Online Appendix to “The Effects of Post-Auction Bargaining between Bidders”

Piotr Dworczak

March 1, 2015

Abstract

This Online Appendix contains the characterization of equilibria that arise in the setting of Section 4 of the paper for $\Delta \neq 1$ (Appendix D). I present the derivation of these equilibria in Appendix E. Propositions 1-3 and 7 are proved as a special case. Appendix F provides proofs of Propositions 5 and 6.

D Appendix D - Equilibrium constructions

In this Appendix I characterize equilibria of the auction-bargaining game in the setting of Section 4 of the paper, allowing the parameter Δ to differ from 1. Appendix E provides proofs of these results, thereby proving Propositions 1-3 from Section 4 of the paper and Proposition 7 from Appendix B as a special case.

D.1 Equilibria in a second-price auction with no announcement

I first analyze a second-price auction with no announcement by the auctioneer.

When s is sufficiently small, the bargaining stage has no impact on equilibrium bidding strategies.

Case 1. When either

1. $\Delta \geq 1$ and $s \leq \frac{2}{3}$, or
2. $\Delta \in [\frac{1}{2}, 1)$ and $s \leq \frac{2}{2\Delta+1}$,

then bidding the true value in the auction (along with sequentially rational behavior of players in the bargaining stage) constitutes an equilibrium. ■

The boundary value of s is determined by the condition that the low type is exactly indifferent between playing her equilibrium strategy and deviating to bidding m (in order to get a strictly profitable offer from the high type in the bargaining stage).

Case 2. When either

1. $\Delta \geq 1$ and $\frac{2}{3} < s \leq 2$, or
2. $\Delta \in [\frac{1}{2}, 1)$ and $\frac{2}{2\Delta+1} < s \leq 2$,

then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type randomizes between l and $b^{\star}(s)$ with probabilities $1 - \theta(s)$, and $\theta(s)$, respectively.

- The medium type bids $b^\star(s)$ with probability one.
- The high type bids h with probability one.

We have

$$b^\star(s) = \begin{cases} m + \left(\frac{3}{8}s - \frac{1}{4}\right)(m-l) & \text{if } \Delta \geq 1 \\ m + \frac{1}{4}\left(2s - \frac{s+2}{\Delta+1}\right)(m-l) & \text{if } \Delta \in [\frac{1}{2}, 1) \end{cases}$$

and

$$\theta(s) = \begin{cases} \frac{3s-2}{2+s} & \text{if } \Delta \geq 1 \\ \frac{s(2\Delta+1)-2}{2+s} & \text{if } \Delta \in [\frac{1}{2}, 1) \end{cases}.$$

■

The (upper) boundary value of s is determined by the condition that the high type is exactly indifferent between offering l and m when observing the bid $b^\star(s)$. In the case $\Delta > 1$, the high type sells for m under the prior, so she continues to offer to sell for m even when the atom $b^\star(s)$ is chosen with probability one by the low type. In this case $\theta(s)$ converges to one as s goes to 2, and for larger s we have Case 3 analyzed below. When $\Delta < 1$, the high type sells for l under the prior, so the medium type has to bid $b^\star(s)$ with sufficiently high probability relative to the low type if the high type is to offer m after seeing $b^\star(s)$. In that case $\theta(s)$ converges to Δ as s goes to 2, and for large s we have Case 4 discussed next.

Case 3. When $\Delta \geq 1$ and $2 < s < 4\Delta$, then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type bids $b^\star(s)$ with probability one.
- The medium type bids $b^\star(s)$ with probability one.
- The high type bids h with probability one.

We have

$$b^\star(s) = m + \frac{1}{4}s(m-l).$$

■

As s goes to 4Δ, $b^\star(s)$ converges to h. Therefore, the expected revenue for the auctioneer approaches h for all $\Delta \geq 1$ when s is sufficiently high.

Case 4. When $\Delta \in [1/2, 1)$ and $2 < s < 2 + 2\Delta$, then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type randomizes between l and $b^\star(s)$ with probabilities $1-\Delta$, and Δ, respectively.
- The medium type bids $b^\star(s)$ with probability one.
- The high type bids h with probability one.

We have

$$b^\star(s) = m + \frac{1}{2}\frac{\Delta}{\Delta+1}s(m-l).$$

After observing a bid $b^\star(s)$ in the auction, the high type is indifferent between offering l and m, and offers to buy for m with conditional probability $(1-\Delta)/2 + (1+\Delta)/s$. ■

As s goes to $2 + 2\Delta$, $b^\star(s)$ converges to h. However, the expected revenue for the auctioneer will no longer converge to h. Instead, it converges to $h-(1-\Delta)l/3$. Even in the limit, the low type bids the atom at l with probability $1-\Delta$. This is necessary so that conditional on observing $b^\star(s)$, the high type is indifferent between the two offers. To keep the low type indifferent between l and $b^\star(s)$, the high type offers m after observing $b^\star(s)$ with proabaility that is decreasing with s.

D.2 Equilibria in a first-price auction with revelation of the winning bid

Case 1. When either

1. $\Delta \in [1,\, 2]$ and $s < 1 + \frac{3}{2\Delta - 1}$, or
2. $\Delta \in [\frac{1}{2},\, 1)$ and $s < \frac{4(\Delta+1)}{\sqrt{(1-\Delta)(3\Delta+1)}+3\Delta-1}$,

then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type bids l with probability one.
- The medium type bids according to a continuous distribution function $\underline{F}_m(b)$ on $(l,\, \underline{b}_h(s))$ with probability $1-\mu(s)$, and according to a continuous distribution function $\bar{F}_m(b)$ on $(\underline{b}_h(s),\, \bar{b}_m(s))$ with probability $\mu(s)$.
- The high type bids according to a continuous distribution function $\underline{F}_h(b)$ on $(\underline{b}_h(s),\, \bar{b}_m(s))$ with probability $\eta(s)$, and according to a continuous distribution function $\bar{F}_h(b)$ on $(\bar{b}_m(s),\, \bar{b}_h(s))$ with probability $1-\eta(s)$.

After observing a bid $b \in (\underline{b}_h(s),\, \bar{b}_m(s))$ in the auction, the low type is indifferent between offering m and h, and offers m with conditional probability $\gamma(b;\, s)$ with $\gamma(\underline{b}_h(s);\, s) = 1$ and $\gamma(\bar{b}_m(s);\, s) = 0$.

We have

$$\mu(s) = \begin{cases} \frac{s}{(2-s)\Delta+s+2} & \text{if } s \geq \frac{2(\Delta^2-1)}{\Delta^2-\Delta+1} \text{ and } \Delta \in [1,\, 2] \\ \frac{s\Delta}{2\Delta+s+2} & \text{if } s < \frac{2(\Delta^2-1)}{\Delta^2-\Delta+1} \text{ and } \Delta \in [1,\, 2] \\ \frac{s\Delta}{(2-s)\Delta+s+2} & \text{if } \Delta \in [\frac{1}{2},\, 1) \end{cases},$$

$$\eta(s) = \frac{\mu(s)}{\Delta},$$

$$\underline{F}_m(b) = \frac{b-l}{(m-b)(1-\mu(s))},$$

$$\bar{F}_m(b) = \underline{F}_h(b) = \frac{\Delta(4-2\mu(s))(b-\underline{b}_h(s))}{2(\Delta+1)\mu(s)(m-b)-\mu(s)s(h-m)},$$

$$\bar{F}_h(b) = \frac{2+\eta(s)}{1-\eta(s)}\,\frac{b-\bar{b}_m(s)}{h-b},$$

$$\underline{b}_h(s) = \frac{1-\mu(s)}{2-\mu(s)}\,m + \frac{1}{2-\mu(s)}\,l,$$

$$\bar{b}_m(s) = m - \frac{(2+\mu(s)s)\Delta}{2\mu(s)+4\Delta}(m-l),$$

$$\bar{b}_h(s) = \frac{1-\eta(s)}{3}\,h + \frac{2+\eta(s)}{3}\,\bar{b}_m(s).$$

■

As s converges to the upper boundary, the inequality $\underline{b}_h(s) < \bar{b}_m(s)$ becomes binding. In Case 2 (below) I construct an equilibrium in which there is an atom in the distribution at $\underline{b}_h(s) = \bar{b}_m(s)$.

It is worth mentioning why we need to consider subcases to determine the value of $\mu(s)$ when $\Delta \geq 1$. When $\Delta > 1$, the high type offers to buy for m under the prior when she makes an offer in the bargaining stage. When the high type wins the auction when bidding in the range $(\bar{b}_m(s),\, \bar{b}_h(s))$, the posterior probability of a low type goes up. This causes the high type to change her offer to l if she wins with a sufficiently low bid in this range. If there exists $b^\star(s) \in (\bar{b}_m(s),\, \bar{b}_h(s))$ such that $\bar{F}_m(b^\star(s)) = 1 - (\Delta-1)/(\Delta\mu(s))$ (the existence or not of such $b^\star(s)$ delineates the two subcases), the high type offers to buy for m after winnning with a bid $b \in (b^\star(s),\, \bar{b}_h(s))$, and offers to buy for l after winning with a bid $b \in (\bar{b}_m(s),\, b^\star(s))$.

Case 2. When either

1. $\Delta \in [1, 2]$ and $1 + \frac{3}{2\Delta - 1} \leq s < 2 + \frac{2}{\Delta}$, or
2. $\Delta \in [\frac{1}{2}, 1)$ and $\frac{4(\Delta+1)}{\sqrt{(1-\Delta)(3\Delta+1)}+3\Delta-1} \leq s < \frac{2(\Delta+1)}{\Delta(2-\Delta)}$,

then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type bids l with probability one.
- The medium type randomizes between a continuous distribution function $\underline{F}_m(b)$ on $(l, \underline{b}_h(s))$ with probability $1 - \mu(s)$, and an atom at $\underline{b}_h(s) = \bar{b}_m(s)$ with probability $\mu(s)$.
- The high type bids randomizes between an atom at $\underline{b}_h(s) = \bar{b}_m(s)$ with probability $\eta(s)$, and a continuous distribution function $\bar{F}_h(b)$ on $(\bar{b}_m(s), \bar{b}_h(s))$ with probability $1 - \eta(s)$.

After observing the bid $b = \underline{b}_h(s) = \bar{b}_m(s)$ in the auction, the low type is indifferent between offering m and h, and offers m with conditional probability $\gamma(s)$.

We have

$$\mu(s) = \begin{cases} \frac{2s\Delta - 2(\Delta+1)}{s\Delta} & \text{if } \Delta \geq 1 \\ \frac{2(s-1)\Delta - 2}{s\Delta} & \text{if } \Delta < 1 \end{cases},$$

$$\eta(s) = \frac{\mu(s)}{\Delta},$$

$$\underline{F}_m(b) = \frac{b - l}{(m - b)(1 - \mu(s))},$$

$$\bar{F}_h(b) = \frac{2 + \eta(s)}{1 - \eta(s)} \frac{b - \bar{b}_m(s)}{h - b},$$

$$\bar{b}_m(s) = \underline{b}_h(s) = \frac{1 - \mu(s)}{2 - \mu(s)} m + \frac{1}{2 - \mu(s)} l,$$

$$\bar{b}_h(s) = \frac{1 - \eta(s)}{3} h + \frac{2 + \eta(s)}{3} \bar{b}_m(s),$$

$$\gamma(s) = \begin{cases} 1 + \mu(s) \left[\frac{(1+\Delta)(\Delta(2-\mu(s))+1)}{s\Delta^2(2-\mu(s))} - \frac{2+\Delta^2}{2\Delta^2} \right] & \text{if } s > \Delta + 1 \text{ and } \Delta \geq 1 \\ \mu(s) \frac{(1+\Delta)(\Delta(2-\mu(s))+1)}{s\Delta^2(2-\mu(s))} & \text{if } s \leq \Delta + 1 \text{ and } \Delta \geq 1 \\ \mu(s) \left[\frac{(1+\Delta)(\Delta(2-\mu)+1)}{s\Delta^2(2-\mu(s))} - \frac{1}{2} \right] & \text{if } \Delta < 1 \end{cases},$$

■

As s converges to the upper boundary of the permitted region, (i) in the case $\Delta \in [1, 2]$, $\underline{b}_h(s)$ approaches l, and (ii) in the case $\Delta \in [1/2, 1)$, $\eta(s)$ approaches 1. Thus, when $\Delta > 1$, the high type continues to bid in the upper interval with positive probability. The expected revenue of the auctioneer does not go all the way down to l (but goes to l when $\Delta \to 1^+$). In the case $\Delta < 1$, the high type abandons bidding in her upper interval altogether, and bids the atom $\bar{b}_m(s)$ with probability one, as shown in the analysis of Case 3 below.

Case 3. When $\Delta \in [1/2, 1)$ and $2(\Delta + 1)/(\Delta(2 - \Delta)) \leq s < 4/\Delta$, then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type bids l with probability one.
- The medium type randomizes between a continuous distribution function $\underline{F}_m(b)$ on $(l, \bar{b}_m(s))$ with probability $1 - \mu(s)$, and an atom at $\bar{b}_m(s)$ with probability $\mu(s)$.
- The high type bids $\bar{b}_m(s)$ with probability one.

After observing the bid $\bar{b}_m(s)$ in the auction, the low type offers m with conditional probability one.

We have

$$\mu(s) = \frac{2s\Delta - 2}{2 + s\Delta},$$

$$\underline{F}_m(b) = \frac{b - l}{(m - b)(1 - \mu(s))},$$

$$\bar{b}_m(s) = \frac{1 - \mu(s)}{2 - \mu(s)}\, m + \frac{1}{2 - \mu(s)}\, l.$$

■

As s converges to the upper boundary, $\mu(s)$ approaches 1, and as a consequence $\bar{b}_m(s)$ goes to l. Thus, in the case $\Delta \leq 1$, expected revenue in a first-price auction with revelation of the winning bid approaches l for sufficiently large s.

D.3 Equilibria in a first-price auction with no announcement

Case 1. When $s \leq 2$ (Δ is arbitrary), the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type bids l with probability one.
- The medium type bids according to a continuous distribution function

$$F_m(b) = \frac{b - l}{m - b}$$

  on $(l,\, \frac{1}{2}(m + l))$;
- The high type bids according to a continuous distribution function

$$F_h(b) = \frac{2b - m - l}{h - b}$$

  on $(\frac{1}{2}(m + l),\, \frac{1}{3}(h + m + l))$. ■

The (upper) boundary value of s is determined by the condition that either the low type (when $\Delta \geq 1$), or the high type (when $\Delta \leq 1$) is exactly indifferent between playing her equilibrium strategy and deviating to bidding in the range of the medium type. Thus, depending on whether Δ is larger or smaller than 1, one of these two types will start bidding in the range of the medium type in order to acquire payoff-relevant information for the bargaining stage. This brings us to either Case 2, or Case 3 below. (When Δ is exactly one, both types start bidding in the range of the medium type when s crosses 2, and we jump immediately to Case 4).

Case 2. When $\Delta \in [1, 2]$ and $2 < s < (6\Delta - 2)/(\Delta + 1)$, then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type randomizes between an atom at l with probability $1 - \theta(s)$, and a continuous distribution function $F_l(b)$ on $(l,\, \bar{b}_l(s))$ with probability $\theta(s)$.
- The medium type bids according to a continuous distribution function $F_m(b)$ on $(l,\, \bar{b}_m(s))$, with $\bar{b}_m(s) = \bar{b}_l(s)$.
- The high type bids according to a continuous distribution function $F_h(b)$ on $(\underline{b}_h(s),\, \bar{b}_h(s))$, with $\underline{b}_h(s) = \bar{b}_m(s)$.

The low type offers to sell for m conditional on winning the auction (and making an offer), and offers to sell for h conditional on losing the auction (and making an offer).

We have

$$\theta(s) = \frac{s-2}{s+2},$$

$$F_l(b) = F_m(b) = \frac{4}{s}\frac{2(b-l)}{s(m-l)+2(m-b)-2(b-l)},$$

$$F_h(b) = 2\frac{b-\underline{b}_h(s)}{h-b},$$

$$\bar{b}_m(s) = \bar{b}_l(s) = \underline{b}_h(s) = l + \frac{s}{4}(m-l),$$

$$\bar{b}_h(s) = \frac{1}{3}\,h + \frac{2}{3}\,\underline{b}_h(s).$$

■

Because $\Delta \geq 1$, the low type benefits relatively more than the high type fom acquiring information by bidding in the range of the medium type. In the equilibrium above, the low type mixes between bidding l and bidding in the support of the medium type. This allows her to condition the offer in the bargaining stage on the outcome of the auction. With sufficiently large s, the high type will also start bidding in this range, which brings us to Case 4.

Case 3. When $\Delta \in [1/2, 1)$ and $2 < s < (6-2\Delta)/(\Delta+1)$, then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type bids l with probability one.
- The medium type bids according to a continuous distribution function $F_m(b)$ on $(l, \bar{b}_m(s))$.
- The high type bids according to a continuous distribution function $\underline{F}_h(b)$ on $(l, \bar{b}_m(s))$ with probability $\eta(s)$, and according to a continuous distribution function $\bar{F}_h(b)$ on $(\bar{b}_m(s), \bar{b}_h(s))$ with probability $1-\eta(s)$.

The high type offers to buy for l conditional on winning the auction (and making an offer), and offers to buy for m conditional on losing the auction (and making an offer).

We have

$$\eta(s) = \frac{s-2}{s+2},$$

$$\underline{F}_h(b) = F_m(b) = \frac{s+2}{s}\frac{2(b-l)}{2(h-b)-2(b-m)-s(h-m)},$$

$$\bar{F}_h(b) = \frac{2+\eta(s)}{1-\eta(s)}\frac{b-\bar{b}_m(s)}{h-b},$$

$$\bar{b}_m(s) = \frac{s(2-s)}{6s+4}\,h + \frac{s(s+2)}{6s+4}\,m + \frac{2(s+2)}{6s+4}\,l,$$

$$\bar{b}_h(s) = \frac{1-\eta(s)}{3}\,h + \frac{2+\eta(s)}{3}\,\bar{b}_m(s).$$

■

Because now $\Delta \leq 1$, it is the high type that benefits relatively more fom acquiring information by bidding in the range of the medium type. In the equilibrium above, the high type mixes between bidding in the interval $(\bar{b}_m(s), \bar{b}_h(s))$ and bidding in the support of the medium type. This allows her to condition the offer in the bargaining stage on the outcome of the auction. With sufficiently large s, the low type will also start bidding in this range, which brings us to Case 5.

Case 4. When $\Delta \in [1,\, 2]$ and

$$\frac{6\Delta - 2}{\Delta + 1} \le s < \min\left\{\frac{10\Delta + 2}{\Delta + 1},\, \frac{2\Delta + 2}{\Delta - 1}\right\},$$

then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type randomizes between an atom at l with probability $1 - \theta(s)$, and a continuous distribution function $F_l(b)$ on $(l,\, \bar{b}_l(s))$ with probability $\theta(s)$.
- The medium type bids according to a continuous distribution function $F_m(b)$ on $(l,\, \bar{b}_m(s))$, with $\bar{b}_m(s) = \bar{b}_l(s)$.
- The high type bids according to a continuous distribution function $\underline{F}_h(b)$ on $(l,\, b^\star(s))$ with probability $\eta(s)$, and according to a continuous distribution function $\bar{F}_h(b)$ on $(\bar{b}_m(s),\, \bar{b}_h(s))$ with probability $1 - \eta(s)$, where $l < b^\star(s) \le \bar{b}_m(s)$.

The low type offers to sell for m conditional on winning the auction (and making an offer), and offers to sell for h conditional on losing the auction (and making an offer). The high type offers to buy for m conditional on losing the auction, or winning with a bid $b \in (\bar{b}_m(s),\, \bar{b}_h(s))$ (and making an offer), and offers to buy for l conditional on winning the auction with a bid $b \in (l,\, b^\star(s))$ (and making an offer).

We have

$$\theta(s) = \frac{(s-2)(\Delta+1)}{8\Delta},$$

$$\eta(s) = \frac{(s-6)\Delta + s + 2}{8\Delta},$$

$$F_l(b^\star(s)) = \frac{\Delta(s-6) + s + 2}{(\Delta+1)(s-2)},$$

$$F_m(b^\star(s)) = \frac{\Delta(s-6) + s + 2}{2\Delta(s-2)},$$

$$\bar{F}_h(b) = \frac{2 + \eta(s)}{1 - \eta(s)} \frac{b - \bar{b}_m(s)}{h - b},$$

$$b^\star(s) = m - \alpha(s,\, \Delta)(m - l),$$

$$\bar{b}_m(s) = m - \beta(s,\, \Delta)(m - l),$$

$$\bar{b}_h(s) = \frac{1 - \eta(s)}{3} h + \frac{2 + \eta(s)}{3} \bar{b}_m(s),$$

where $\alpha(s,\, \Delta) \ge \beta(s,\, \Delta)$, and $\alpha(s,\, \Delta)$, $\beta(s,\, \Delta)$, $F_l(b)$, $F_m(b)$, and $\underline{F}_h(b)$ are defined in Appendix E. ■

The equilibrium has a fairly complicated structure. When $\Delta > 1$, the high type cannot gain payoff-relevant information by bidding near the upper end of the support of the medium type. Indeed, offering to buy for m would be optimal for the high type in the bargaining stage regardless of the outcome of the auction. To maximize the "informational gain", the high type wants to bid close to the lower end of the interval $(l,\, b_m(s))$, i.e. she wants to place her bid "between" the equilibrium bids of the low and medium types. As a result, the high type bids in $(l,\, b^\star(s))$ in equilibrium. When $\Delta = 1$, we have $b^\star(s) = \bar{b}_m(s)$. As s approaches the upper boundary of the permitted region, (i) $\theta(s)$ approaches 1, and (ii) $b^\star(s)$ approaches l.

Case 5. When $\Delta \in [1/2,\, 1)$ and $(6 - 2\Delta)/(\Delta + 1) \le s < (10 + 2\Delta)/(\Delta + 1)$, then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type randomizes between an atom at l with probability $1 - \theta(s)$, and a continuous distribution function $F_l(b)$ on $(b^\star(s),\, \bar{b}_m(s))$ with probability $\theta(s)$, where $l \le b^\star(s) < \bar{b}_m(s)$.
- The medium type bids according to a continuous distribution function $F_m(b)$ on $(l,\, \bar{b}_m(s))$.

- The high type bids according to a continuous distribution function $\underline{F}_h(b)$ on $(l,\, \bar{b}_m(s))$ with probability $\eta(s)$, and according to a continuous distribution function $\bar{F}_h(b)$ on $(\bar{b}_m(s),\, \bar{b}_h(s))$ with probability $1-\eta(s)$.

The high type offers to buy for l conditional on winning the auction (and making the offer), and offers to buy for m conditional on losing the auction (and making the offer). The low type offers to sell for h conditional on losing the auction with a bid $b \in (b^\star(s),\, \bar{b}_m(s))$ (and making the offer), and offers to sell for m conditional on losing the auction with a bid $b = l$, or winning the auction with a bid $b \in (b^\star(s),\, \bar{b}_m(s))$ (and making the offer).

We have

$$\theta(s) = \frac{(s-6)+\Delta(2+s)}{8},$$

$$\eta(s) = \frac{(s-2)(\Delta+1)}{8},$$

$$F_m(b^\star(s)) = \frac{(1-\Delta)(s+2)}{2(s-2)}$$

$$\underline{F}_h(b^\star(s)) = \frac{4(1-\Delta)}{(\Delta+1)(s-2)},$$

$$\bar{F}_h(b) = \frac{2+\eta(s)}{1-\eta(s)}\frac{b-\bar{b}_m(s)}{h-b},$$

$$b^\star(s) = m - \alpha'(s,\, \Delta)(m-l),$$

$$\bar{b}_m(s) = m - \beta'(s,\, \Delta)(m-l),$$

$$\bar{b}_h(s) = \frac{1-\eta(s)}{3}\,h + \frac{2+\eta(s)}{3}\,\bar{b}_m(s),$$

where $\alpha'(s,\, \Delta) \geq \beta'(s,\, \Delta)$, and $\alpha'(s,\, \Delta)$, $\beta'(s,\, \Delta)$, $F_l(b)$, $F_m(b)$, and $\underline{F}_h(b)$ are defined in Appendix E. ■

The equilibrium above is a symmetric image of the equilibrium in Case 4. When $\Delta < 1$, the low type cannot gain payoff-relevant information by bidding near the lower end of the support of the medium type. Offering to sell for m would be optimal for the low type in the bargaining stage regardless of the outcome of the auction. Instead, the low type wants to bid close to the upper end of the interval $(l,\, \bar{b}_m(s))$, i.e. she wants to place her bid "between" the equilibrium bids of the high and medium types. As a result, the low type bids in $(b^\star(s),\, \bar{b}_m(s))$ in equilibrium.

D.4 Equilibria in a first-price auction with revelation of all bids

Case 1. When $s \leq 2$, then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type randomizes between an atom at l with probability $1-\theta(s)$, and a continuous distribution function $F_l(b)$ on $(l,\, \bar{b}_l(s))$ with probability $\theta(s)$.
- The medium type bids according to a distribution function $\underline{F}_m(b)$ on $(l,\, \bar{b}_l(s))$ with probability $\mu_0(s)$, $\hat{F}_m(b)$ on $(\bar{b}_l(s),\, \underline{b}_h(s))$ with probability $\mu_1(s)$, and $\bar{F}_m(b)$ on $(\underline{b}_h(s),\, \bar{b}_m(s))$ with probability $1-\mu_0(s)-\mu_1(s)$.
- The high type bids according to a distribution function $\underline{F}_h(b)$ on $(\underline{b}_h(s),\, \bar{b}_m(s))$ with probability $\eta(s)$, and according to a distribution function $\bar{F}_h(b)$ on $(\bar{b}_m(s),\, \bar{b}_h(s))$ with probability $1-\eta(s)$.

We have $l < \bar{b}_l(s) < \underline{b}_h(s) < \bar{b}_m(s) < \bar{b}_h(s)$. After observing a bid $b \in (\underline{b}_h(s), \bar{b}_m(s))$ in the auction, the low type is indifferent between offering m and h, and offers m with conditional probability $\gamma(b;\, s)$ with $\gamma(\underline{b}_h(s);\, s) = 1$ and $\gamma(\bar{b}_m(s);\, s) = 0$ in the bargaining stage. After observing a bid $b \in (l, \bar{b}_l(s))$ in the auction, the high type is indifferent between offering m and l, and offers m with conditional probability $\delta(b;\, s)$ with $\delta(l;\, s) = 0$ and $\delta(\bar{b}_l(s);\, s) = 1$ in the bargaining stage. Moreover,

$$\theta(s) = \eta(s) = \mu_0(s) = \frac{s}{4},$$

$$\mu_1(s) = 1 - \frac{s}{2},$$

$$\bar{b}_l(s) = m - \frac{4-s}{4+s}(m-l),$$

$$\underline{b}_h(s) = m - \frac{4-s}{8-s}(m-l),$$

$$\bar{b}_m(s) = m - \frac{4-s}{8+s}(m-l),$$

$$\bar{b}_h(s) = \frac{1-\eta(s)}{3}\, h + \frac{2+\eta(s)}{3}\, \bar{b}_m(s),$$

$$F_l(b) = \underline{F}_m(b) = \frac{4-s}{2s}\frac{b-l}{m-b},$$

$$\hat{F}_m(b) = \frac{4(b-l) - s(m-b) - s(m-l)}{2(b-m)(s-2)},$$

$$\bar{F}_m(b) = \underline{F}_h(b) = \frac{4(m+l-2b) + (b-l)s}{2s(b-m)},$$

$$\bar{F}_h(b) = \frac{2+\eta(s)}{1-\eta(s)}\frac{b-\bar{b}_m(s)}{h-b}.$$

■

As s approaches 2, $\bar{b}_l(s)$ goes to $\underline{b}_h(s)$. Thus, the interval $(\bar{b}_l(s), \underline{b}_h(s))$, in which only the medium type bids, collapses to a point. It turns out that for larger s, all types bid in this middle region, as discussed in Case 2 below.

Case 2. When $s \in (2, 6)$, then the following bidding strategies (along with sequentially rational behavior of players in the bargaining stage) constitute an equilibrium:

- The low type randomizes between an atom at l with probability $1 - \theta_0(s) - \theta_1(s)$, a continuous distribution function $\underline{F}_l(b)$ on $(l, \underline{b}_h(s))$ with probability $\theta_0(s)$, and a continuous distribution function $\bar{F}_l(b)$ on $(\underline{b}_h(s), \bar{b}_l(s))$ with probability $\theta_1(s)$.
- The medium type bids according to a distribution function $\underline{F}_m(b)$ on $(l, \underline{b}_h(s))$ with probability $\mu_0(s)$, $\hat{F}_m(b)$ on $(\underline{b}_h(s), \bar{b}_l(s))$ with probability $\mu_1(s)$, and $\bar{F}_m(b)$ on $(\bar{b}_l(s), \bar{b}_m(s))$ with probability $1-\mu_0(s)-\mu_1(s)$.
- The high type bids according to a distribution function $\underline{F}_h(b)$ on $(\underline{b}_h(s), \bar{b}_l(s))$ with probability $\eta_0(s)$, $\hat{F}_h(b)$ on $(\bar{b}_l(s), \bar{b}_m(s))$ with probability $\eta_1(s)$, and $\bar{F}_h(b)$ on $(\bar{b}_m(s), \bar{b}_h(s))$ with probability $1-\eta_0(s)-\eta_1(s)$.

We have $l < \underline{b}_h(s) < \bar{b}_l(s) < \bar{b}_m(s) < \bar{b}_h(s)$. After observing a bid $b \in (\underline{b}_h(s), \bar{b}_m(s))$ in the auction, the low type is indifferent between offering m and h, and offers m with conditional probability $\gamma(b;\, s)$ with $\gamma(\underline{b}_h(s);\, s) = 1$ and $\gamma(\bar{b}_m(s);\, s) = 0$ in the bargaining stage. After observing a bid $b \in (l, \bar{b}_l(s))$ in the auction, the high type is indifferent between offering m and l, and offers m with conditional probability $\delta(b;\, s)$ with $\delta(l;\, s) = 0$ and $\delta(\bar{b}_l(s);\, s) = 1$ in the bargaining stage. Moreover,

$$\theta_0(s) = \eta_1(s) = \mu_0(s) = \frac{3}{4} - \frac{s}{8},$$

$$\theta_1(s) = \eta_0(s) = \mu_1(s) = \frac{s}{4} - \frac{1}{2},$$

$$\underline{b}_h(s) = \frac{1}{3} + \frac{2}{3}m,$$

$$\bar{b}_l(s) = m - \frac{6-s}{3s+6}(m-l),$$

$$\bar{b}_m(s) = m - \frac{6-s}{s+18}(m-l),$$

$$\bar{b}_h(s) = \frac{1-\eta(s)}{3}h + \frac{2+\eta(s)}{3}\bar{b}_m(s),$$

$$\underline{F}_l(b) = \underline{F}_m(b) = \frac{b-l}{2(m-b)},$$

$$\bar{F}_l(b) = \hat{F}_m(b) = \underline{F}_h(b) = \frac{6-s}{s-2}\frac{2(m-b)-(b-l)}{6(b-m)},$$

$$\bar{F}_m(b) = \hat{F}_h(b) = \frac{1}{6-s}\frac{6(b-l)+3s(b-m)-s(m-l)}{2(b-m)},$$

$$\bar{F}_h(b) = \frac{2+\eta(s)}{1-\eta(s)}\frac{b-\bar{b}_m(s)}{h-b}.$$

■

As a result of these bidding strategies, the expected revenue of the auctioneer converges to m as s goes to 6.

E Appendix E - Proofs

In this Appendix I provide details of the derivation of equilibria presented in Appendix D. In the interest of space,[1] parts of the proofs are omitted, and the emphasis is on the elements of the derivation that lead to binding equilibrium restrictions on parameters. For brevity of notation, I suppress the dependence of variables on s. I do not explicitly define all off-equilibrium beliefs, unless they are directly important for the derivation.

E.1 Second price auction with no announcement

Case 1.

Assume first that $\Delta \geq 1$, i.e. $h - m \geq m - l$. The equilibrium payoff of the low type is $s(h-l)/6$. Deviating to a bid in the interval (l, m) would give the same expected payoff (we specify off-equilibrium beliefs of the high type so that she offers l when she observes such a bid). Deviating to a bid of m results in the following (sequentially rational) outcome in the bargaining stage: when the low type loses, she offers to sell for h, and when she wins, she offers to sell for m. The expected payoff is

$$\frac{1}{6}\left[s(h-l) + (\frac{3}{2}s - 1)(m-l)\right],$$

therefore $s \leq 2/3$ is necessary for equilibrium. Deviating to a bid in (m, h) yields at most (depending on off-equilibrium beliefs)

$$\frac{1}{6}\left[s(h-l) + (2s-2)(m-l)\right]$$

which is less than the equilibrium payoff if $s \leq 2/3$. Bidding h or above is also worse for the low type.

The optimality for the medium and high type under the restriction $s \leq 2/3$ can be checked analogously.

[1] As well as for the sanity of the reader.

Now assume that $\Delta \in [1/2,\, 1]$, so that $h - m \leq m - l$. Now the low type gets $s(m-l)/3$ in equilibrium. Bidding m would give

$$\frac{1}{6}\left[(\frac{3}{2}s - 1)(m-l) + s(h-l)\right].$$

To have an equilibrium we therefore need to assume that

$$s \leq \frac{m-l}{h-l-\frac{1}{2}(m-l)}$$

which is exactly the restriction that I imposed. The medium type gets

$$\frac{1}{6}(2+s)(m-l)$$

in expectation in equilibrium, and there are no profitable deviations.

Optimality of the high type's strategy can be checked analogously, and it does not lead to additional restrictions on parameters.

Case 2.

Consider first the case $\Delta \geq 1$. The equilibrium expected payoff of the low type (i.e. when she bids l) is $s(h-l)/6$. If she bids $b^\star$, she obtains

$$\frac{1}{6}\left[s(h-l) + \frac{3}{2}s(m-l) - (1+\theta)(b^\star - l)\right].$$

Indifference requires

$$\theta = 2 - \frac{3}{2}s\frac{m-l}{b^\star - l}. \tag{E.1}$$

Any bid in $(l,\, b^\star)$ yields the same or smaller payoff depending on off-equilibrium beliefs. Bidding in $(b^\star,\, h)$ gives at most

$$\frac{1}{6}\left[s(h-l) + 2s(m-l) - (2+2\theta)(b^\star - l)\right].$$

Therefore, we need

$$s(m-l) \leq (3+\theta)(b^\star - l)$$

which is satisfied given the indifference condition (E.1). Bidding h or above is clearly worse for the low type.

The equilibrium payoff of the medium type is

$$\frac{1}{6}\left[s(h-m) + (1+\theta)(m-b^\star) + (2-\theta)\frac{1}{2}s(m-l) + 2(1-\theta)(m-l)\right].$$

If she bids below $b^\star$, she gets

$$\frac{1}{6}\left[s(h-m) + 2(1-\theta)(m-l) + (1-\theta)s(m-l)\right].$$

Thus, equilibrium requires

$$(1+\theta)(m-b^\star) \geq -\frac{1}{2}\theta s(m-l). \tag{E.2}$$

If the medium type bids in $(b^\star,\, h)$, she gets

$$\frac{1}{6}\left[s(h-m) + (2+2\theta)(m-b^\star) + s(m-l) + 2(1-\theta)(m-l)\right].$$

We must therefore have

$$(1+\theta)(m-b^\star) \leq -\frac{1}{2}\theta s(m-l). \tag{E.3}$$

Equations (E.2) and (E.3) imply that

$$(1+\theta)(b^\star - m) = \frac{1}{2}\theta s(m-l).$$

It is easily verified that these conditions are sufficient for equilibrium if $s \in (2/3,\, 2]$. We obtain

$$b^\star = m + \left(\frac{3}{8}s - \frac{1}{4}\right)(m-l),$$

and

$$\theta = \frac{3s-2}{2+s}.$$

Optimality of the high type's strategy can be verified easily.

Now consider the case $\Delta \in [1/2,\, 1]$. The equilibrium payoff of the low type (from bidding l) is $\frac{1}{3}s(m-l)$, and bidding $b^\star$ yields

$$\frac{1}{6}\left[s(h-l) + \frac{3}{2}s(m-l) - (1+\theta)(b^\star - l)\right],$$

as long as $(1+\theta)/\theta \geq (h-l)/(m-l)$ which guarantees that the high type offers m when she wins against a bid of $b^\star$. Indifference requires

$$1+\theta = s\left[\frac{h-l}{b^\star - l} - \frac{1}{2}\frac{m-l}{b^\star - l}\right]. \tag{E.4}$$

A deviation to a bid in $(l,\, b^\star)$ yields the same expected payoff, and bidding in $(b^\star,\, h)$ would yield at most (depending on off-equilibrium beliefs)

$$\frac{1}{6}\left[s(h-l) + 2s(m-l) + (2+2\theta)(l-b^\star)\right],$$

hence we need $s(h-l) \leq (2+2\theta)(b^\star - l)$ which is satisfied given the indifference condition (E.4).

Now consider the strategy of the medium type. In equilibrium she gets

$$\frac{1}{6}\left[s(h-m) + (2-\theta)\frac{1}{2}s(m-l) - (1+\theta)(b^\star - m) + 2(1-\theta)(m-l)\right],$$

as long as $(h-m)/(m-l) \geq \theta/2$ (which always holds in the case we are analyzing). Bidding below $b^\star$ would give

$$\frac{1}{6}\left[s(h-m) + 2(1-\theta)(m-l) + (1-\theta)s(m-l)\right]$$

as long as $(h-m)/(m-l) \geq \theta$. And if she bids in $(b^\star,\, h)$, she gets

$$\frac{1}{6}\left[s(h-m) + (2+2\theta)(m-b^\star) + s(m-l) + 2(1-\theta)(m-l)\right]$$

which together implies (just as in the previous case)

$$(1+\theta)(b^\star - m) = \frac{1}{2}\theta s(m-l).$$

Thus, we obtain

$$b^\star = m + \left[\frac{1}{2}s - \left(\frac{1}{2} + \frac{1}{4}s\right)\frac{m-l}{h-l}\right](m-l),$$

and

$$1-\theta = 2 - \frac{s\left(4\frac{h-l}{m-l} - 2\right)}{(2+s)\left(2 - \frac{m-l}{h-l}\right)}.$$

The high type cannot deviate profitably either.

Lastly, we need to verify that θ is between 0 and 1 and that $(1+\theta)/\theta \geq (h-l)/(m-l)$ and $(h-m)/(m-l) \geq \theta$. Note that in the case we are considering we have $(m-l)/(h-l) \in [1/2,\, 2/3]$. For θ to be between 0 and 1 we must have

$$\frac{4-2\frac{m-l}{h-l}}{4\frac{h-l}{m-l}+\frac{m-l}{h-l}-4} \leq s \leq \frac{4-2\frac{m-l}{h-l}}{2\frac{h-l}{m-l}+\frac{m-l}{h-l}-3}.$$

Finally, the condition $(1+\theta)/\theta \geq (h-l)/(m-l)$ is satisfied since $(h-l)/(m-l) \in [3/2,\, 2]$ and the condition $(h-m)/(m-l) \geq \theta$ gives us

$$\frac{4-2s\Delta}{2+s} \geq 1-\Delta.$$

This last condition boils down to $s \leq 2$, and the lower bound $s > 2/(2\Delta+1)$ comes from requiring that $b^\star > m$.

Case 3.

The equilibrium expected payoff of the low type is

$$\frac{1}{6}\left[s(h-l)+\frac{3}{2}s(m-l)+2(l-b^\star)\right].$$

A deviation to bidding in $[l,\, b^\star)$ would give $s(h-l)/6$ (where we specify off-equilibrium beliefs of the high type so that she offers l when she wins against a bid below $b^\star$). This gives us the condition $(3/2)s(m-l) \geq 2(b^\star-l)$. A deviation to a higher bid cannot be strictly profitable (regardless of off-equilibrium beliefs).

As for the medium type, in equilibrium she gets

$$\frac{1}{6}\left[s(h-m)+\frac{1}{2}s(m-l)+2(m-b^\star)\right].$$

Bidding above $b^\star$ would yield

$$\frac{1}{6}\left[s(h-m)+s(m-l)+4(m-b^\star)\right],$$

so we require $2(m-b^\star) \geq \frac{1}{2}s(m-l)$. Bidding below $b^\star$ gives

$$\frac{1}{6}s(h-m),$$

so we need $(1/2)s(m-l) \geq 2(b^\star-m)$. Hence

$$b^\star = m+\frac{1}{4}s(m-l).$$

The high type cannot deviate profitably either. The only condition we need to verify is that $b^\star < h$ and that $(3/2)s(m-l) \geq 2(m-l+s(m-l)/4)$. This gives us the restriction $2 \leq s \leq 4\Delta$.

Case 4.

The expected payoff for the low type from bidding l is

$$\frac{1}{3}s(m-l).$$

Because the low type bids $b^\star$ with probability Δ in equilibrium, the high type is exactly indifferent between offering m and l in the bargaining stage after observing a bid of $b^\star$. Let ζ denote the probability that the high type offers m in that case. Then, the expected payoff for the low type from bidding $b^\star$ is

$$\frac{1}{6}\left[s(h-l)+\zeta s(m-l)+(1+\Delta)(l-b^\star)+\frac{1}{2}s(m-l)\right].$$

The indifference condition yields

$$\frac{3}{2}s(m-l) = s(h-l) + \zeta s(m-l) + (1+\Delta)(l-b^\star). \tag{E.5}$$

The equilibrium payoff of the medium type is

$$\frac{1}{6}\left[s(h-m) + (1+\Delta)(m-b^\star) + (2-\Delta)\frac{1}{2}s(m-l) + 2(1-\Delta)(m-l)\right].$$

If she bids below $b^\star$, she gets

$$\frac{1}{6}\left[s(h-m) + 2(1-\Delta)(m-l) + (1-\Delta)s(m-l)\right].$$

Thus, equilibrium requires

$$(1+\Delta)(m-b^\star) \geq -\frac{1}{2}\Delta s(m-l). \tag{E.6}$$

If the medium type bids in $(b^\star, h)$, she gets

$$\frac{1}{6}\left[s(h-m) + (2+2\Delta)(m-b^\star) + s(m-l) + 2(1-\Delta)(m-l)\right].$$

We must therefore have

$$(1+\Delta)(m-b^\star) \leq -\frac{1}{2}\Delta s(m-l). \tag{E.7}$$

Equations (E.6) and (E.7) imply that

$$b^\star = m + \frac{1}{2}\frac{\Delta}{\Delta+1}s(m-l).$$

From this, we can compute ζ using equation (E.5):

$$\zeta = \frac{1-\Delta}{2} + \frac{1+\Delta}{s}.$$

It is easy to verify optimality of the high type's strategy. The only thing we have to check is that (i) ζ lies between 0 and 1, and (ii) $b^\star \leq h$. From the first restriction, we get the condition

$$s \geq 2,$$

and the second yields

$$2 + 2\Delta \geq s.$$

E.2 First price auction with revelation of the winning bid

Case 1.

Subcase $\Delta \in [1/2, 1]$.

Consider the high type first. If she bids in her upper range $(\bar{b}_m, \bar{b}_h)$, she gets an expected payoff equal to

$$\frac{1}{6}\left[\left(4 + 2\eta + 2(1-\eta)\bar{F}_h(b)\right)(h-b) + s(h-l)\right].$$

Because the high type must be indifferent between all bids $b \in (\bar{b}_m, \bar{b}_h)$, we obtain

$$\bar{F}_h(b) = \frac{2+\eta}{1-\eta}\frac{b-\bar{b}_m}{h-b},$$

and

$$\frac{\bar{b}_h - \bar{b}_m}{h - \bar{b}_h} = \frac{1-\eta}{2+\eta}.$$

Bidding in the lower range $(\underline{b}_h, \bar{b}_m)$ yields

$$\frac{1}{6}\left[\left(4-2\mu+2\eta\underline{F}_h(b)+2\mu\bar{F}_m(b)\right)(h-b)+\mu(1-\bar{F}_m(b))s(h-m)+s(h-l)+\gamma(b)s(h-m)\right],$$

where recall that $\gamma(b)$ is the probability that the low type offers to sell for m (conditional on making an offer) after observing a bid $b \in (\underline{b}_h, \bar{b}_m)$.

Consider the medium type. Bidding in the lower range $(l, \underline{b}_h)$ gives her an expected profit of

$$\frac{1}{6}\left[(2+2(1-\mu)\underline{F}_m(b))(m-b)+s(h-l)\right]$$

which yields

$$\underline{F}_m(b)=\frac{b-l}{(m-b)(1-\mu)},$$

and

$$\underline{b}_h=\frac{(1-\mu)m+l}{2-\mu}.$$

Bidding in the upper range $(\underline{b}_h, \bar{b}_m)$ yields

$$\frac{1}{6}\left[\left(2+2\eta\underline{F}_h(b)+2(1-\mu)+2\mu\bar{F}_m(b)\right)(m-b)+s(m-l)+(1-\eta+\eta(1-\underline{F}_h(b)))s(h-m)\right].$$

In equilibrium, the low type must be indifferent between the prices m and h after observing a bid b in $(\underline{b}_h, \bar{b}_m)$ which implies that

$$\underline{F}_h(b)=\frac{\mu}{\eta\Delta}\bar{F}_m(b)$$

for all b in this range. This yields (to simplify notation, I denote $F \equiv \bar{F}_m$)

$$\left[4-2\mu+2\frac{\mu}{\Delta}F(b)+2\mu F(b)\right](h-b)+\mu(1-F(b))s(h-m)+\gamma(b)s(h-m)=\text{const.}$$

$$\left[2+2\frac{\mu}{\Delta}F(b)+2(1-\mu)+2\mu F(b)\right](m-b)+(1-\eta+(\eta-\frac{\mu}{\Delta}F(b)))s(h-m)=\text{const.}$$

We can solve the second equation for F:

$$F(b)=\frac{\alpha-(4-2\mu)(m-b)}{(\frac{1}{\Delta}+1)2\mu(m-b)-\frac{\mu}{\Delta}s(h-m)},$$

where α is a constant to be determined. For the function F to be a well defined cumulative distribution function, we need

$$2(\Delta+1)>s\Delta(2-\mu).$$

To determine α, we use the condition $F(\underline{b}_h)=0$ which yields

$$F(b)=\frac{\Delta(4-2\mu)(b-\underline{b}_h)}{(\Delta+1)2\mu(m-b)-\mu s(h-m)}.$$

We can now recover $\gamma(b)$ as

$$\gamma(b)=\frac{\beta-\left[4-2\mu+2\frac{\mu}{\Delta}F(b)+2\mu F(b)\right](h-b)-\mu(1-F(b))s(h-m)}{s(h-m)},$$

for some constant β to be determined. Because it must be that

$$\lim_{b\nearrow\underline{b}_h}\gamma(b)=1,$$

we obtain

$$\beta=(1+\mu)s(h-m)+(4-2\mu)(h-m)+2(m-l).$$

Similarly, we have

$$\lim_{b \searrow \bar{b}_m} \gamma(b) = 0,$$

which gives the condition

$$(1+\mu)s\Delta + (4-2\mu)\Delta + 2 = \left[4 + 2\frac{\mu}{\Delta}\right] \frac{h-\bar{b}_m}{m-l}.$$

The condition $F(\bar{b}_m) = 1$ yields

$$\frac{m-\bar{b}_m}{m-l} = \frac{(2+\mu s)\Delta}{2\mu+4\Delta}.$$

From the above expressions, we can determine μ as

$$\mu = \frac{2+(4+s)\Delta - 4\frac{h-\bar{b}_m}{m-l}}{(2-s)\Delta + \frac{2}{\Delta}\frac{h-\bar{b}_m}{m-l}}.$$

Moreover, we have

$$\frac{h-\bar{b}_m}{m-l} = \frac{(2+\mu s)\Delta}{2\mu+4\Delta} + \Delta$$

Simplifying, solving the quadratic equation that arises, and choosing the positive root yields

$$\mu = \frac{s\Delta}{(2-s)\Delta + (s+2)}.$$

The obtained number is always positive and is less than 1 if $(2s-2)\Delta < (s-2)$. Observing that $\mu = \eta\Delta$, we get

$$\eta = \frac{s}{(2-s)\Delta + (s+2)}.$$

To guarantee that $\eta \in [0,\, 1]$, we need a slightly stronger condition $s \leq 2 + 2/\Delta$.

We can now calculate $\underline{b}_h$. We obtain

$$\underline{b}_h = \frac{1}{2}\frac{(2-2s)\Delta + (s+2)}{(2-\frac{3}{2}s)\Delta + (s+2)} m + \frac{1}{2}\frac{(2-s)\Delta + (s+2)}{(2-\frac{3}{2}s)\Delta + (s+2)} l$$

which is a weighted average of m and l. Then we determine $\bar{b}_m$ as

$$\bar{b}_m = \frac{1}{2}\frac{2(2-s)\Delta + 2(s+2) + (2s - s^2\Delta)}{2(2-s)\Delta + 2(s+2) + s} m + \frac{1}{2}\frac{2(2-s)\Delta + 2(s+2) + s^2\Delta}{2(2-s)\Delta + 2(s+2) + s} l,$$

again a weighted average of m and l. To verify that $\bar{b}_m > \underline{b}_h$, we only need to show that the weight on m is larger in $\bar{b}_m$. This restriction yields the condition that

$$\frac{1}{2-\mu} > \frac{(2+\mu s)\Delta}{2\mu + 4\Delta}. \tag{E.8}$$

Finally, we can recover $\bar{b}_h$ as

$$\bar{b}_h = \frac{1}{3}\frac{(2-s)\Delta + 2}{(2-s)\Delta + (s+2)} h + \frac{2}{3}\frac{(2-s)\Delta + (2s+2)}{(2-s)\Delta + (s+2)} \bar{b}_m.$$

Because $\bar{b}_h$ is a weighted average of $\bar{b}_m$ and $h > \bar{b}_m$, we know that $\bar{b}_h > \bar{b}_m$.

Solving (E.8) yields the condition

$$s < \frac{4(\Delta+1)}{\sqrt{(1-\Delta)(3\Delta+1)} + 3\Delta - 1}.$$

It can be verified that under this condition there are no profitable deviations for any of the types. The condition implies all other parameter restrictions imposed in the derivation above.

Subcase $\Delta \in [1,\, 2]$

This derivation is similar to the previous case, except for one additional difficulty. Players condition their offer in the bargaining stage on their bid when they win after bidding in the range $(\underline{b}_h,\, \bar{b}_m)$. More precisely, for the high type, the offer changes at $\hat{b}_h \in (\underline{b}_h,\, \bar{b}_m)$ such that

$$\bar{F}_m(\hat{b}_h) = 1 - \frac{\Delta - 1}{\Delta\mu},$$

i.e. the high type will offer to sell for m if she won by bidding $b > \hat{b}_h$. For the medium type, an analogous threshold bid $\hat{b}_m$ is defined by

$$\underline{F}_h(\hat{b}_m) = \frac{1}{\eta\Delta} \tag{E.9}$$

i.e. if the medium type wins and observes a bid above this threshold, she offers to sell for h. I conjecture and then verify that in equilibrium the offer of the medium type will not switch, that is, there is no solution to equation (E.9).

Consider the high type first. If she bids in her upper range $(\bar{b}_m,\, \bar{b}_h)$, she receives the same expected payoff as in the previous subcase. Bidding in the lower range $(\underline{b}_h,\, \bar{b}_m)$ yields

$$\frac{1}{6}\left[\left(4 - 2\mu + 2\eta\underline{F}_h(b) + 2\mu\bar{F}_m(b)\right)(h-b) + \mu(1-\bar{F}_m(b))s(h-m) + s(h-l) + \gamma(b)s(h-m)\right]$$

as long as $b < \hat{b}_h$, and

$$\frac{1}{6}\left[\left(4 - 2\mu + 2\eta\underline{F}_h(b) + 2\mu\bar{F}_m(b)\right)(h-b) + 2s(h-m) + \gamma(b)s(h-m)\right],$$

otherwise.

Consider the medium type. Bidding in her lower range $(l,\, \underline{b}_h)$ gives her an expected payoff of

$$\frac{1}{6}\left[\left(2 + 2(1-\mu)\underline{F}_m(b)\right)(m-b) + s(h-l)\right]$$

which yields

$$\underline{F}_m(b) = \frac{b-l}{(m-b)(1-\mu)},$$

and

$$\underline{b}_h = \frac{(1-\mu)m + l}{2-\mu}.$$

Bidding in the upper range $(\underline{b}_h,\, \bar{b}_m)$ yields

$$\frac{1}{6}\left[\left(2 + 2\eta\underline{F}_h(b) + 2(1-\mu) + 2\mu\bar{F}_m(b)\right)(m-b) + s(m-l) + (1-\eta+\eta(1-\underline{F}_h(b)))s(h-m)\right].$$

The condition that the low type is indifferent between offering m and h after observing a bid $b \in (\underline{b}_h,\, \bar{b}_m)$ entails

$$\underline{F}_h(b) = \frac{\mu}{\eta\Delta}\bar{F}_m(b).$$

for all b in this range. This allows us to verify the conjecture about the medium type. Denoting for simplicity $F = \bar{F}_m$, in the range $b < \hat{b}_h$ we obtain

$$\left[4 - 2\mu + 2\frac{\mu}{\Delta}F(b) + 2\mu F(b)\right](h-b) + \mu\left(1-F(b)\right)s(h-m) + \gamma(b)s(h-m) = \text{const.},$$

$$\left[2 + 2\frac{\mu}{\Delta}F(b) + 2(1-\mu) + 2\mu F(b)\right](m-b) + \left(1 - \eta + (\eta - \frac{\mu}{\Delta}F(b))\right)s(h-m) = \text{const.},$$

and in the range $b > \hat{b}_h$ we get

$$\left[4 - 2\mu + 2\frac{\mu}{\Delta}F(b) + 2\mu F(b)\right](h-b) + \gamma(b)s(h-m) = const,$$

$$\left[2+2\frac{\mu}{\Delta}F(b)+2(1-\mu)+2\mu F(b)\right](m-b)+\left(1-\eta+(\eta-\frac{\mu}{\Delta}F(b))\right)s(h-m)=\text{const.}$$

We can solve the second equation in each system (the same in both cases) for F:

$$F(b)=\frac{\alpha-(4-2\mu)(m-b)}{(\frac{1}{\Delta}+1)2\mu(m-b)-\frac{\mu}{\Delta}s(h-m)},$$

for some constant α to be determined. For the function F to be a well defined cdf we need

$$2(\Delta+1)>s\Delta(2-\mu).$$

To determine α, we use $F(\underline{b}_h)=0$ which yields

$$F(b)=\frac{\Delta(4-2\mu)(b-\bar{b}_h)}{(\Delta+1)2\mu(m-b)-\mu s(h-m)}.$$

Suppose that the condition $\Delta\mu>\Delta-1$ holds (which implies that the high type will indeed condition her offer on the observed bid). We can then recover $\gamma(b)$ as

$$\gamma(b)=\frac{\beta-\left[4-2\mu+2\frac{\mu}{\Delta}F(b)+2\mu F(b)\right](h-b)-\mu(1-F(b))s(h-m)}{s(h-m)},$$

for some constant β to be determined , for $b<\hat{b}_h$, and

$$\gamma(b)=\frac{\beta'-\left[4-2\mu+2\frac{\mu}{\Delta}F(b)+2\mu F(b)\right](h-b)}{s(h-m)}$$

for some constant β' to be determined, for $b>\hat{b}_h$. We know that

$$\lim_{b\searrow\underline{b}_h}\gamma(b)=1 \tag{E.10}$$

which yields

$$\beta=(1+\mu)s(h-m)+(4-2\mu)(h-m)+2(m-l).$$

We also have

$$\lim_{b\nearrow\bar{b}_m}\gamma(b)=0 \tag{E.11}$$

which gives

$$\beta'=(4+2\frac{\mu}{\Delta})(h-\bar{b}_m).$$

Moreover, $\gamma(b)$ needs to be continuous (this is necessary for the optimality of the high type's strategy), so we get

$$\mu s\Delta+(4-2\mu)\Delta+2+s=(4+2\frac{\mu}{\Delta})\frac{h-\bar{b}_m}{m-l},$$

if $\Delta\mu>\Delta-1$.

Now suppose that the opposite is true, i.e. $\Delta\mu\leq\Delta-1$. Then $\gamma(b)$ takes the form

$$\gamma(b)=\frac{\tilde{\beta}-\left[4-2\mu+2\frac{\mu}{\Delta}F(b)+2\mu F(b)\right](h-b)}{s(h-m)},$$

for all $b\in(\underline{b}_h,\bar{b}_m)$, and for some constant $\tilde{\beta}$ to be determined. From (E.10) we get

$$\tilde{\beta}=s(h-m)+[4-2\mu](h-\underline{b}_h).$$

Equation (E.11) yields

$$s\Delta+(4-2\mu)\frac{h-\underline{b}_h}{m-l}=(4+2\frac{\mu}{\Delta})\frac{h-\bar{b}_m}{m-l}.$$

The condition $F(\bar{b}_m) = 1$ gives

$$\bar{b}_m = m - \frac{(2+\mu s)\Delta}{2\mu + 4\Delta}(m-l).$$

Summarizing, we have established that

$$\begin{cases} \mu s\Delta + (4-2\mu)\Delta + 2 + s = (4+2\frac{\mu}{\Delta})\frac{h-\bar{b}_m}{m-l} & if\ \Delta\mu > \Delta - 1 \\ s\Delta + (4-2\mu)\Delta + 2 = (4+2\frac{\mu}{\Delta})\frac{h-\bar{b}_m}{m-l} & \text{otherwise} \end{cases}.$$

We also have $\mu = \eta\Delta$. Gathering all the formulas, solving a quadratic equation to determine μ, and choosing the positive root, yields an expression for μ in the two cases:

$$\mu(s) = \begin{cases} \frac{s}{(2-s)\Delta + s + 2} & \text{if } \Delta\mu > \Delta - 1 \\ \frac{s\Delta}{2\Delta + s + 2} & \text{otherwise} \end{cases},$$

Verifying that all probabilities lie between 0 and 1, and that all cumulative distribution functions are well defined yields the restriction

$$s < 1 + \frac{3}{2\Delta - 1}.$$

Under this condition, it is straightforward to check that none of the deviations for either type can be strictly profitable.

Case 2.

Subcase $\Delta \in [1/2, 1]$.

If the high type bids $\underline{b}_h$, she gets the expected payoff of

$$\frac{1}{6}\left[(4+\eta-\mu)(h-\underline{b}_h) + s(h-l) + (\gamma + \frac{\mu}{2})s(h-m)\right],$$

where $\gamma \in [0, 1]$ is the probability that the low type offers m after observing the bid $\underline{b}_h$ (equal to $\bar{b}_m$). In order to make the low type indifferent between offering m and h, we need $\eta\Delta = \mu$. If the high type bids in $(\underline{b}_h, \bar{b}_h)$, she receives

$$\frac{1}{6}\left[\left(4 + 2\eta + 2(1-\eta)\bar{F}_h(b)\right)(h-b) + s(h-l)\right].$$

From this, we can compute the cumulative distribution function

$$\bar{F}_h(b) = \frac{2+\eta}{1-\eta}\left[\frac{b-\underline{b}_h}{h-b}\right],$$

and

$$\bar{b}_h = \frac{1-\eta}{3}h + \frac{2+\eta}{3}\underline{b}_h.$$

The indifference condition for the high type reads

$$(\gamma + \frac{\mu}{2})s(h-m) = (\eta+\mu)(h-\underline{b}_h).$$

Consider a deviation of the high type to bidding $b \in (l, \underline{b}_h)$. Such a deviation would yield

$$\frac{1}{6}\left[\left(2 + 2(1-\mu)\underline{F}_m(b)\right)(h-b) + s(h-l) + s(h-m) + \left(\mu + (1-\mu)(1-\underline{F}_m(b))\right)s(h-m)\right].$$

Plugging in the distribution $\underline{F}_m$ (that will be derived later) yields the following condition

$$(4+\eta-\mu)(h-\underline{b}_h) \geq 2\frac{h-b}{m-b}(m-l) + \left((1-\frac{\mu}{2}) - \frac{b-l}{m-b}\right)s(h-m).$$

By calculating the derivative with respect to b on $(l, \underline{b}_h)$, we show that the best deviation is to $b = l$,[2] as long as $s > 2$ (which holds by assumption). This yields the condition

$$(4 + \eta - \mu)(h - \underline{b}_h) \geq 2(h - l) + (1 - \frac{\mu}{2})s(h - m).$$

Consider the medium type now. By bidding $\underline{b}_h = \bar{b}_m$, she gets

$$\frac{1}{6}\left[(4 + \eta - \mu)(m - \bar{b}_m) + s(m - l) + (1 - \frac{\eta}{2})s(h - m)\right].$$

Bidding in $(l, \bar{b}_m)$ gives

$$\frac{1}{6}\left[(2 + 2(1 - \mu)\underline{F}_m(b))(m - b) + s(h - l)\right].$$

The indifference condition for the medium type, and $\underline{F}_m(l) = 0$ yield

$$\underline{F}_m(b) = \frac{b - l}{(m - b)(1 - \mu)}.$$

Indifference between a bid in $(l, \bar{b}_m)$ and $\bar{b}_m$ yields

$$\frac{\eta}{2}s(h - m) = (\eta + \mu)(m - \bar{b}_m).$$

From $\underline{F}_m(\bar{b}_m) = 1$ we get

$$\bar{b}_m = \frac{m(1 - \mu) + l}{2 - \mu} > l.$$

Consider the deviation of the medium type to bidding $b \in (\bar{b}_m, \bar{b}_h)$. It would yield

$$\frac{1}{6}\left[\left(4 + 2\eta + 2(1 - \eta)\bar{F}_h(b)\right)(m - b) + s(m - l) + (1 - \eta)(1 - \bar{F}_h(b))s(h - m)\right].$$

Plugging in the distribution $\bar{F}_h(b)$ yields

$$\frac{1}{6}\left[\left(4 + 2\eta + 2(2 + \eta)\frac{b - \bar{b}_m}{h - b}\right)(m - b) + s(m - l) + ((1 - \eta) - (2 + \eta)\frac{b - \bar{b}_m}{h - b})s(h - m)\right].$$

Again by calculating the derivative, we verify that the most profitable deviation is $b \nearrow \bar{b}_m$. We thus obtain the condition

$$(\eta + \mu)(m - \bar{b}_m) \leq \frac{\eta}{2}s(h - m).$$

Gathering all the equation, we can calculate

$$\eta = \frac{2(s - 1)\Delta - 2}{s\Delta^2}.$$

For η to lie in $[0, 1]$, we need

$$s \geq 1 + \frac{1}{\Delta},$$

and

$$s \leq \frac{2(\Delta + 1)}{\Delta(2 - \Delta)}.$$

This produces the upper bound on s, and the obtained lower bound holds vacuously given the assumptions of the proposition.

Solving for γ yields

$$\gamma = \mu\left[\frac{(1 + \Delta)(\Delta(2 - \mu) + 1)}{s\Delta^2(2 - \mu)} - \frac{1}{2}\right].$$

[2] Formally, this is a deviation to $l + \epsilon$, for an arbitrarily small $\epsilon > 0$.

Finally, it is trivial to verify optimality for the low type.

Subcase $\Delta \in [1, 2]$.

In this subcase, the offer made by the high type in the bargaining stage depends on the value of μ. More specifically, the equilibrium expected payoff of the high type is

$$\frac{1}{6}\left[(4+\eta-\mu)(h-\underline{b}_h)+s(h-l)+(\gamma+\frac{\mu}{2})s(h-m)\right],$$

if $\mu \geq 2-2/\Delta$, where $\gamma \in [0, 1]$, and $\eta\Delta = \mu$, and

$$\frac{1}{6}\left[(4+\eta-\mu)(h-\underline{b}_h)+2s(h-m)+\gamma s(h-m)\right],$$

otherwise. If the high type bids $b \in (\underline{b}_h, \bar{b}_h)$, she obtains

$$\frac{1}{6}\left[\left(4+2\eta+2(1-\eta)\bar{F}_h(b)\right)(h-b)+2s(h-m)\right].$$

The indifference condition between bids $b \in (\underline{b}_h, \bar{b}_h)$ yields

$$\bar{F}_h(b)=\frac{2+\eta}{1-\eta}\left[\frac{b-\underline{b}_h}{h-b}\right],$$

and

$$\bar{b}_h=\frac{1-\eta}{3}h+\frac{2+\eta}{3}\underline{b}_h.$$

Moreover, the indifference condition between $\underline{b}_h$ and bids in $(\underline{b}_h, \bar{b}_h)$ gives us

$$(\gamma+\frac{\mu}{2}-1)s(h-m)+s(m-l)=(\eta+\mu)(h-\underline{b}_h),$$

if $\mu > 2-2/\Delta$, and

$$\gamma s(h-m)=(\eta+\mu)(h-\underline{b}_h),$$

otherwise.

Consider the medium type. Bidding $\bar{b}_m = \underline{b}_h$ gives her an expected payoff of

$$\frac{1}{6}\left[(4+\eta-\mu)(m-\bar{b}_m)+s(m-l)+(1-\frac{\eta}{2})s(h-m)\right],$$

if $\eta \leq 2/\Delta$, and

$$\frac{1}{6}\left[(4+\eta-\mu)(m-\bar{b}_m)+s(h-m)\right],$$

otherwise. Bidding $b \in (l, \bar{b}_m)$ yields

$$\frac{1}{6}\left[(2+2(1-\mu)\underline{F}_m(b))(m-b)+s(h-l)\right].$$

The indifference condition between bids $b \in (l, \bar{b}_m)$ yields

$$\underline{F}_m(b)=\frac{b-l}{(m-b)(1-\mu)}.$$

Indifference with $\bar{b}_m$ yields

$$\frac{\eta}{2}s(h-m)=(\eta+\mu)(m-\bar{b}_m),$$

if $\eta \leq 2/\Delta$, and

$$s(m-l)=(\eta+\mu)(m-\bar{b}_m),$$

otherwise. From $\underline{F}_m(\bar{b}_m) = 1$, we get

$$\bar{b}_m = \frac{m(1-\mu)+l}{2-\mu} > l.$$

Summarizing, we have (recall that $\mu = \eta\Delta$)

$$\begin{cases} (\gamma + \frac{\eta\Delta}{2} - 1)s\Delta + s = (\eta+\eta\Delta)(\Delta + \frac{1}{2-\eta\Delta}) \text{ and } \frac{\eta}{2}s\Delta = (\eta+\eta\Delta)\frac{1}{2-\eta\Delta} & \text{if } \mu > 2 - \frac{2}{\Delta} \text{ and } \eta \le \frac{2}{\Delta}, \\ \gamma s\Delta = (\eta+\eta\Delta)(\Delta + \frac{1}{2-\eta\Delta}) \text{ and } \frac{\eta}{2}s\Delta = (\eta+\eta\Delta)\frac{1}{2-\eta\Delta} & \text{if } \mu \le 2 - \frac{2}{\Delta} \text{ and } \eta \le \frac{2}{\Delta}, \\ (\gamma + \frac{\eta\Delta}{2} - 1)s\Delta + s = (\eta+\eta\Delta)(\Delta + \frac{1}{2-\eta\Delta}) \text{ and } s = (\eta+\eta\Delta)\frac{1}{2-\eta\Delta} & \text{if } \mu > 2 - \frac{2}{\Delta} \text{ and } \eta > \frac{2}{\Delta}, \\ \gamma s\Delta = (\eta+\eta\Delta)(\Delta + \frac{1}{2-\eta\Delta}) \text{ and } s = (\eta+\eta\Delta)\frac{1}{2-\eta\Delta} & \text{if } \mu \le 2 - \frac{2}{\Delta} \text{ and } \eta > \frac{2}{\Delta}. \end{cases}$$

In each case the second equation identifies η. We have

$$\begin{cases} \eta = \frac{2s\Delta - 2(\Delta+1)}{s\Delta^2} & \text{if } \mu > 2 - \frac{2}{\Delta} \text{ and } \eta \le \frac{2}{\Delta}, \\ \eta = \frac{2s\Delta - 2(\Delta+1)}{s\Delta^2} & \text{if } \mu \le 2 - \frac{2}{\Delta} \text{ and } \eta \le \frac{2}{\Delta}, \\ \eta = \frac{2s}{1+\Delta+s\Delta} & \text{if } \mu > 2 - \frac{2}{\Delta} \text{ and } \eta > \frac{2}{\Delta}, \\ \eta = \frac{2s}{1+\Delta+s\Delta} & \text{if } \mu \le 2 - \frac{2}{\Delta} \text{ and } \eta > \frac{2}{\Delta}. \end{cases}$$

The last two cases produce a contradiction, the first case is valid when $\Delta + 1 \le s$, and the second case is valid when $\Delta + 1 > s$. This means that the medium type always makes an offer to the low type when she wins at $\bar{b}_m$. But the offer made by the high type may vary, depending on μ.

Corresponding to the two cases, we can calculate γ as

$$\begin{cases} \gamma = 1 + \mu \left(\frac{(1+\Delta)(\Delta(2-\mu)+1)}{s\Delta^2(2-\mu)} - \frac{2+\Delta^2}{2\Delta^2} \right) & \text{if } s > \Delta + 1, \\ \gamma = \mu \left(\frac{(1+\Delta)(\Delta(2-\mu)+1)}{s\Delta^2(2-\mu)} - \frac{1}{2} \right) & \text{if } s \le \Delta + 1. \end{cases}$$

Finally, notice that in both cases

$$\mu = \frac{2s\Delta - 2(\Delta+1)}{s\Delta}.$$

Requiring that $\mu \le 1$ yields the upper bound on s.

Under these conditions, it is now a routine exercise to check that none of the types can deviate profitably.

Case 3.

The equilibrium expected payoff of the high type is

$$\frac{1}{6}\left[(5-\mu)(h-\bar{b}_m) + (1+\frac{\mu}{2})s(h-m) + s(h-l)\right].$$

Bidding above (where I specify off-equilibrium beliefs to be monotone, i.e. assign probability one to the high type if a bid above $\bar{b}_m$ is observed) would yield

$$(h-\bar{b}_m) + \frac{1}{6}s(h-l),$$

so we obtain a condition

$$(1+\frac{\mu}{2})s(h-m) \ge (1+\mu)(h-\bar{b}_m).$$

The medium type's indifference condition on $(l, \bar{b}_m)$ yields (as in the previous cases)

$$\underline{F}_m(b) = \frac{b-l}{(m-b)(1-\mu)},$$

and

$$\bar{b}_m = \frac{m(1-\mu)+l}{2-\mu}.$$

The expected payoff of the medium type is

$$\frac{1}{6}\left[(4-2\mu)(m-\bar{b}_m)+s(h-l)\right].$$

Bidding $\bar{b}_m$ gives

$$\frac{1}{6}\left[(5-\mu)(m-\bar{b}_m)+s(m-l)+\frac{1}{2}s(h-m)\right].$$

Bidding above $\bar{b}_m$ would give

$$(m-\bar{b}_m)+\frac{1}{6}s(m-l).$$

The indifference condition together with the restriction that the deviation is not profitable yield

$$\mu=\frac{2s\Delta-2}{2+s\Delta}.$$

For this quantity to be a well defined probability, we need $s\Delta\in[1,\,4]$. This gives us the upper bound on s.

Now, consider a deviation of the high type to bidding $b\in(l,\,\bar{b}_m)$. In that case, the high type receives

$$\frac{1}{6}\left[\left(2+2\frac{b-l}{m-b}\right)(h-b)+s(h-l)+s(h-m)+\left(\mu+(1-\mu)-\frac{b-l}{m-b}\right)s(h-m)\right].$$

The sign of the derivative of the expected payoff with respect to b is determined by the sign of the expression

$$(2-s)(m-l)(h-m)$$

so for $s<2$ we need to consider the deviation to the right endpoint $\bar{b}_m$, and for $s>2$ to the left endpoint l. This gives us two conditions

$$\begin{cases}(1+\mu)\Delta+\frac{1-\mu}{2-\mu}\geq\frac{\mu}{2}s\Delta & \text{if}\, s<2\\ (5-\mu)\Delta+\frac{5-\mu}{2-\mu}\geq 2\Delta+2+(1-\frac{\mu}{2})s\Delta & \text{if}\, s>2\end{cases}.$$

Plugging in μ, we obtain a restriction on the parameters that is vacuously satisfied given the assumptions of the proposition.

Checking optimality for the low type is a routine exercise (and it does not produce additional constraints on the parameters).

E.3 First price auction with no announcement

Case 1.

Subcase $\Delta\in[1,\,2]$.

In the bargaining stage, in equilibrium, the low type always offers to sell for h. The medium type offers to sell for h when she loses, and offers to buy for l when she wins. The high type always offers to buy for m.

Consider the high type. In equilibrium, she bids according to

$$F_h(b)=\frac{2b-m-l}{h-b},$$

and gets the expected payoff of

$$\frac{1}{6}\left[4h-2m-2l+2s(h-m)\right].$$

The only deviation we have to check is to bidding in the range of the medium type. If the high type bids $b\in(l,\,(l+m)/2)$, she obtains

$$\frac{1}{6}\left[(2+2F_m(b))\,(h-b)+s(h-l)+(1-F_m(b))s(h-m)\right],$$

if $F_m(b) < 1/\Delta$, and

$$\frac{1}{6}\left[\left(2+2F_m(b)\right)(h-b)+2s(h-m)\right],$$

otherwise. Using the fact that $F_m(b) = (b-l)/(m-b)$, we can simplify the two expressions to

$$\frac{1}{6}\left[2\frac{h-b}{m-b}(m-l)+s(h-l)+\frac{m+l-2b}{m-b}s(h-m)\right],$$

if $b < l + (m-l)/(\Delta+1)$, and

$$\frac{1}{6}\left[2\frac{h-b}{m-b}(m-l)+2s(h-m)\right],$$

otherwise. In the first case, the derivative of the expression with respect to b is positive if $s \leq 2$, which shows that the deviation is not profitable. An analogous conclusion holds for the second case, establishing optimality of the strategy of the high type under $s \leq 2$.

Consider the medium type. Because she cannot increase her expected payoff in the bargaining stage, the optimality of the bidding strategy is trivial to verify.

Finally, consider the low type. In equilibrium she gets an expected payoff of

$$\frac{1}{6}\left[s(h-l)+s(m-l)\right].$$

The only deviation we have to check is to a bid which would allow her to distinguish between the medium and high type. Bidding in the range of the medium type yields

$$\frac{1}{6}\left[s(h-l)+\left(2+2\frac{b-l}{m-b}\right)(l-b)+\left(1+\frac{b-l}{m-b}\right)s(m-l)\right],$$

which simply boils down to $s \leq 2$.

Subcase $\Delta \in [1/2, 1)$.

This case is fully analogous but I nevertheless provide some details below.

In equilibrium, the low type always offers to sell for m. The medium type offers to sell for h when she loses, and offers to buy for l when she wins. The high type always offers to buy for l.

Consider the high type. In equilibrium, she bids according to

$$F_h(b) = \frac{2b-m-l}{h-b},$$

and gets an expected payoff of $\left[4h-2m-2l+s(h-l)\right]/6$. We verify that a deviation to bidding in the range of the medium type is not profitable under the restriction that $s \leq 2$.

The optimality for the medium type is again trivial to check.

As for the low type, in equilibrium, she gets an expected payoff of $s(m-l)/3$. The only deviation we have to check is to a bid which would allow her to distinguish between the medium and the high type. Bidding b in the support of the medium type yields an expected payoff

$$\frac{1}{6}\left[s(h-l)+\left(2+2\frac{b-l}{m-b}\right)(l-b)+\frac{b-l}{m-b}s(m-l)\right],$$

if $F_m(b) \geq 1-\Delta$, and

$$\frac{1}{6}\left[2s(m-l)+\left(2+2\frac{b-l}{m-b}\right)(l-b)\right],$$

otherwise. Analyzing the derivatives of these expressions with respect to b, we conclude the deviation cannot be profitable if $s \leq 2$.

Case 2.

Consider the low type first. Her equilibrium payoff (from bidding l) is $[s(h-l)+s(h-m)]/6$. The indifference condition between bidding l and $b \in (l, \bar{b}_m)$ yields

$$2\left[F_m(b)+1-\theta+\theta F_l(b)\right](l-b)+s(h-l)+s(h-m)+F_m(b)s(m-l)=s(h-l)+s(h-m),$$

or simplifying

$$2\left[F_m(b)+1-\theta+\theta F_l(b)\right](b-l)=F_m(b)s(m-l).$$

The indifference condition for the medium type is

$$\begin{aligned}2\left[F_m(b)+1-\theta+\theta F_l(b)\right](m-b)+\left[1-\theta+\theta F_l(b)\right]s(m-l)+s(h-m)\\ =2(1-\theta)(m-l)+(1-\theta)s(m-l)+s(h-m),\end{aligned}$$

or simplifying

$$2\left[F_m(b)+1-\theta+\theta F_l(b)\right](m-b)+\theta F_l(b)s(m-l)=2(1-\theta)(m-l).$$

We can now calculate $\bar{b}_m$. From the low type's indifference condition we get

$$\bar{b}_m=l+\frac{1}{4}s(m-l).$$

Combining with the indifference condition for the medium type we obtain

$$\theta=\frac{s-2}{s+2},$$

and

$$F_l(b)=F_m(b)=\frac{4}{s}\frac{b-l}{\frac{s}{2}(m-l)+(m-b)-(b-l)}.$$

Having specified these distributions, it is a straightforward exercise to verify that there is no profitable deviation for either of the two types under the imposed assumptions.

Now consider the high type. Her equilibrium expected payoff from bidding $b \in (\underline{b}_h, \bar{b}_h)$ is

$$\frac{1}{6}\left[\left(4+2F_h(b)\right)(h-b)+2s(h-m)\right],$$

yielding the equilibrium distribution

$$F_h(b)=2\left[\frac{h-\bar{b}_m}{h-b}-1\right].$$

We can then calculate $\bar{b}_h$ as

$$\bar{b}_h=\frac{1}{3}h+\frac{2}{3}\bar{b}_m.$$

The last thing we need to check is the possibility of a profitable deviation for the high type. Without loss of generality, we can assume that the high type will offer to buy for m when she loses and offer to buy for l when she wins (if it is not true, then the deviation cannot be profitable). Hence, a deviation to $b \in (l, \bar{b}_m)$ yields (after some simplifications)

$$\frac{1}{6}\left[2\left(1-\theta+(\theta+1)F_l(b)\right)(h-b)+\left(1-\theta+\theta F_l(b)\right)s(h-l)+(2\theta+1)(1-F_l(b))s(h-m)\right].$$

Therefore, we need the condition that, for all $b \in (l, \bar{b}_m)$,

$$\begin{aligned}2\left(1-\theta+(\theta+1)F_l(b)\right)(h-b)+\left(1-\theta+\theta F_l(b)\right)s(h-l)+(2\theta+1)(1-F_l(b))s(h-m)\\ \leq 4(h-a)+2s(h-m).\end{aligned}$$

Taking the derivative of the left hand side with respect to b, we see that the most profitable deviation is bidding $b \nearrow l$, which gives us the condition

$$s\leq\frac{6\Delta-2}{\Delta+1}.$$

This is the condition imposed by the assumptions of the proposition.

Case 3.

This case is analogous to the Case 2 above (with the roles of the low and high type exchanged). First, we write down indifference conditions for the medium and high type when bidding $b \in (l, \bar{b}_m)$:

$$2\left[\eta \underline{F}_h(b) + F_m(b) + 1\right](h-b) + s(h-l) + s(h-m) + (1 - F_m(b))s(h-m) \\ = 2(h-l) + s(h-l) + 2s(h-m),$$

and

$$2\left[\eta \underline{F}_h(b) + F_m(b) + 1\right](m-b) + \left[1 - \eta + \eta(1 - \underline{F}_h(b))\right] s(h-m) + s(m-l) \\ = 2(m-l) + s(h-m) + s(m-l).$$

The high type gets the following expected payoff when bidding in her upper range $(\bar{b}_m, \bar{b}_h)$:

$$\frac{1}{6}\left[2\left(2 + \eta + (1-\eta)\bar{F}_h(b)\right)(h-b) + s(h-l) + s(h-m)\right],$$

which yields

$$\bar{F}_h(b) = \frac{2+\eta}{1-\eta}\left(\frac{h - \bar{b}_m}{h-b} - 1\right).$$

The indifference condition for the high type yields

$$2(2+\eta)(h - \bar{b}_m) = 2(h-l) + s(h-m).$$

Combining and simplifying the obtained equations we conclude that

$$\eta = \frac{s-2}{s+2},$$

and

$$\bar{b}_m = (1-x)m + xl,$$

where

$$x = \frac{1}{2} + (s-2)\frac{s\Delta - 1}{6s+4}.$$

Moreover, we can solve for the equilibrium distributions

$$F_m(b) = \underline{F}_h(b) = \frac{s+2}{s}\frac{2(b-l)}{2(h-b) - 2(b-m) - s(h-m)}.$$

We can also determine $\bar{b}_h$ from the equation

$$\frac{h - \bar{b}_m}{h - \bar{b}_h} = \frac{3s+6}{3s+2}.$$

This guarantees that $\bar{b}_h$ is always greater than $\bar{b}_m$. Ruling out deviations of the high and medium types does not produce binding constraints on parameters, but we need to consider deviations for the low type. The only deviation we have to check is to bidding in the range $(l, \bar{b}_m)$. By analyzing the derivative of the profit with respect to b, we conclude that the most attractive deviation is to bid $\bar{b}_m$ which yields the restriction

$$2s(m-l) \geq 2(2+\eta)(l - \bar{b}_m) + 2\eta s(m-l) + s(m-l) + (1-\eta)s(h-l),$$

or, simplifying,

$$s \leq \frac{6 - 2\Delta}{\Delta + 1}.$$

Case 4.

The strategies for the low and medium type are defined similarly as in Case 2. However, we have to account for a more complicated strategy of the high type. The high type offers to buy for l after winning the auction, and offers to buy for m after losing the auction (with a bid $b \in (l, b^\star)$). (Here, we skip a more detailed description of the strategies in the bargaining stage, see the analogous Case 5 for more details.)

The indifference conditions now take the following form. For the high type, when she bids $b \in (l, b^\star)$, we must have

$$2\left(\eta \underline{F}_h(b) + F_m(b) + \theta F_l(b) + 1 - \theta\right)(h - b) + \left(1 - F_m(b) + 2\theta(1 - F_l(b))\right) s(h - m) \\ + \left(1 - \theta + \theta F_l(b)\right) s(h - l) = (2 + s)(1 - \theta)(h - l) + (1 + 2\theta)s(h - m).$$

For the medium type, for $b \in (l, b^\star)$,

$$2\left(\eta \underline{F}_h(b) + F_m(b) + \theta F_l(b) + 1 - \theta\right)(m - b) + \left(1 - \eta + \eta(1 - \underline{F}_h(b))\right) s(h - m) \\ + \left(1 - \theta + \theta F_l(b)\right) s(m - l) = 2(1 - \theta)(m - l) + s(h - m) + (1 - \theta)s(m - l),$$

and for $b \in (b^\star, \bar{b}_m)$,

$$2\left(\eta + F_m(b) + \theta F_l(b) + 1 - \theta\right)(m - b) + (1 - \eta)s(h - m) + \\ \left(1 - \theta + \theta F_l(b)\right) s(m - l) = 2(1 - \theta)(m - l) + s(h - m) + (1 - \theta)s(m - l).$$

For the low type: when $b \in (l, b^\star)$, we require

$$2\left(\eta \underline{F}_h(b) + F_m(b) + \theta F_l(b) + 1 - \theta\right)(l - b) + \left(2\eta \underline{F}_h(b) + F_m(b) + 1 - \eta\right) s(m - l) \\ + \left(1 - \eta + \eta(1 - \underline{F}_h(b))\right) s(h - l) = s(h - l) + (1 - \eta)s(m - l),$$

and for $b \in (b^\star, \bar{b}_m)$,

$$2\left(\eta + F_m(b) + \theta F_l(b) + 1 - \theta\right)(l - b) + \left(\eta + F_m(b) + 1\right) s(m - l) + (1 - \eta)s(h - l) \\ = s(h - l) + (1 - \eta)s(m - l).$$

We get two sets of conditions. First, setting $b = b^\star$, we obtain

$$2\left(\eta + F_m(b^\star) + \theta F_l(b^\star) + 1 - \theta\right)(h - b^\star) + \left(1 - F_m(b^\star) + 2\theta(1 - F_l(b^\star))\right) s(h - m) + \\ \left(1 - \theta + \theta F_l(b^\star)\right) s(h - l) = (2 + s)(1 - \theta)(h - l) + (1 + 2\theta)s(h - m),$$

$$2\left(\eta + F_m(b^\star) + \theta F_l(b^\star) + 1 - \theta\right)(m - b^\star) + (1 - \eta)s(h - m) + \left(1 - \theta + \theta F_l(b^\star)\right) s(m - l) \\ = 2(1 - \theta)(m - l) + s(h - m) + (1 - \theta)s(m - l),$$

$$2\left(\eta + F_m(b^\star) + \theta F_l(b^\star) + 1 - \theta\right)(l - b^\star) + \left(\eta + F_m(b^\star) + 1\right) s(m - l) + (1 - \eta)s(h - l) \\ = s(h - l) + (1 - \eta)s(m - l).$$

This is a system of linear equations in $F_m(b^\star)$, $F_l(b^\star)$ and $b^\star$. We can show that

$$\Delta = \frac{1 - \theta + \theta F_l(b^\star)}{F_m(b^\star)}.$$

Similarly, in order to calculate η and θ, setting $b = \bar{b}_m$, we obtain

$$2(\eta + 2)\beta = 2(1 - \theta) - \theta s + \eta s \Delta$$

$$2(\eta + 2)(\beta - 1) + s = \eta s(\Delta - 1),$$

where

$$\beta = \frac{m - \bar{b}_m}{m - l}.$$

And finally we have the indifference condition for the high type,

$$2(2+\eta)(\Delta+\beta) = (2+s)(1-\theta)(\Delta+1) + (2\theta-1)s\Delta.$$

Solving this system of equations, we obtain

$$\eta = \frac{s - 6\Delta + \Delta s + 2}{8\Delta},$$

$$\theta = \frac{(s-2)(\Delta+1)}{8\Delta},$$

$$\beta = \frac{\Delta^2 s^2 - 6\Delta^2 s + 2\Delta s + 20\Delta - s^2 + 4}{2(10\Delta + s + \Delta s + 2)},$$

$$F_m(b^\star) = \frac{s - 6\Delta + \Delta s + 2}{2\Delta(s-2)},$$

$$F_l(b^\star) = \frac{s - 6\Delta + \Delta s + 2}{(\Delta+1)(s-2)}.$$

In order to make sure that these quantities are well defined, we impose the restrictions

$$s \geq \frac{6\Delta - 2}{\Delta + 1},$$

and

$$s \leq \min\left\{\frac{10\Delta+2}{\Delta+1}, \frac{2\Delta+2}{\Delta-1}\right\}.$$

Note that we allow β to be below 0. Similarly, define

$$\alpha = \frac{m - b^\star}{m - l}.$$

Using a symbolic equation solver, we can show that

$$\alpha = \frac{s-2}{2}\frac{\Delta^2 s^2 - 6\Delta^2 s + 6\Delta s + 20\Delta - s^2 - 4s + 4}{8s - 20\Delta + \Delta s^2 + s^2 - 4}.$$

The constraint $s \leq (10\Delta+2)/(\Delta+1)$ comes from $\theta \leq 1$, and $s \leq (2\Delta+2)/(\Delta-1)$ comes from $\alpha \leq 1$ (or $b^\star \geq l$).

When the high type bids in the upper range $(\bar{b}_m, \bar{b}_h)$, she gets

$$2\left(2+\eta+(1-\eta)\bar{F}_h(b)\right)(h-b) + 2s(h-m),$$

which yields

$$\bar{F}_h(b) = \frac{2+\eta}{1-\eta}\left(\frac{h-\bar{b}_m}{h-b} - 1\right),$$

and

$$\frac{h-\bar{b}_m}{h-\bar{b}_h} = \frac{3}{2+\eta}.$$

Finally, the distributions $F_l(b)$, $F_m(b)$, and $\underline{F}_h(b)$ can be calculated from the system of linear equations presented above but the analytic expressions are too complicated to be useful (I thus do not report them here). Under these parameter restriction, there are no profitable deviations for either type.

Case 5.

This case is analogous to Case 4 but with the roles of the high and low types exchanged.

Consider the high type first. In her upper range $(\bar{b}_m, \bar{b}_h)$, she always wins (against other types) and offers to sell for l. In her lower range $(l, \bar{b}_m)$, she offers to sell for l when she wins, and offers to sell for m when she loses. We therefore need $(1 - F_m(b))\Delta \geq \theta$ and $F_m(b)\Delta \leq (1-\theta)$, if $b < b^\star$, which boils down to the condition

$$(1 - F_m(b^\star))\Delta \geq \theta.$$

Similarly, in the range $b \in (b^\star, \bar{b}_m)$ we need

$$(1 - F_m(b))\Delta \geq \theta(1 - F_l(b)).$$

The medium type offers to sell when she loses, and offers to buy when she wins. This gives us the following condition that needs to hold for all $b \in (l, \bar{b}_m)$,

$$\eta \underline{F}_h(b) \leq \Delta + \theta F_l(b) - \theta.$$

And finally, the low type offers to sell for m after bidding l, and offers to sell for h when she loses after bidding $b \in (b^\star, \bar{b}_m)$, which yields the restriction, for all $b \in (b^\star, \bar{b}_m)$,

$$(1 - \eta \underline{F}_h(b))\Delta \geq 1 - F_m(b).$$

Now we are ready to calculate the payoffs from these strategies. The indifference condition for the high type when she bids $b \in (l, b^\star)$ is

$$\begin{aligned} 2\left(\eta \underline{F}_h(b) + F_m(b) + 1 - \theta\right)(h - b) + \left(2 - F_m(b) + \theta\right) s(h-m) + (1-\theta)s(h-l) \\ = (2+s)(1-\theta)(h-l) + (2+\theta)s(h-m), \end{aligned}$$

and for $b \in (b^\star, \bar{b}_m)$,

$$\begin{aligned} 2\left(\eta \underline{F}_h(b) + F_m(b) + 1 - \theta + \theta F_l(b)\right)(h - b) + \left(2 - F_m(b) + \theta - 2\theta F_l(b)\right) s(h-m) + \\ \left(1 - \theta + \theta F_l(b)\right) s(h-l) = (2+s)(1-\theta)(h-l) + (2+\theta)s(h-m). \end{aligned}$$

The indifference condition for the medium type when she bids $b \in (l, b^\star)$ is

$$\begin{aligned} 2\left(\eta \underline{F}_h(b) + F_m(b) + 1 - \theta\right)(m - b) + \left(1 - \eta \underline{F}_h(b)\right) s(h-m) + (1-\theta)s(m-l) \\ = 2(1-\theta)(m-l) + s(h-m) + (1-\theta)s(m-l), \end{aligned}$$

and for $b \in (b^\star, \bar{b}_m)$,

$$\begin{aligned} 2\left(\eta \underline{F}_h(b) + F_m(b) + \theta F_l(b) + 1 - \theta\right)(m - b) + \left(1 - \eta \underline{F}_h(b)\right) s(h-m) + \\ \left(1 - \theta + \theta F_l(b)\right) s(m-l) = 2(1-\theta)(m-l) + s(h-m) + (1-\theta)s(m-l). \end{aligned}$$

Finally, the indifference condition for the low type when she bids $b \in (b^\star, \bar{b}_m)$,

$$\begin{aligned} 2\left(\eta \underline{F}_h(b) + F_m(b) + \theta F_l(b) + 1 - \theta\right)(l - b) + \left(2\eta \underline{F}_h(b) + F_m(b)\right) s(m-l) \\ + \left(1 - \eta \underline{F}_h(b)\right)) s(h-l) = 2s(m-l). \end{aligned}$$

Setting $b = \bar{b}_m$ in the equations above (whenever applicable), we obtain

$$2(\eta + 2)(\Delta + \beta') = 2(1-\theta)(\Delta + 1) - \theta s + (1+\theta)s\Delta,$$

$$2(\eta + 2)\beta' = 2(1-\theta) - \theta s + \eta s \Delta,$$

$$2(\eta + 2)(\beta' - 1) + \eta s + (1-\eta)s\Delta = 0,$$

where

$$\beta' = \frac{m - \bar{b}_m}{m - l}.$$

Solving this system we obtain,

$$\theta = \frac{s - 6 + \Delta(2 + s)}{8},$$

$$\eta = \frac{(s-2)(\Delta+1)}{8},$$

$$\beta' = \frac{(\Delta^2 s - \Delta^2 s)(s-2) - 6\Delta s - 4\Delta - s^2 + 4s + 28}{2(s + 14 + \Delta(s-2))}.$$

Setting $b = b^\star$, we obtain

$$2\left(\eta \underline{F}_h(b^\star) + F_m(b^\star) + 1 - \theta\right)(h - b^\star) + \left(2 - F_m(b^\star) + \theta\right)s(h-m) + (1-\theta)s(h-l) \\ = (2+s)(1-\theta)(h-l) + (2+\theta)s(h-m) + (1-\theta)s(h-l),$$

$$2\left(\eta \underline{F}_h(b^\star) + F_m(b^\star) + 1 - \theta\right)(m - b^\star) + \left(1 - \eta \underline{F}_h(b^\star)\right)s(h-m) + (1-\theta)s(m-l) \\ = 2(1-\theta)(m-l) + s(h-m) + (1-\theta)s(m-l),$$

$$2\left(\eta \underline{F}_h(b^\star) + F_m(b^\star) + 1 - \theta\right)(l - b^\star) + \left(2\eta \underline{F}_h(b^\star) + F_m(b^\star)\right)s(m-l) \\ + \left(1 - \eta \underline{F}_h(b^\star)\right)s(h-l) = 2s(m-l).$$

This yields

$$\underline{F}_h(b^\star) = \frac{4(1-\Delta)}{(\Delta+1)(s-2)},$$

$$F_m(b^\star) = \frac{(1-\Delta)(s+2)}{2(s-2)},$$

$$\alpha' = \frac{(s-2)(2\Delta + s - \Delta s + 2\Delta^2 s - 14)}{8\Delta s - 24s - 4\Delta + \Delta s^2 + s^2 + 28},$$

where

$$\alpha' = \frac{m - b^\star}{m - l}.$$

In order to make sure that all of the defined coefficients are in the permitted ranges, we impose the conditions

$$s \geq \frac{6 - 2\Delta}{\Delta + 1},$$

and

$$s \leq \frac{2\Delta + 10}{\Delta + 1}.$$

The first condition corresponds to $\theta = 0$, and the second one corresponds to $\eta = 1$.

The distribution for the high type in the range $(\bar{b}_m, \bar{b}_h)$ can be determined in an analogous way as in Case 4.

Finally, the distributions $F_l(b)$, $F_m(b)$, and $\underline{F}_h(b)$ can be calculated from the system of linear equations presented above but the analytic expressions are too complicated to be useful (I thus do not report them here). Strictly profitable deviations do not exist for either type under the imposed parameter restrictions.

E.4 First price auction with revelation of all bids

The analysis of this auction design is similar to the analysis of the auction with the revelation of the winning bid only, and I therefore make the proofs more compact by skipping some of the details.

Case 1.

Consider the low type. In the bargaining stage she offers to sell for h after observing a bid above $\bar{b}_l$, and offers to sell for m after observing a bid below $\bar{b}_l$. When she bids $b \in (l, \bar{b}_l)$, the indifference condition reads

$$2\left(\mu_0 \underline{F}_m(b) + 1 - \theta + \theta F_l(b)\right)(l - b) + s(h - l) + \mu_0 s(m - l) + \delta(b) s(m - l) = s(h - l) + \mu_0 s(m - l),$$

where recall that $\delta(b)$ denotes the conditional probability with which the high type offers to buy for m after observing a bid $b \in (l, \bar{b}_l)$.

Consider the medium type: when she bids $b \in (l, \bar{b}_l)$, the indifference conditions becomes

$$2\left(\mu_0 \underline{F}_m(b) + 1 - \theta + \theta F_l(b)\right)(m - b) + s(h - m) + s(m - l) = 2(1 + \mu_0)(m - \bar{b}_l) + s(h - m) + s(m - l).$$

Finally in this interval of bids, we have to make sure that the high type is indifferent between offering m and l after observing $b \in (l, \bar{b}_l)$. This requires, for all $b \in (l, \bar{b}_l)$,

$$\mu_0 \underline{F}_m(b) = \theta F_l(b),$$

and thus in particular

$$\mu_0 = \theta,$$

and

$$\underline{F}_m(b) = F_l(b) =: \underline{F}(b).$$

From the above conditions we can calculate $\underline{F}(b)$ and $\delta(b)$.

Consider the interval $(\bar{b}_l, \underline{b}_h)$. Here, we only need to determine the distribution of bids $\hat{F}_m$ for the medium type. The indifference condition for the medium type gives us

$$2\left(\mu_0 + \mu_1 \hat{F}_m(b) + 1\right)(m - b) + s(h - m) + s(m - l) = 2(\mu_0 + 1)(m - \bar{b}_l) + s(h - m) + s(m - l).$$

Consider the interval $(\underline{b}_h, \bar{b}_m)$. The high type's indifference condition is, for every $b \in (\underline{b}_h, \bar{b}_m)$,

$$\begin{aligned} 2\left(\eta \underline{F}_h(b) + \mu_0 + \mu_1 + (1 - \mu_0 - \mu_1)\bar{F}_m(b) + 1\right)(h - b) + s(h - m) + (1 - \theta)s(h - l) + \gamma(b)s(h - m) \\ = 2(\eta + 2)(h - \bar{b}_m) + s(h - m) + (1 - \theta)s(h - l), \end{aligned}$$

where recall that $\gamma(b)$ denotes the conditional probability with which the low type offers to sell for m after observing a bid $b \in (\underline{b}_h, \bar{b}_m)$.

Analogously, for the medium type, for every $b \in (\underline{b}_h, \bar{b}_m)$, we have

$$\begin{aligned} 2\left(\eta \underline{F}_h(b) + \mu_0 + \mu_1 + (1 - \mu_0 - \mu_1)\bar{F}_m(b) + 1\right)(m - b) + s(h - m) + s(m - l) \\ = 2\left(\mu_0 + \mu_1 + 1\right)(m - \underline{b}_h) + s(h - m) + s(m - l). \end{aligned}$$

We also need to make sure that the low type is indifferent between offering m and h after observing a bid $b \in (\underline{b}_h, \bar{b}_m)$ which leads to the condition, for every $b \in (\underline{b}_h, \bar{b}_m)$,

$$(1 - \mu_0 - \mu_1)\bar{F}_m(b) = \eta \underline{F}_h(b).$$

Therefore, we have

$$1 - \mu_0 - \mu_1 = \eta,$$

and

$$\bar{F}_m(b) = \underline{F}_h(b) =: \bar{F}(b).$$

From the above conditions we can determine $\bar{F}(b)$ and $\gamma(b)$.

Finally, in the interval $(\bar{b}_m, \bar{b}_h)$ we have the following indifference condition for the high type:

$$\begin{aligned} 2\left(2 + \eta + (1 - \eta)\bar{F}_h(b)\right)(h - b) + s(h - m) + (1 - \theta)s(h - l) \\ = 2(2 + \eta)(h - \bar{b}_m) + s(h - m) + (1 - \theta)s(h - l). \end{aligned}$$

Denote $x = (m - \bar{b}_l)/(m - l)$, $y = (m - \underline{b}_h)/(m - l)$, and $z = (m - \bar{b}_m)/(m - l)$. Then, considering the above equations at the cutoff bids $b = \bar{b}_l, \underline{b}_h, \bar{b}_m$, we obtain

$$2(\mu_0 + 1)(x - 1) + s = 0,$$

$$2(1 - \mu_0) = 2(1 + \mu_0)x,$$

$$2(\mu_0 + \mu_1 + 1)y = 2(1 + \mu_0)x,$$

$$2\left(3 - \mu_0 - \mu_1\right) z = 2\left(1 + \mu_0 + \mu_1\right) y,$$

$$2\left(1 + \mu_0 + \mu_1\right)(1 + y) + s = 2\left(3 - \mu_0 - \mu_1\right)(1 + z).$$

Solving, we get

$$\mu_0 = \theta = \eta = \frac{s}{4},$$

$$\mu_1 = 1 - \frac{s}{2},$$

$$x = \frac{4 - s}{4 + s},$$

$$y = \frac{4 - s}{8 - s},$$

$$z = \frac{4 - s}{8 + s}.$$

These quantities are well defined if and only if $s \leq 2$. For $s = 2$, we have $\mu_1 = 0$, and $x = y$.

Plugging in these solutions, and solving the system consisting of remaining equations, we get

$$\underline{F}(b) = \frac{b - l}{m - b} \frac{4 - s}{2s},$$

$$\hat{F}_m(b) = \frac{1 + \mu_0}{\mu_1} \left(\frac{m - a}{m - b} - 1 \right),$$

$$\bar{F}(b) = \frac{4(2b - m - l) - (b - l)s}{2s(m - b)},$$

$$\bar{F}_h(b) = \frac{2 + \eta}{1 - \eta} \left(\frac{h - \bar{b}_m}{h - b} - 1 \right),$$

$$\delta(b) = \frac{b - l}{m - b} \frac{4 - s}{2s},$$

$$\gamma(b) = \frac{4\left(3m - 2b - h\right) - (h - b)s}{2s(m - b)},$$

$$\frac{h - \bar{b}_m}{h - \bar{b}_h} = \frac{3}{2 + \eta}.$$

This defines all of the equilibrium quantities. Under the imposed condition $s \leq 2$, all the functions and distributions are well defined. It is also a straightforward exercise to check that there are no profitable deviations for any of the types.

Case 2.

We first consider bids in the interval $(l,\, \underline{b}_h)$. In the bargaining stage the low type offers to sell for h after observing a bid above $\bar{b}_l$, and offers to sell for m after observing a bid below $\bar{b}_l$. The indifference condition for the low type is

$$2\left(\mu_0\underline{F}_m(b) + 1 - \theta_0 - \theta_1 + \theta_0\underline{F}_l(b)\right)(l-b) + s(h-l) + \mu_0 s(m-l) + \delta(b)s(m-l) \\ = s(h-l) + \mu_0 s(m-l).$$

For the medium type, the indifference condition reads, for every $b \in (l,\, \underline{b}_h)$,

$$2\left(\mu_0\underline{F}_m(b) + 1 - \theta_0 - \theta_1 + \theta_0\underline{F}_l(b)\right)(m-b) + s(h-m) + s(m-l) \\ = 2\left(\mu_0 + 1 - \theta_1\right)(m - \underline{b}_h) + s(h-m) + s(m-l).$$

To make sure that the high type is indifferent between offering l and m after observing $b \in (l,\, \underline{b}_h)$, we require,

$$\mu_0\underline{F}_m(b) = \theta_0\underline{F}_l(b),$$

and so

$$\mu_0 = \theta_0,$$

and

$$\underline{F}_m(b) = \underline{F}_l(b) = \underline{F}(b).$$

From this, we can calculate $\underline{F}(b)$ and $\delta(b)$ on $(l,\, \underline{b}_h)$.

Consider the interval $(\underline{b}_h,\, \bar{b}_l)$. The indifference condition for the low type is

$$2\left(\mu_0 + \mu_1\hat{F}_m(b) + 1 - \theta_1 + \theta_1\bar{F}_l(b) + \eta_0\underline{F}_h(b)\right)(l-b) + s(h-l) + \mu_0 s(m-l) + \delta(b)s(m-l) \\ = s(h-l) + \mu_0 s(m-l),$$

and for the medium type we have, for all $b \in (\underline{b}_h,\, \bar{b}_l)$,

$$2\left(\mu_0 + \mu_1\hat{F}_m(b) + 1 - \theta_1 + \theta_1\bar{F}_l(b) + \eta_0\underline{F}_h(b)\right)(m-b) + s(h-m) + s(m-l) \\ = 2\left(\mu_0 + \mu_1 + 1 + \eta_0\right)(m - \bar{b}_l) + s(h-m) + s(m-l).$$

Finally, the high type's indifference condition in this range is

$$2\left(\mu_0 + \mu_1\hat{F}_m(b) + 1 - \theta_1 + \theta_1\bar{F}_l(b) + \eta_0\underline{F}_h(b)\right)(h-b) + s(h-m) + (1-\theta_0-\theta_1)s(h-l) + \gamma(b)s(h-m) \\ = 2\left[\mu_0 + \mu_1 + 1 + \eta_0\right](h - \bar{b}_l) + s(h-m) + (1-\theta_0-\theta_1)s(h-l) + \beta\gamma(\bar{b}_l)s(h-m).$$

To make sure that the high type is indifferent between offering m and l after observing $b \in (\underline{b}_h,\, \bar{b}_l)$, we require,

$$\mu_1\hat{F}_m(b) = \theta_1\bar{F}_l(b),$$

and so

$$\mu_1 = \theta_1.$$

Similarly, the low type should be indifferent between offering m and h after observing $b \in (\underline{b}_h,\, \bar{b}_l)$, which yields

$$\mu_1\hat{F}_m(b) = \eta_0\underline{F}_h(b),$$

so that

$$\mu_1 = \eta_0.$$

Moreover, we obtain

$$\hat{F}_m(b) = \underline{F}_h(b) = \bar{F}_l(b) =: \hat{F}(b).$$

From the above equations, we can calculate $\hat{F}(b)$, $\gamma(b)$ and $\delta(b)$ on $(\bar{b}_l, \bar{b}_m)$.

Consider the interval $(\bar{b}_l, \bar{b}_m)$. The high type's indifference condition reads, for all $b \in (\bar{b}_l, \bar{b}_m)$,

$$2\left(\eta_0 + \eta_1 F_h(b) + \mu_0 + \mu_1 + (1-\mu_0-\mu_1)\bar{F}_m(b) + 1\right)(h-b) + s(h-m) + (1-\theta_0-\theta_1)s(h-l) + \gamma(b)s(h-m) = 2\left(2+\eta_0+\eta_1\right)(h-\bar{b}_m) + s(h-m) + (1-\theta_0-\theta_1)s(h-l).$$

For the medium type, we obtain the condition, for all $b \in (\bar{b}_l, \bar{b}_m)$,

$$2\left(\eta_0 + \eta_1 \hat{F}_h(b) + \mu_0 + \mu_1 + (1-\mu_0-\mu_1)\bar{F}_m(b) + 1\right)(m-b) + s(h-m) + s(m-l) = 2\left(2+\eta_0+\eta_1\right)(m-\bar{b}_m) + s(h-m) + s(m-l).$$

Moreover, to make the low type indifferent between offering m and h after observing $b \in (\bar{b}_l, \bar{b}_m)$, we need

$$(1-\mu_0-\mu_1)\bar{F}_m(b) = \eta_1 \hat{F}_h(b),$$

which implies

$$1-\mu_0-\mu_1 = \eta_1,$$

and

$$\bar{F}_m(b) = \hat{F}_h(b) = \bar{F}(b).$$

From the above equations, we can calculate $\bar{F}(b)$ and $\gamma(b)$ on $(\bar{b}_l, \bar{b}_m)$.

Finally, in the interval $(\bar{b}_m, \bar{b}_h)$, we have the following indifference condition for the high type:

$$2\left(2+\eta_0+\eta_1 + (1-\eta_0-\eta_1)\bar{F}_h(b)\right)(h-b) + s(h-m) + (1-\theta_0-\theta_1)s(h-l) = 2(2+\eta_0+\eta_1)(h-\bar{b}_m) + s(h-m) + (1-\theta_0-\theta_1)s(h-l).$$

Defining $x = (m-\underline{b}_h)/(m-l)$, $y = (m-\bar{b}_l)/(m-l)$, and $z = (m-\bar{b}_m)/(m-l)$, we can obtain the following system of equations (by setting b equal to the cutoff bids $b = \underline{b}_h, \bar{b}_l, \bar{b}_m$),

$$2(\mu_0+1-\theta_1)(x-1) + \delta(\underline{b}_h)s = 0,$$

$$1-\theta_0-\theta_1 = (\mu_0+1-\theta_1)x,$$

$$2\left(\mu_0+\mu_1+1+\eta_0\right)(y-1) + s = 0,$$

$$\left(\mu_0+1-\theta_1\right)x = \left(\mu_0+\mu_1+1+\eta_0\right)y,$$

$$2\left(\mu_0+1-\theta_1\right)(1+x) + s = 2\left(\mu_0+\mu_1+1+\eta_0\right)(1+y) + \gamma(\bar{b}_l)s,$$

$$2\left(\eta_0+\mu_0+\mu_1+1\right)(1+y) + \gamma(\bar{b}_l)s = 2\left(\eta_0+\eta_1+2\right)(1+z),$$

$$\left(\eta_0+\mu_0+\mu_1+1\right)y = \left(\eta_0+\eta_1+2\right)z.$$

Solving, we get the quantities defined in the statement of the proposition. To make sure that all probabilities and distribution functions are well defined, we require that $s \in [2, 6]$. When $s = 6$, we have $\mu_1 = 1$, $\theta_1 = 1$, $\eta_0 = 1$, $y = z$. Under $s \in [2, 6]$, we can verify that there are no profitable deviations for any of the types.

F Appendix F

In this Appendix I present proofs of Propositions 5 and 6 from Section 5 of the paper.

F.1 Proof of Proposition 5

I give the proof for the case $\Delta \geq 1$ (the case $\Delta < 1$ is analogous). Let χ_j denote the equilibrium expected payoff of type $j \in \{l, m, h\}$, and denote by χ_{ji} the expected payoff of type j who deviates to choosing bundle $i \neq j$ (and then behaves optimally in the bargaining stage given the beliefs).

Consider the Bundling Mechanism with no signals, i.e. $\mathcal{B}((t_j, p_j, q_j)_{j\in\{l,m,h\}})$ with $p_j = q_j = 0$ for all $j \in \{l, m, h\}$. We have to prove that choosing the equilibrium bundle is optimal for all types for some choice of $\{t_j\}_{j\in\{l,m,h\}}$, given that the other player is also choosing the equilibrium bundle, and fixing the sequentially rational behavior of players in the bargaining subgame. We also have to show that equilibrium payoffs are larger than the outside options (for all types) to prove that the mechanism is individually rational.

The equilibrium (expected) payoffs are given by

$$\chi_l = \frac{1}{6}l + \frac{s}{6}\left[(h-l) + (m-l)\right] - t_l,$$

$$\chi_m = \frac{1}{2}m + \frac{s}{6}\left[(m-l) + (h-m)\right] - t_m,$$

$$\chi_h = \frac{5}{6}h + \frac{s}{3}(h-m) - t_h.$$

Let o_j denote the outside option of type j, i.e. the expected payoff of type j if she does not participate in the mechanism. Using the assumptions imposed in Subsection C.1.1 of Appendix C of the paper, we have

$$o_l = \frac{s}{6}\left[(h-l) + (m-l)\right],$$

$$o_m = \frac{s}{6}(h-m),$$

$$o_h = \frac{s}{3}(h-m).$$

I now calculate the expected payoffs from deviations to choosing other bundles (omitting for now the deviation of a high type to a low bundle, and the deviation of a low type to a high bundle).

$$\chi_{lm} = \frac{1}{2}l + \frac{s}{6}\left[\frac{1}{2}(m-l) + (h-l) + (m-l)\right] - t_m,$$

$$\chi_{ml} = \frac{1}{6}m + \frac{s}{6}\left[\frac{1}{2}(m-l) + (h-m)\right] - t_l,$$

$$\chi_{mh} = \frac{5}{6}m + \frac{s}{6}\left[\frac{1}{2}(h-m) + \frac{1}{2}(h-m)\mathbf{1}_{\{\Delta>2\}} + (m-l)\mathbf{1}_{\{\Delta\leq 2\}}\right] - t_h,$$

$$\chi_{hm} = \frac{1}{2}h + \frac{s}{6}\left[2(h-m)\mathbf{1}_{\{\Delta>2\}} + \left(\frac{1}{2}(h-m) + (h-l)\right)\mathbf{1}_{\{\Delta\leq 2\}}\right] - t_m.$$

First, let us consider the incentive compatibility and individual rationality constraints of the medium and low type. After simplifications, we get

$$-\frac{1}{3}l - \frac{s}{12}(m-l) \geq t_l - t_m, \tag{IC_{lm}}$$

$$\frac{1}{3}m + \frac{s}{12}(m-l) \geq t_m - t_l, \tag{IC_{ml}}$$

$$\frac{1}{6}l \geq t_l, \tag{IR_l}$$

$$\frac{1}{2}m + \frac{s}{6}(m-l) \geq t_m. \qquad (IR_m)$$

To satisfy these constraints, we can set

$$t_l = \frac{1}{6}l,$$

and

$$t_m = \frac{1}{3}m + \frac{1}{6}l + \frac{s}{12}(m-l).$$

Now, consider the IC and IR constraints for the high and medium type. After substituting t_m and t_l and simplifying:

$$\frac{1}{3}h + \frac{1}{3}m + \frac{1}{6}l + \frac{s}{3}(h-m) + \frac{s}{12}(m-l) - \frac{s}{6}\left[2(h-m)\mathbf{1}_{\{\Delta>2\}} + \left(\frac{1}{2}(h-m) + (h-l)\right)\mathbf{1}_{\{\Delta\leq 2\}}\right] \geq t_h, \quad (IC_{hm})$$

$$-\frac{1}{6}l - \frac{2}{3}m + \frac{s}{12}(m-l) + \frac{s}{6}\left[\frac{1}{2}(h-m) - \frac{1}{2}(h-m)\mathbf{1}_{\{\Delta>2\}} - (m-l)\mathbf{1}_{\{\Delta\leq 2\}}\right] \geq -t_h, \quad (IC_{mh})$$

$$\frac{5}{6}h \geq t_h. \qquad (IR_h)$$

I deal with the two cases (i) $\Delta \leq 2$ and (ii)$\Delta > 2$ separately. Assuming that $\Delta \leq 2$, the three conditions above become

$$\frac{1}{3}h + \frac{1}{3}m + \frac{1}{6}l + \frac{s}{12}(h-m) - \frac{s}{12}(m-l) \geq t_h,$$

$$t_h \geq \frac{1}{6}l + \frac{2}{3}m + \frac{s}{12}(m-l) - \frac{s}{12}(h-m),$$

$$\frac{5}{6}h \geq t_h.$$

Setting

$$t_h = \min\left\{\frac{1}{3}h + \frac{1}{3}m + \frac{1}{6}l + \frac{s}{12}(h-m) - \frac{s}{12}(m-l),\ \frac{5}{6}h\right\}$$

satisfies all of the constraints. With these transfers, it is easy to check that the low type does not want to deviate to the high bundle, and *vice versa*.

When $\Delta > 2$, we have

$$\frac{1}{3}h + \frac{1}{3}m + \frac{1}{6}l + \frac{s}{12}(m-l) \geq t_h,$$

$$t_h \geq \frac{1}{6}l + \frac{2}{3}m - \frac{s}{12}(m-l),$$

$$\frac{5}{6}h \geq t_h.$$

In this case, I set

$$t_h = \min\left\{\frac{1}{3}h + \frac{1}{3}m + \frac{1}{6}l + \frac{s}{12}(m-l),\ \frac{5}{6}h\right\},$$

and it is straightforward to check that all constraints hold.

F.2 Proof of Proposition 6a

By Proposition 4 in Section 5 of the paper, if a Bundling Mechanism cannot implement efficiency in the entire game, then there are no mechanisms that can achieve it.[3] As a normalization, we can assume without loss of generality that $p_m \geq \frac{1}{2}$ and $q_m \geq \frac{1}{2}$.

First, I find conditions on the distribution of signals that are necessary for achieving efficiency in the entire game. We need (i) the low type to make an offer with price m with conditional probability one if the other player chose the medium bundle, and (ii) the high type to make an offer with price m with conditional probability one if the other player chose the medium bundle. From now on, I discuss the case $\Delta \geq 1$ (the other case is fully analogous and thus the proof is skipped).

Under $\Delta \geq 1$, to satisfy condition (i), we need $p_m = 1$ and $p_h \leq 1/\Delta$. To satisfy condition (ii), we need either $q_m = 1$ (the high type offers m if and only if she observes the signal), or $q_l \geq 1 - \Delta(1 - q_m)$ (the high type offers m regardless of the realization of the signal).

A necessary condition for implementation can be dervied as follows. I set $p_l = p_h$ and $q_l = q_h$. This choice is inconsequential for the equilibrium payoffs of the low and high type,[4] respectively, but minimizes the payoff of the medium type after deviating to the low or high bundle (keeping everything else fixed). Indeed, a signal specified in this way is completely uninformative to the medium type. If under some condition on parameters the incentive compatibility constraints (omitting the constraint that the low type does not want to deviate to the high bundle, and vice versa) yield a contradiction, we will know that there can be no incentive compatible mechanism that implements full efficiency under that condition.

Below I analyze the case $q_m = 1$. The other case, $q_l \geq 1 - \Delta(1 - q_m)$, is very similar and thus omitted (it leads to exactly the same restriction).

The equilibrium expected payoffs are given by

$$\chi_l = \frac{1}{6}l + \frac{s}{6}\left[(1 + p_h)(m - l) + (1 - p_h)(h - l) + q_l(m - l)\right] - t_l,$$

$$\chi_m = \frac{1}{2}m + \frac{s}{6}\left[(h - m) + (m - l)\right] - t_m,$$

$$\chi_h = \frac{5}{6}h + \frac{s}{6}\left[(1 + q_l)(h - m) + (1 - q_l)(h - l) + p_h(h - m)\right] - t_h.$$

Assuming that $\Delta \geq 2$, we consider expected payoffs from deviations to choosing other bundles:

$$\chi_{hm} = \frac{1}{2}h + \frac{s}{3}(h - m) + \frac{s}{6}(h - m) - t_m,$$

$$\chi_{lm} = \frac{1}{2}l + \frac{s}{12}(m - l) + \frac{s}{6}(h - l) + \frac{s}{6}(m - l) - t_m,$$

$$\chi_{mh} = \frac{5}{6}m + \frac{s}{6}(h - m) - t_h,$$

$$\chi_{ml} = \frac{1}{6}m + \frac{s}{6}(h - m) + \frac{s}{12}(m - l) - t_l.$$

Thus, we obtain the following incentive compatibility constraints:

$$\frac{s}{6}\left(-\frac{1}{2} + q_l\right)(m - l) - \frac{s}{6}p_h(h - m) - \frac{1}{3}l \geq t_l - t_m, \qquad (IC_{lm})$$

$$\frac{1}{3}m + \frac{s}{12}(m - l) \geq t_m - t_l, \qquad (IC_{ml})$$

$$-\frac{1}{3}m + \frac{s}{6}(m - l) \geq t_m - t_h, \qquad (IC_{mh})$$

[3] Formally, the allocation rule in Proposition 4 was restricted to be symmetric. But it is easy to show that allowing asymmetric mechanisms does not change anything.

[4] The only quantity that matters for the low (high) type is the relative likelihood of the medium type to the high (low) type after observing the signal.

$$\frac{1}{3}h - \frac{s}{6}(1-p_h)(h-m) + \frac{s}{6}(1-q_l)(m-l) \geq t_h - t_m. \qquad (IC_{hm})$$

Combining equations (IC_{lm}) and (IC_{ml}), as well as equations (IC_{mh}) and (IC_{hm}), we obtain the conditions

$$q_l s \geq s p_h \Delta - 2,$$

$$2\Delta + 2s - (1-p_h)s\Delta \geq q_l s,$$

so that a necessary condition is

$$2(\Delta + s + 1) \geq s\Delta.$$

We have thus shown that if the above condition fails, we cannot implement efficiency in the entire game. Finally, notice that the failure of this condition implies $\Delta \geq 2$ (so our earlier assumption that $\Delta \geq 2$ was inconsequential).

F.3 Proof of Proposition 6b

To prove that the condition is sufficient to achieve efficiency in the entire game, I look at a specific Bundling Mechanism. I focus on the case $\Delta \in [1, 2]$, omitting the fully analogous case $\Delta \in [1/2, 1)$.

Let $p_m = 1$ and $p_h \leq 1/\Delta$. This is a necessary and sufficient condition for the low type to offer m when the medium bundle is chosen by the other player (which is necesary for full efficiency). By choosing $p_h = 1/\Delta$, we maximize the probability that the high type receives a strictly profitable offer from the low type in equilibrium (which relaxes the IC constraint for the high type). Next, we choose $q_l = q_m = q_h = 1$ for the high bundle. That is, the high type does not observe a signal in equilibrium and always offers to buy for m (because $\Delta \geq 1$). Finally, we chose $p_l = p_h$ so that the signal in the low bundle is uninformative for the medium type.

With this information structure, the equilibrium (expected) payoffs are given by:

$$\chi_l = \frac{1}{6}l + \frac{s}{6}\left[(2+p_h)(m-l) + (1-p_h)(h-l)\right] - t_l,$$

$$\chi_m = \frac{1}{2}m + \frac{s}{6}\left[(h-m) + (m-l)\right] - t_m,$$

$$\chi_h = \frac{5}{6}h + \frac{s}{6}(2+p_h)(h-m) - t_h.$$

We consider all possible deviations to other bundles:

$$\chi_{lm} = \frac{1}{2}l + \frac{s}{12}(m-l) + \frac{s}{6}(h-l) + \frac{s}{6}(m-l) - t_m,$$

$$\chi_{ml} = \frac{1}{6}m + \frac{s}{6}(h-m) + \frac{s}{12}(m-l) - t_l,$$

$$\chi_{mh} = \frac{5}{6}m + \frac{s}{6}(m-l) + \frac{s}{12}(h-m) - t_h,$$

$$\chi_{hm} = \frac{1}{2}h + \frac{s}{6}(h-l) + \frac{s}{12}(h-m) + \frac{s}{6}(h-m) - t_m,$$

$$\chi_{lh} = \frac{5}{6}l + \frac{s}{6}\left[2(m-l) + \frac{1}{2}(h-l)\right] - t_h,$$

$$\chi_{hl} = \frac{1}{6}h + \frac{s}{6}\left[(1+\frac{3}{2}p_h)(h-m) + (1-\frac{1}{2}p_h)(h-l)\right] - t_l.$$

Thus, we obtain six IC constraints plus three IR constraints:

$$\frac{s}{6}\left[\frac{1}{2}(m-l) - p_h(h-m)\right] - \frac{1}{3}l \geq t_l - t_m, \qquad (IC_{lm})$$

$$\frac{1}{3}m + \frac{s}{12}(m-l) \geq t_m - t_l, \qquad (IC_{ml})$$

$$\frac{1}{3}h + \frac{s}{6}\left[(p_h - \frac{1}{2})(h-m) - (m-l)\right] \geq t_h - t_m, \qquad (IC_{hm})$$

$$-\frac{1}{3}m + \frac{s}{12}(h-m) \geq t_m - t_h, \qquad (IC_{mh})$$

$$-\frac{2}{3}l + \frac{s}{6}\left[\frac{1}{2}(m-l) + (\frac{1}{2} - p_h)(h-m)\right] \geq t_l - t_h, \qquad (IC_{lh})$$

$$\frac{2}{3}h - \frac{s}{6}(1 - \frac{1}{2}p_h)(m-l) \geq t_h - t_l, \qquad (IC_{hl})$$

$$\frac{1}{6}l + \frac{s}{6}\left[(m-l) - p_h(h-m)\right] \geq t_l, \qquad (IR_l)$$

$$\frac{1}{2}m + \frac{s}{6}(m-l) \geq t_m, \qquad (IR_m)$$

$$\frac{5}{6}h + \frac{s}{6}p_h(h-m) \geq t_h. \qquad (IR_h)$$

I set

$$t_l = \frac{1}{6}l,$$

$$t_m = \frac{1}{3}m + \frac{1}{6}l + \frac{s}{12}(m-l).$$

After substituting $p_h = 1/\Delta$, the remaining constraints are:

$$\frac{1}{3}h + \frac{1}{3}m + \frac{1}{6}l + \frac{s}{12}(m-l) - \frac{s}{12}(h-m) \geq t_h, \qquad (IC_{hm})$$

$$t_h \geq \frac{2}{3}m + \frac{1}{6}l + \frac{s}{12}\left[(m-l) - (h-m)\right], \qquad (IC_{mh})$$

$$\frac{2}{3}h + \frac{1}{6}l - \frac{s}{6}\left(1 - \frac{1}{2\Delta}\right)(m-l) \geq t_h, \qquad (IC_{hl})$$

$$\frac{5}{6}h + \frac{s}{6}(m-l) \geq t_h. \qquad (IR_h)$$

I set

$$t_h = \frac{2}{3}m + \frac{1}{6}l + \frac{s}{12}\left[(m-l) - (h-m)\right].$$

Given these transfers, the equations collapse to a single condition

$$s \leq \frac{8\Delta}{3 - (\frac{1}{\Delta} + \Delta)}.$$

If that condition is satisfied, then all IC and IR constraints are also satisfied, and thus there exists a mechanism that implements full efficiency.