Online Appendix of "Mechanism Design with Aftermarkets: Optimal Mechanisms under Binary Actions"

Piotr Dworczak

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OA.1 The optimal mechanism for Section 5.1 in the case $N \geq 2$

In this appendix, I formally derive the surplus-maximizing mechanism for the resale example from Section 5.1 in the case of multiple agents ($N \geq 2$), under the assumptions of Appendix A.

Claim OA.1. *Suppose that* $\mathbb{E}_f[\theta_N^{(1)}] < (\theta_H - \theta_L)/(v - \theta_L)$, *where* $\theta_N^{(1)}$ *denotes the first order statistic of* $(\theta_1, ..., \theta_N)$.[1] *Then, one of the following two mechanisms is optimal when* $N > 1:$

(a) For some type $\theta^\star \in (0, 1)$[2]

$$x(\theta) = \begin{cases} (1/N)F^{N-1}(\theta^\star) & \theta < \theta^\star \\ F^{N-1}(\theta) & \theta \geq \theta^\star, \end{cases}$$

and

$$x(\theta)(1 - \pi_h(\theta)) = (1/N)F^{N-1}(\theta^\star);$$

(b) $x(\theta) = F^{N-1}(\theta)\mathbf{1}_{\{\theta \geq r^\star\}}$ *and* $\pi_h(\theta) = 1$, *where* $r^\star$ *is defined by*

$$\mathbb{E}_f[\theta_N^{(1)} \,|\, \theta_N^{(1)} \geq r^\star] = \frac{\theta_H - \theta_L}{v - \theta_L}.$$

Mechanism (a) is optimal whenever a regularity condition (see equation OA.1.30) holds. The regularity condition is satisfied, for example, by all distributions $F(\theta) = \theta^\kappa$ *for* $\kappa > 0$.

[1] If this assumption does not hold, then the first-best is implementable by any auction that allocates the good to the highest type and reveals no information.

[2] See equation (OA.1.27).

I focus on case *(a)* in the discussion. (Appendix OA.1 contains the discussion of case *b*.) The following indirect implementation of the optimal mechanism from case *(a)* is possible. The designer first selects, secretly and uniformly at random, a "favored" agent. Next, the designer names a price $p^\star$, and agents simultaneously accept or reject. If all non-favored agents reject, then regardless of the decision of the favored agent, she receives the object for free. In the opposite case, the designer runs a tie-breaking auction with reserve price $p^\star$ among agents who accepted (if only one non-favored agent accepted, she pays $p^\star$). The designer reveals a binary signal that states whether the auction took place or not (equivalently, whether the price was 0 or strictly positive). Note that this is a cutoff mechanism because the signal is determined by the second highest type and the outcome of the randomization that selects the favored agent.

The price $p^\star$ is chosen so that exactly types above $\theta^\star$ want to accept it. Then, since the favored agent is chosen uniformly at random, the probability of acquiring the object for an agent with type $\theta < \theta^\star$ is exactly $1/N$ times the probability $F^{N-1}(\theta^\star)$ that all other agents also have types below $\theta^\star$. For an agent with type $\theta \geq \theta^\star$, the probability is equal to the probability of being the highest type, $F^{N-1}(\theta)$. Finally, the low signal ("the auction did not take place") is sent with probability one conditional on the winner's type being below $\theta^\star$; and with probability $(1/N)F^{N-1}(\theta^\star)$ otherwise (the auction does not take place in this case only if that agent is favored and all other agents have values below $\theta^\star$).

Intuitively, in order to induce a high price in the aftermarket with positive probability, the designer runs a two-step procedure, and announces whether the second step (the auction) was reached. The auction is a signal of high value of the winner of the object. The threshold type $\theta^\star$ is set in such a way that conditional on announcing that the auction took place, the third party is indifferent between the high and low price (and quotes the high price). At the same time, for any type θ, there is always some positive probability that the low signal is sent (that induces a low price in the aftermarket). This provides the necessary separation between the high and low types since high types derive a higher utility from acquiring the object only when the aftermarket price is low.

To gain intuition for the optimal design of the allocation rule, suppose that no agent accepts the initial price $p^\star$. Conditional on this event, the low price θ_L will be offered in the second stage. To maximize the probability of resale (which is consistent with social surplus maximization because the third party has a higher value than the agent), the designer should allocate to the lowest type. However, incentive-compatibility constraints make it impossible to allocate to low types more often than to high types. Therefore, the mechanism allocates the object by a uniform lottery (through the selection of the favored agent).

Proof of Claim OA.1 By Theorem 4 in Appendix A, a cutoff mechanism is optimal. I let $y(\theta) = x(\theta)(1-\pi_h(\theta))$ denote the probability that the good is allocated and a low price is recommended. In a cutoff mechanism, by Fact 2, both $x(\theta)$ and $y(\theta)$ are non-decreasing in θ. I let $\phi(\theta) \equiv (v-\theta_H) - (v-\theta_L)(1-\theta)$. I consider a relaxed problem (by omitting constraint OB_l), and then verify that the solution is feasible. The relaxed problem of maximizing surplus takes the form:

$$\max_{x,\, y} \left\{ v \int_0^1 x(\theta) f(\theta) d\theta - (v-\theta_H) \int_0^1 \theta y(\theta) f(\theta) d\theta \right\} \tag{OA.1.1}$$

subject to

$$0 \leq y(\theta) \leq x(\theta) \leq 1,\ \forall \theta, \tag{OA.1.2}$$

$$x,\ y \text{ are non-decreasing}, \tag{OA.1.3}$$

$$\int_0^1 x(\theta)\phi(\theta) f(\theta) d\theta \geq \int_0^1 y(\theta)\phi(\theta) f(\theta) d\theta, \tag{OA.1.4}$$

$$\int_\tau^1 x(\theta) f(\theta) d\theta \leq \frac{1}{N}(1-F^N(\tau)),\ \forall \tau \in [0,\, 1]. \tag{OA.1.5}$$

The objective function (OA.1.1) is equal to per-agent total expected surplus.

I will solve the problem (OA.1.1)-(OA.1.5) in two steps. In the first step, I optimize over y treating x as given. In the second, I optimize over x.

Step 1. Optimization over y for fixed x. For a fixed non-decreasing function x, the first-step problem can be expressed as (terms in the objective not depending on y can be omitted):

$$\min_y \int_0^1 \theta y(\theta) f(\theta) d\theta \tag{OA.1.6}$$

subject to

$$0 \leq y(\theta) \leq x(\theta),\ \forall \theta, \tag{OA.1.7}$$

$$y \text{ is non-decreasing}, \tag{OA.1.8}$$

$$\int_0^1 x(\theta)\phi(\theta) f(\theta) d\theta \geq \int_0^1 y(\theta)\phi(\theta) f(\theta) d\theta. \tag{OA.1.9}$$

This problem has been solved in the proof of Theorem 2. The optimal solution takes the form

$$y^\star(\theta) = \begin{cases} x(\theta) & \theta < \theta^\star \\ \bar{x} & \theta \geq \theta^\star \end{cases}, \tag{OA.1.10}$$

for some $\theta^\star \in [0, 1]$, and $\bar{x} \in [x^-(\theta^\star), x^+(\theta^\star)]$, where $x^-(\theta^\star)$ and $x^+(\theta^\star)$ denote the left and the right limit of x at $\theta^\star$, respectively. If x is continuous at $\theta^\star$, $\bar{x} = x(\theta^\star)$. Indeed, $y^\star$ crosses any other y satisfying (OA.1.7), (OA.1.8), and $\int_0^1 y(\theta)f(\theta)d\theta = \int_0^1 y^\star(\theta)f(\theta)d\theta$ once and from above, so it is first-order stochastically dominated (in the sense defined in Lemma 2) by any such y.

There are two cases, depending on the properties of the fixed function x. In case *(1)*, x satisfies $\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq 0$. Then $y^\star(\theta) = 0$, for all θ ($\bar{x} = 0$, $\theta^\star = 0$), achieves the global minimum. In case *(2)*, $\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta < 0$, and $\bar{x} \in [x^-(\theta^\star), x^+(\theta^\star)]$ is pinned down by a binding constraint (OA.1.9):

$$\int_{\theta^\star}^1 (x(\theta) - \bar{x})\phi(\theta)f(\theta)d\theta = 0.$$

If there are multiple $(\bar{x}, \theta^\star)$ satisfying these restrictions, then it must be that $x(\theta) = \bar{x}$ in some (possibly one-sided) neighborhood of $\theta^\star$, so $y^\star$ is defined uniquely.

Step 2. Optimization over x. Having solved for the optimal y given x, in step 2, I optimize over x. I proceed by finding the optimal x separately for cases *(1)* and *(2)* defined above. At the end, I compare the two constrained optima to find the globally optimal mechanism.

Case 1: $\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq 0$.

Because in this case a high price is always quoted in the second stage, the problem (OA.1.1) - (OA.1.5) becomes

$$\max_x \int_0^1 x(\theta)f(\theta)d\theta \tag{OA.1.11}$$

subject to

$$0 \leq x(\theta) \leq 1, \; \forall\theta, \tag{OA.1.12}$$

$$x \text{ is non-decreasing,} \tag{OA.1.13}$$

$$\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq 0, \tag{OA.1.14}$$

$$\int_\tau^1 x(\theta)f(\theta)d\theta \leq \frac{1}{N}(1 - F^N(\tau)), \; \forall\tau \in [0, 1]. \tag{OA.1.15}$$

By a similar argument as before, an optimal x should first-order stochastically dominate any x' satisfying conditions (OA.1.12), (OA.1.13), and (OA.1.15). Informally, optimality requires that x "shifts mass as much as possible to the right," subject to constraints. Thus, an optimal x satisfies (OA.1.15) with equality for all $\tau \geq \beta$, and is zero on $[0, \beta]$, where

$\beta \geq 0$ is the smallest number such that constraint (OA.1.14) holds. Either

$$\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq 0, \tag{OA.1.16}$$

in which case $\beta = 0$, or $\beta > 0$ is defined by

$$\int_\beta^1 x(\theta)\phi(\theta)f(\theta)d\theta = 0. \tag{OA.1.17}$$

Since x satisfies the Matthews-Border condition (OA.1.15) with equality on $[\beta, 1]$, it is induced by a joint rule that gives the good to the agent with the highest type, conditional on at least one agent having a type above β. That is

$$x(\theta) = \begin{cases} 0 & \theta < \beta \\ F^{N-1}(\theta) & \theta \geq \beta \end{cases}.$$

If condition (OA.1.16) holds, $\beta = 0$. With the above x, (OA.1.16) is equivalent to $\mathbb{E}_f[\theta_N^{(1)}] \geq (\theta_H - \theta_L)/(v - \theta_L)$. Thus, under this condition, the optimal mechanism for case *(1)* is an efficient auction with no information revelation.

If condition (OA.1.16) does not hold, then $\beta > 0$, and the mechanism is an auction with a positive reserve price and no information revelation.

Case 2: $\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta < 0$.

In case *(2)*, problem (OA.1.1) - (OA.1.5), in a relaxed version, becomes

$$\max_{x,\theta^\star,\bar{x}} \left\{ \int_0^{\theta^\star} [v - (v-\theta_H)\theta]x(\theta)f(\theta)d\theta + v\int_{\theta^\star}^1 x(\theta)f(\theta)d\theta - \bar{x}(v-\theta_H)\int_{\theta^\star}^1 \theta f(\theta)d\theta \right\} \tag{OA.1.18}$$

subject to

$$0 \leq x(\theta) \leq \bar{x},\ \forall \theta \leq \theta^\star, \tag{OA.1.19}$$

$$\bar{x} \leq x(\theta) \leq 1,\ \forall \theta \geq \theta^\star, \tag{OA.1.20}$$

$$x \text{ is non-decreasing}, \tag{OA.1.21}$$

$$\int_{\theta^\star}^1 (x(\theta) - \bar{x})\phi(\theta)f(\theta)d\theta \geq 0, \tag{OA.1.22}$$

$$\int_\tau^1 x(\theta)f(\theta)d\theta \leq \frac{1}{N}(1 - F^N(\tau)),\ \forall \tau \in [0, 1]. \tag{OA.1.23}$$

The problem is relaxed because condition (OA.1.22) should in fact be an equality. For any fixed $\bar{x}$ and $\theta^\star$, x should be maximized point-wise on $[\theta^\star,\, 1]$, which means that Matthews-Border condition (OA.1.23) will bind everywhere on $[\theta^\star,\, 1]$ (point-wise maximization in this interval does not interact with any other constraint). Thus, $x(\theta) = F^{N-1}(\theta)$ for $\theta \in [\theta^\star,\, 1]$.

Now, consider x on $[0,\, \theta^\star]$. In the objective function, x multiplies the function $v-(v-\theta_H)\theta$ that is positive decreasing. Because x cannot be decreasing (due to constraint OA.1.21), the optimal x must be constant on $[0,\, \theta^\star]$, equal to some $\gamma \leq \bar{x}$ such that condition (OA.1.23) is satisfied. Overall, the problem reduces to

$$\max_{\gamma\leq\bar{x},\, \theta^\star} \left\{\gamma \int_0^{\theta^\star} [v-(v-\theta_H)\theta] f(\theta) d\theta - \bar{x}(v-\theta_H) \int_{\theta^\star}^1 \theta f(\theta) d\theta \right\} \tag{OA.1.24}$$

subject to

$$\int_{\theta^\star}^1 (F^{N-1}(\theta) - \bar{x})\phi(\theta) f(\theta) d\theta \geq 0, \tag{OA.1.25}$$

$$\gamma(F(\theta^\star) - t) \leq \frac{1}{N}(F^N(\theta^\star) - t^N),\ \forall t \in [0,\, F(\theta^\star)]. \tag{OA.1.26}$$

Constraint (OA.1.26) can only be binding at the ends of the interval because the function on the left hand side is affine in t, and the function on the right is concave in t. Thus, (OA.1.26) becomes $\gamma \leq (1/N)F^{N-1}(\theta^\star)$. Because the objective function is increasing in γ, it is optimal to set γ to its upper bound: $\gamma = \max(\bar{x},\, (1/N)F^{N-1}(\theta^\star))$. The objective is also strictly increasing in $\theta^\star$. This means that constraint (OA.1.25) must bind. Suppose that $\bar{x} > (1/N)F^{N-1}(\theta^\star)$. Then, by decreasing $\bar{x}$ slightly, we increase the objective function and preserve constraint (OA.1.25). Thus, $\gamma = \bar{x} = (1/N)F^{N-1}(\theta^\star)$ at the optimal solution. Because the objective function is increasing in $\theta^\star$, the solution is obtained by finding the highest $\theta^\star$ for which equation (OA.1.25) binds, that is,

$$\int_{\theta^\star}^1 (F^{N-1}(\theta) - \frac{1}{N}F^{N-1}(\theta^\star))\phi(\theta) f(\theta) d\theta = 0. \tag{OA.1.27}$$

Because we are in case *(2)*, by assumption, (OA.1.25) is violated with $\theta^\star = 0$. Thus, $\theta^\star$ is strictly positive, and hence $\bar{x}$ is also strictly positive.

Summarizing, the solution takes the form

$$x(\theta) = \begin{cases} (1/N)F^{N-1}(\theta^\star) & \theta < \theta^\star \\ F^{N-1}(\theta) & \theta \geq \theta^\star \end{cases},$$

and $y(\theta) = (1/N)F^{N-1}(\theta^\star)$, for all θ. The functions x and y are easily seen to be feasible for the original (unrelaxed) problem. From this, one can directly derive the form of the optimal disclosure rule π_h from the statement of Claim OA.1 (point a).

Comparing case *(1)* and case *(2)*.

Assumption (5.2) in the current setting means that $\mathbb{E}_f[\theta_N^{(1)}] < (\theta_H - \theta_L)/(v - \theta_L)$. When $\mathbb{E}_f[\theta_N^{(1)}] < (\theta_H - \theta_L)/(v - \theta_L)$, then $\beta > 0$ in case *(1)*, so the optimal mechanism in case *(1)* can be implemented as an auction with a reserve price and no information revelation. That mechanism corresponds to the mechanism described in point (b) in Claim OA.1 (with $r^\star = \beta$). I have shown that the optimal mechanism is either the one form case *(1)* (corresponding to point (b) in Claim OA.1) or the one from case *(2)* (corresponding to point (a) in Claim OA.1). What remains to be shown is that the mechanism from case *(2)* is optimal under a regularity condition to be defined.

Given the optimal mechanism for case *(1)*, I will construct an alternative mechanism that is feasible and yields a strictly higher value of objective (OA.1.1) under a regularity condition. This will mean that the mechanism from case *(2)* must be optimal.

Fix the optimal mechanism in case *(1)* with $\beta > 0$. Consider an alternative mechanism, indexed by $\epsilon \geq 0$ with $y_\epsilon(\theta) = \epsilon$, for all θ, and

$$x_\epsilon(\theta) = \begin{cases} \epsilon & \theta < \beta_\epsilon \\ F^{N-1}(\theta) & \theta \geq \beta_\epsilon \end{cases},$$

where β_ϵ is defined by

$$\int_{\beta_\epsilon}^{1} (F^{N-1}(\theta) - \epsilon)\phi(\theta) f(\theta) d\theta = 0. \tag{OA.1.28}$$

At $\epsilon = 0$, $\beta(0) = \beta > 0$ (because β is defined by equation OA.1.17), so for small ϵ, there exists a strictly positive solution β_ϵ to equation (OA.1.28). Intuitively, I constructed a mechanism that takes a small step ϵ towards the optimal mechanism from case *(2)*. For small enough ϵ, constraint (OA.1.5) holds, and constraint (OA.1.4) is satisfied with equality given that equation (OA.1.28) holds. Thus, the pair (x_ϵ, y_ϵ) is feasible for small enough ϵ.

For $\epsilon = 0$, (x_0, y_0) is the optimal solution for case *(1)*. Therefore, it is enough to show that the objective function (OA.1.1) is strictly increasing in ϵ in the neighborhood of $\epsilon = 0$. Because the objective function is differentiable in ϵ (in particular, β_ϵ is differentiable in ϵ by the implicit function theorem), it is enough to show that the derivative is strictly positive at 0. Using the implicit function theorem to differentiate β_ϵ using equation (OA.1.28), the right derivative of (OA.1.1) under the mechanism (x_ϵ, y_ϵ) at $\epsilon = 0$ can be shown to be

$$vF(\beta) + v\frac{\int_\beta^1 \phi(\theta) f(\theta) d\theta}{\phi(\beta)} - (v - \theta_H)\int_0^1 \theta f(\theta) d\theta. \tag{OA.1.29}$$

Equation (OA.1.17) defining β can be written as

$$\frac{v-\theta_H}{v-\theta_L} = 1 - \mathbb{E}_f[\theta_N^{(1)} | \, \theta_N^{(1)} \geq \beta] \equiv 1 - \theta_\beta^{(1)}.$$

Given that $\theta_L > 0$, we have $(v-\theta_H)/v < 1-\theta_\beta^{(1)}$. Moreover, recalling that $\phi(\theta) \equiv (v - \theta_H) - (v-\theta_L)(1-\theta)$, we have $\phi(\theta) = (v-\theta_L)[\theta - \theta_\beta^{(1)}]$. Using these relations, to show that (OA.1.29) is strictly positive, it is enough to show that

$$F(\beta) \geq \frac{\int_\beta^1 (\theta - \theta_\beta^{(1)}) f(\theta) d\theta}{\theta_\beta^{(1)} - \beta} + (1-\theta_\beta^{(1)}) \int_0^1 \theta f(\theta) d\theta.$$

Rearranging terms, we get

$$\theta_\beta^{(1)} - (\theta_\beta^{(1)} - \beta)(1-\theta_\beta^{(1)}) \int_0^1 \theta f(\theta) d\theta - \beta F(\beta) \geq \int_\beta^1 \theta f(\theta) d\theta.$$

Using integration by parts, and rearranging again,

$$(1-\theta_\beta^{(1)}) \left[1 + (\theta_\beta^{(1)} - \beta) \int_0^1 \theta f(\theta) d\theta \right] \leq \int_\beta^1 F(\theta) d\theta. \tag{OA.1.30}$$

If inequality (OA.1.30) holds for all $\beta \in [0, 1]$, I will say that the distribution F satisfies the *regularity condition*. Under the regularity condition, I have shown that the mechanism from case *(1)* cannot be optimal, therefore the mechanism from case *(2)* must be optimal.

In the remainder of the proof, I show that $F(\theta) = \theta^\kappa$ satisfies the regularity condition for any $\kappa > 0$. I will show that a more restrictive inequality holds:

$$\int_\beta^1 F(\theta) d\theta - (1 - \theta_{\beta,2}^{(1)}) \left[1 + (1-\beta) \int_0^1 \theta f(\theta) d\theta \right] \geq 0,$$

where $\theta_{\beta,2}^{(1)}$ denotes the expectation of the first order statistic conditional on exceeding β when $N = 2$ (the smaller N, the harder it is to satisfy OA.1.30). By brute-force calculation, one can check that the left hand side of the above inequality is a concave function of β. Thus, it is enough to check that the inequality holds at the two endpoints. When $\beta = 0$, we have

$$\int_0^1 F(\theta) d\theta - (1-\theta_{0,2}^{(1)}) \left[1 + \int_0^1 \theta f(\theta) d\theta \right] = \frac{1}{1+\kappa} - \left(1 - \frac{2\kappa}{2\kappa+1}\right)\left(1 + \frac{\kappa}{\kappa+1}\right) = 0.$$

On the other hand, for $\beta = 1$, we have $\theta_{\beta,2}^{(1)} = 1$, and the inequality is trivially satisfied.

Discussion – what if the regularity condition fails?

In this subsection, I briefly explain the trade-off between the optimal mechanism when the regularity condition holds (point a of Claim OA.1), and the mechanism that may be optimal when the condition fails (point b of Claim OA.1).

The former of the two mechanisms uses a non-trivial announcement policy. Using two signals is beneficial because it allows to always allocate the object in the mechanism while still inducing the high price in the aftermarket under the high signal. However, in a cutoff mechanism, the low signal has to be sent for higher types with at least the probability that it is sent for lower types (by Fact 2). Thus, the low signal is sent with positive probability for types above the threshold $\theta^\star$. Under the low signal, these types will not resell with a relatively high probability (because the price in the aftermarket is low, and the probability of high value is relatively high for these types). An alternative mechanism is to only send the high signal (and hence always induce a high price conditional on allocating the object) at the cost of not allocating the good to low types. The comparison between the two mechanisms depends on the shape of the distribution F. If F fails the regularity condition, the latter mechanism may sometimes be optimal.