

Quantum Effects in White Dwarfs

by Wilson M. Liu '00

All stars, including our own sun, produce energy by the process of nuclear fusion. After several billion years, when our sun has exhausted all its nuclear fuel, it will consist of a dense core surrounded by an envelope of hydrogen and helium. Eventually the outer envelope is lost into space and only the stellar core remains. This core is called a white dwarf and is the final evolutionary stage for all stars with a core mass less than 1.4 times the mass of our sun. In general, a white dwarf is comprised of two components: a gas of dense electrons and the positively charged nuclei, which are the product of nuclear fusion. The internal structure of a white dwarf, and indeed its very existence, depends upon the quantum nature of its constituent electrons which are dense enough to form a degenerate gas. Quantum effects also influence the behavior of the nuclei found in the white dwarf and affect the way the star cools. This paper will begin with a brief discussion on how degeneracy pressure arises from the Pauli exclusion principle. A discussion on how this supports the star from gravitational collapse and affects its structure follows. Finally, quantum mechanical behavior of the thermal ions and its effects on the cooling of the white dwarf are discussed.

1. Pauli Exclusion, Electron Gases, & the Degenerate Equation of State

In quantum mechanics, a particle like an electron is not considered a classical object, like a billiard ball, but is instead expressed as a wavefunction, usually denoted by the symbol ψ . This wavefunction represents the probability that the particle will be found at a point in space. At first this sounds like an abstract idea, but its implications are very real, as we will see. Given a two particle system, the total wavefunction of the system can be represented by the product of the individual wavefunctions of each particle:

$$\psi(r_1, r_2) = \psi_a(r_1)\psi_b(r_2)$$

(Griffiths, 1995) Eq. 5.9

The left side of the equation represents the com-

bined wavefunction of the system, while the right is a product of particle 1 in state 'a' and particle two in state 'b'. For electrons, which are indistinguishable fermions, the wavefunction must be modified to account for the fact that we cannot know which particle is in which state, so the wavefunction now takes the general form:

$$\psi(r_1, r_2) = A[\psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1)]$$

(Griffiths, 1995) Eq. 5.10

One can see that the particles cannot be in identical states, $\psi_a = \psi_b$, otherwise the total wavefunction would be zero. This is the Pauli exclusion principle.

Electrons in a dense gas, like those found in a white dwarf, must obey the exclusion principle. If the density, n , of the electron gas exceeds the quantum concentration, given by

$$n_Q = \left(\frac{2\pi mkT}{h^2}\right)^{3/2}$$

(Phillips, 1994) Eq. 2.22

the gas is said to be degenerate. Here, m is the mass of the electron, kT is the thermal energy of the gas, and h is Planck's constant. In this case, the energy distribution of the electrons is no longer a thermal distribution, but one governed by the exclusion principle. The electrons fill states up to a certain energy, E_F , called the Fermi energy

$$E_F = \frac{h^2}{8\pi^2 m} (3n\pi^2)^{2/3}$$

(Griffiths, 1995) Eq. 5.43

This energy has a pressure associated with it; we can see that work must be done (energy must be added) on a degenerate gas to change its volume (and hence its density, n). This pressure, called the degeneracy pressure, is what keeps the white dwarf from collapsing under its own gravity and is given by the degenerate equation of state:

$$P = \frac{h^2}{5m} \left(\frac{3}{8\pi}\right)^{2/3} n^{5/3}$$

(Phillips, 1994) Eq. 2.31

¹ Adapted from Griffiths (1995), Sections 5.1.1 and 5.3.1 and Phillips (1994), Section 2.2

An alternative explanation of degeneracy pressure, presented by Shu (1982), can be made in terms of the Heisenberg uncertainty principle. At the high densities inside a white dwarf, the electrons are confined to a small volume, corresponding to a small Δx . In order to satisfy the uncertainty principle, $\Delta x \Delta p > h/4\pi$, the momenta of the particles must be large. The momenta of the particles results in a degeneracy pressure which is much greater than the thermal pressure.

II. The Structure of White Dwarfs

The fact that the electrons inside a white dwarf are described by a degenerate equation of state has several implications regarding its structure. First, it can be found that white dwarfs of increasing mass are smaller in size. This is counterintuitive because we expect that more massive objects are larger. However, in white dwarfs the opposite is true. This is the case because a greater mass will result in a greater gravitational force, necessitating a greater degeneracy pressure to counteract it. A greater degeneracy pressure requires a greater density of electrons, hence a smaller volume (Griffiths, 1995; Shu, 1982). A typical white dwarf has a radius about one hundredth that of our sun (Griffiths, 1995).

Another implication of the degenerate equation of state is the existence of a theoretical mass limit for white dwarfs. Above a certain mass ($M \approx 1.4 M_{\text{sun}}$), the Chandrasekhar mass, the internal pressure of the white dwarf is high enough for the electrons to have sufficient momenta that relativistic effects become significant. Relativistic electrons follow a different equation of state,

$$P = \frac{hc}{4} \left(\frac{3}{8\pi} \right)^{1/3} n^{4/3}$$

(Phillips, 1994) Eq. 2.34

where the degeneracy pressure is less than the non-relativistic case. Gravitational forces will therefore no longer be counteracted by a sufficient degeneracy pressure, and the white dwarf will collapse under self-gravitation. In some cases, this collapse and subsequent release of gravitational potential energy will ignite the fusion of oxygen and carbon in the stellar interior, eventually resulting in a supernova, the explosive death of a star (Binney, 1998; Phillips, 1994). Another end state of this collapse is a neutron star, which is comprised of neutrons at nuclear densities and is supported from further collapse by neutron degeneracy pressure in a situation similar to electron degeneracy (Griffiths, 1995).

In addition to preventing a white dwarf's gravitational collapse, the electrons in a white dwarf have other effects on the structure of the star. The primary energy source in a white dwarf is the thermal (non-degenerate) ions in the star's interior. The degenerate electron gas efficiently conducts this energy throughout the star, making the star nearly isothermal (the star is the same temperature throughout). Near the surface of the star, however, the density of stellar material is not high enough to satisfy the conditions necessary for electron degeneracy. In this layer, called the envelope, energy transfer takes place through radiative transfer and convection and is not as efficient as in the interior. Hence this layer is characterized by a steep temperature gradient where the temperature drops from $\sim 10^7$ K to $\sim 10^4$ K over 50 km. The envelope insulates the star from rapid energy loss (Kawaler & Winget, 1987). Along with certain quantum effects in the star's internally contained ions, this has a significant effect on the timescales of white dwarf cooling.

III. Ion Interactions and White Dwarf Cooling

Up to this point, we have concentrated on electrons and the effect that their degeneracy has on the structure of a white dwarf. However, the interiors of white dwarfs are also comprised of ions. The properties of these ions cannot be ignored, since it is their thermal energy which is the primary source of energy in the star. As the thermal energy of the white dwarf is slowly lost to space and the stellar material cools, the ions begin to behave differently. Their behavior can be understood in a classical manner, but under certain conditions, this behavior is significantly modified by quantum effects. Let us first examine the classical picture of white dwarf cooling.

Initially, the thermal energy of the ions is great enough to overcome Coulomb forces between the ions, and thus the gas can be treated as almost ideal. As the gas cools, however, Coulomb interactions (electric repulsion) become important. The ions first form a Coulomb liquid, and as more energy is lost, the ions form a lattice within the star's interior, a crystalline Coulomb solid (Kawaler & Winget, 1987). The strength of Coulomb interactions between ions can be quantified by the "classical coupling parameter" (Chabrier et al., 1992):

$$\Gamma = \frac{(Ze)^2}{a} \cdot \frac{1}{kT}$$

This parameter is the ratio of the potential to thermal energy, where Ze is the ionic charge and a is the aver-

age interionic distance. It has been shown through numerical simulations that Coulomb crystallization occurs at a value of about $\Gamma = 180$ (Chabrier et al., 1992).

Normally, quantum effects are treated as insignificant in the derivation of the freezing point from a Coulomb liquid to a solid. It has been shown by Chabrier et al., however, that quantum mechanical effects in ion interaction can change the characteristics of this phase transition in white dwarfs with a mass greater than one solar mass. A “quantum solid” is a solid in which the oscillatory motion of ions around the lattice sites is dominated by quantum zero-point energy rather than by the classical thermal energy of the particle (Chabrier et al., 1992; Kittel, 1996). It was found by Chabrier et al. (1992) that this characteristic is not only observed in the solid phase of white dwarf ions, but in the liquid phase as well. Thus, the crystallization can no longer be considered classical, but instead as a transition from a quantum liquid to a quantum solid. The average zero-point energy per particle is therefore greater than the value predicted by a classical model. For example, given a mean interionic distance of $10^{-2} a_0$, where $a_0 = 5.29 \times 10^{-11}$ meters (the Bohr radius), the ratio of zero-point (quantum) energy to thermal (classical) energy, E_0/kT , is ~ 2 . If the distance between ions is decreased to $10^{-3} a_0$, then E_0/kT increases to $\sim 5 - 8$. The increased energy per particle due to quantum zero-point energy impedes crystallization, resulting in a depression of the freezing point. For the above interionic distances of $10^{-2} a_0$ and $10^{-3} a_0$, the values of Γ become 183 and 199, respectively, an increase from the aforementioned classical value of 180. For massive white dwarfs ($M > 1 M_{\text{sun}}$) the freezing temperature can be depressed by up to 10%, though for the typical white dwarf ($M \approx 0.6 M_{\text{sun}}$) the freezing temperature is nearly unaffected (Chabrier et al., 1992).

Quantum effects in the crystallization of white dwarfs change the timescale over which a white dwarf cools. During crystallization of the ions, the release of latent heat temporarily delays energy loss in the star (Chabrier et al., 1992; Kawaler & Winget, 1987). However, a decrease in freezing temperature will result in a corresponding decrease in the cooling time of the star. For a $1.3 M_{\text{sun}}$ white dwarf, the decrease in cooling time is about 30% (Chabrier, 1993).

IV. Summary

Electron degeneracy in white dwarf matter has several implications for its structure. First, the degeneracy pressure allows the star to balance itself

against gravitation, preventing its collapse. It is found that for white dwarfs of increasing mass, the size decreases. Additionally, a theoretical upper limit of the mass of a white dwarf, the Chandrasekhar mass, can be calculated to be $M \approx 1.4 M_{\text{sun}}$. Above this mass, electron degeneracy pressure is insufficient to support the star from gravitational collapse. The interior of a white dwarf is nearly isothermal, with a temperature of $\sim 10^7$ K, due to the highly efficient thermal conduction properties of the degenerate electron gas. This isothermal region is surrounded by an insulating, non-degenerate outer envelope characterized by a steep temperature gradient. The envelope regulates energy loss, and thus controls the rate at which the white dwarf cools.

The cooling mechanism of a white dwarf is modified by quantum effects on the non-degenerate ions. These effects are found to depress the temperature of ion crystallization and shorten the timescale over which the star cools. This change in the theory of white dwarf cooling could have an affect on our estimation of galactic age, since observations of white dwarfs correlated to theoretical white dwarf cooling regimes are used as an indicator of galactic age. ■

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ABOUT THE AUTHOR

Wilson Liu '00 is a physics major and an astronomy minor. He is currently completing an honors thesis involving the dating of star clusters and the formation of the Milky Way. He will be attending the University of Arizona next fall as a graduate student in astronomy.