \( I_t \cap I_s = \emptyset \) for \( t \neq s \), (2) it is continuously differentiable, because \( f \) is a twice continuously differentiable function, and (3) the Jacobian determinant is negative on \( \tilde{\Omega} \),
\[
J(t, l) = \det \begin{pmatrix} \frac{\partial \omega_1}{\partial t} & \frac{\partial \omega_2}{\partial t} \\ \frac{\partial \omega_1}{\partial l} & \frac{\partial \omega_2}{\partial l} \end{pmatrix} = \begin{pmatrix} 1 & f'(t) - f''(t)l \\ 1 & -f'(t) \end{pmatrix} = -(2f'(t) - f''(t)l) < 0,
\]
where the inequality follows from the second order condition for the second property in Proposition 2 on the (relatively) open set \( I_t \). Thus, by the Change of Variables Theorem (Theorem 13.49 in Aliprantis and Border, 2006 and Remark 1.3 in Villani, 2009), we have, for all \( t \in [\underline{x}_1, \overline{x}_1] \),
\[
G(t) = \int_{\underline{x}_1}^{t} \int_{L_s}^{l_s} \left| J(s, l) \right| g(s + l, f(s) - f'(s)l) \, dl \, ds
\]
\[
= \int_{\underline{x}_1}^{t} \int_{L_s}^{l_s} (2f'(s) - f''(s)l) g(s + l, f(s) - f'(s)l) \, dl \, ds,
\]
and
\[
\int_{\underline{x}_1}^{t} \, s \, dG(s) = \int_{\underline{x}_1}^{t} \int_{L_s}^{l_s} (s + l) \left| J(s, l) \right| g(s + l, f(s) - f'(s)l) \, dl \, ds
\]
\[
= \int_{\underline{x}_1}^{t} \int_{L_s}^{l_s} (s + l)(2f'(s) - f''(s)l) g(s + l, f(s) - f'(s)l) \, dl \, ds.
\]
Substituting \( G \) from the first equation to the last equation, we have, for all \( t \in [\underline{x}_1, \overline{x}_1] \),
\[
\int_{\underline{x}_1}^{t} \int_{L_s}^{l_s} l(2f'(s) - f''(s)l) g(s + l, f(s) - f'(s)l) \, dl \, ds = 0,
\]
which holds if and only if the inner integral is 0 for almost all \( s \in [\underline{x}_1, \overline{x}_1] \).