Modular forms seminar lecture 4

Note Title

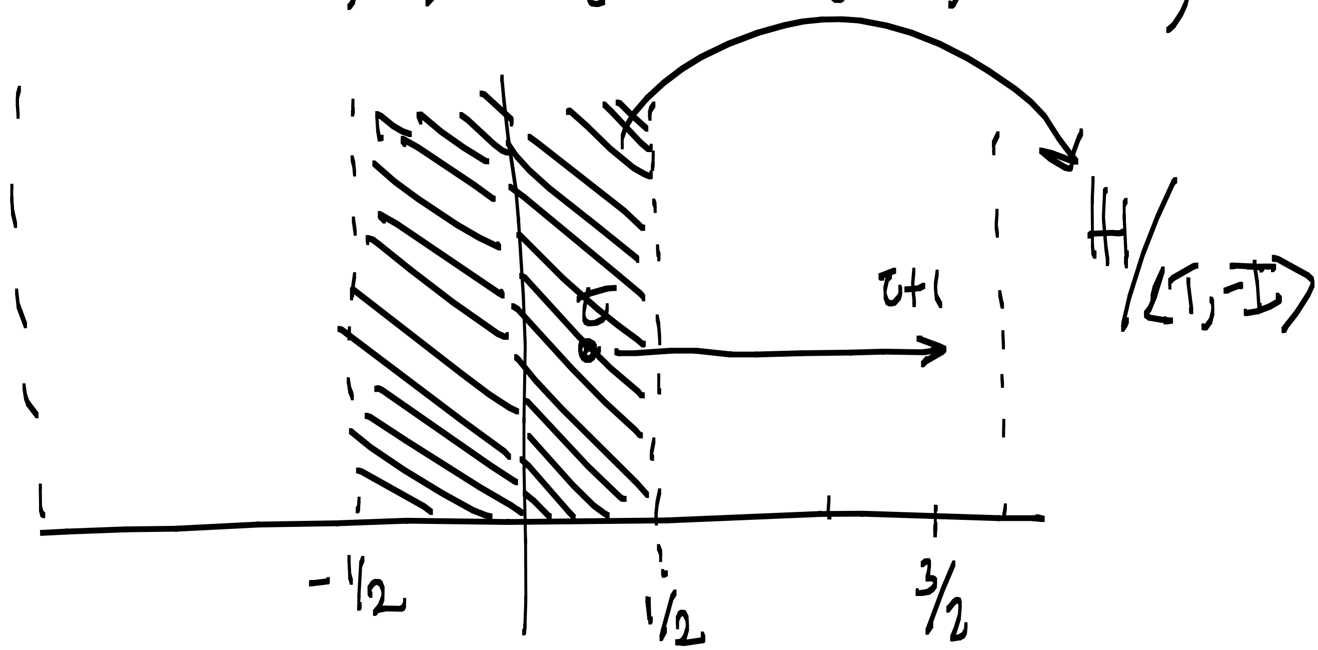
Recall: a modular form $f: \mathbb{H} \to \mathbb{C}$ of weight k and level $\Gamma \subseteq SL_2(\mathbb{Z})$ (finite-index) is a holomorphic function s.t.

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

The vector space of all such m.f. is ∞-dimensional, denoted by $M_k^!(\Gamma)$.

- Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$. Consider the subgroup

$$\langle -I, T \rangle = \left\{ \pm T^n = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \simeq \mathbb{Z}$$



- Then $\Gamma \cap \langle T, -I \rangle = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^n : h \in \mathbb{Z}_{\geq 0} \right\}$

for some h. In particular $\overset{"}{T^h}$

$$T^h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$$

$\Rightarrow \quad f(\tau + h) = f(\tau)$ for any $f \in M_k^!(\Gamma)$

$\Rightarrow \quad f$ has a Fourier expansion "at ∞"

$$f = \sum_{n \in \mathbb{Z}} a_n q_h^n \qquad q_h = e^{2\pi i \tau / h}$$

E.g. $\Gamma(1,2) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$ the "theta group" then $h = 2$ $\quad q_h = e^{\pi i \tau}$

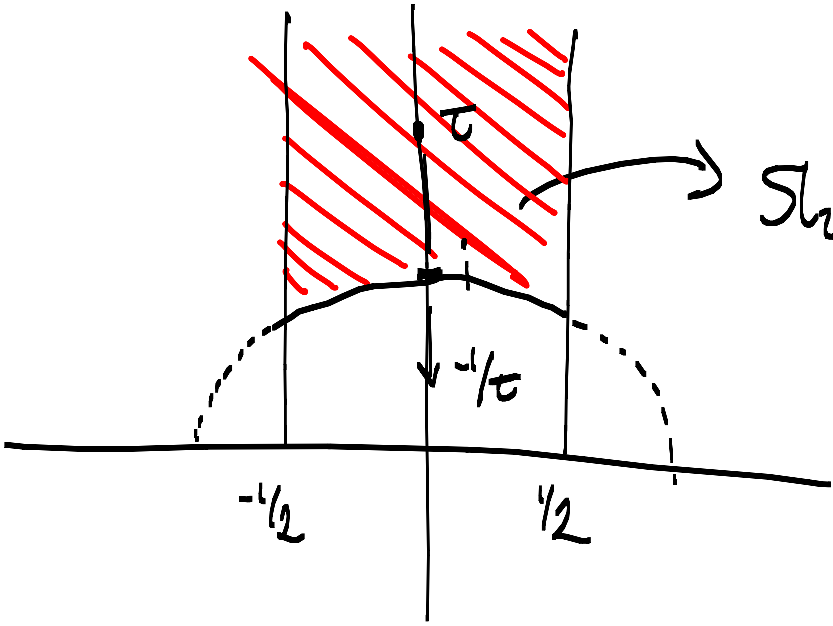
$$\Theta(\tau) = 1 + \sum_{n=1}^{\infty} 2 q_h^{n^2}$$

E.g. $SL_2(\mathbb{Z}) \ni T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad h = 1$ $\qquad q_h = q = e^{2\pi i \tau}$

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q_h^n$$

Geometric picture for $\Gamma = SL_2(\mathbb{Z})$

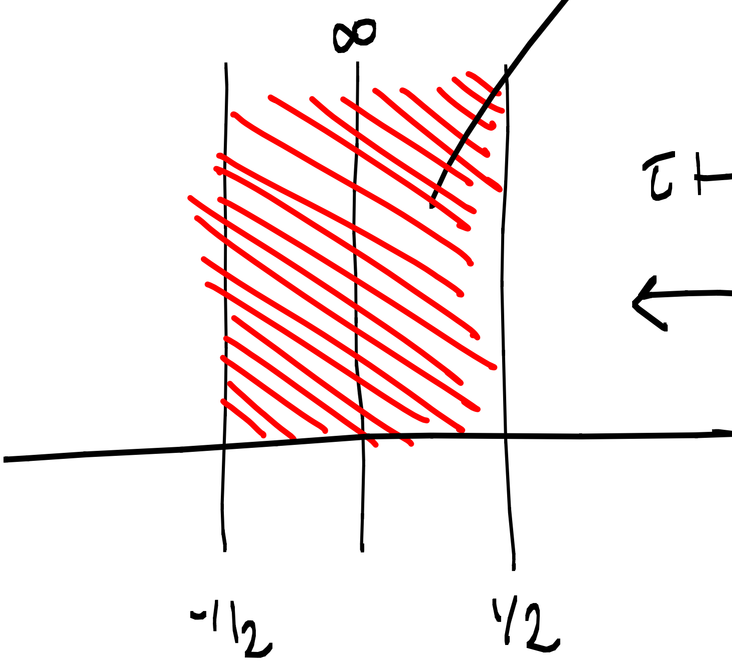
$$= \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$



$$SL_2(\mathbb{Z})\backslash\mathbb{H} = Y(1)$$

("the" modular curve)

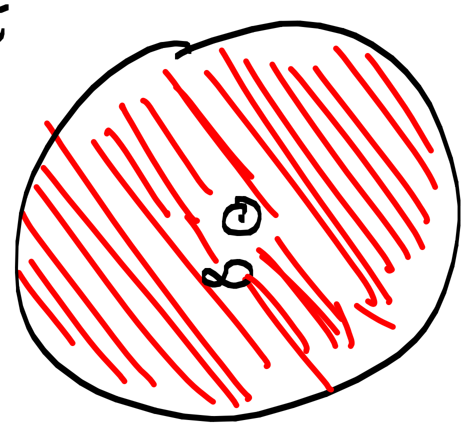
$$\langle -I, T\rangle\backslash\mathbb{H} \simeq \mathbb{D}^\times$$



$$\tau \mapsto q = e^{2\pi i \tau}$$

$$\longleftrightarrow$$

$$\simeq$$



So we have:

$$\begin{array}{ccc} SL_2(\mathbb{Z}) & \longrightarrow & \mathcal{O}_{\mathbb{H}}^{\times} \\ \Psi & & \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & (c\tau + d)^k \end{array}$$

defines a line bundle $\mathcal{L}_k$ on $Y(1) = {}_{SL_2(\mathbb{Z})}\backslash \mathbb{H}$:

$$\mathcal{L}_k = \mathbb{H} \times \mathbb{C} / SL_2(\mathbb{Z})$$

$$\gamma(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^k z\right)$$

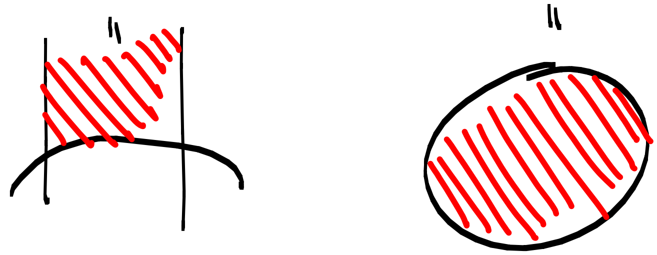
$$\{\text{holomorphic sections of } \mathcal{L}_k\} = M_k^!(SL_2(\mathbb{Z}))$$

$$\left(\text{Fourier expansion: } \sum_{n \in \mathbb{Z}} a_n q^n\right)$$

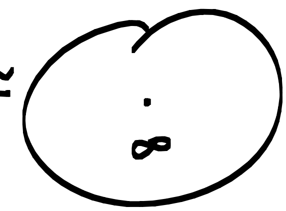
- The examples of mf's that we saw actually have "holomorphic" Fourier expansions. $\subsetneq M_k^!(SL_2(\mathbb{Z}))$

To make sense of that:

1) $X(1) :=$ glue $SL_2(\mathbb{Z})\backslash\mathbb{H}$ + $\mathbb{D}$



along $\langle -I, T\rangle\backslash\mathbb{H} \simeq$



$\simeq \mathbb{D}^{\times}$

2) Extend $\mathcal{L}_k$ to a line bundle $\overline{\mathcal{L}}_k$ on $X(1)$ by letting $\overline{\mathcal{L}}_k\big|_{\mathbb{D}}$ = "trivial"

3) $\left\{ \text{Sections of } \overline{\mathcal{L}}_k \text{ over } X(1) \right\} = \left\{ \text{m.f. of weight } k, \text{ level } SL_2(\mathbb{Z}), \text{ Fourier expansion } \sum_{n=0}^{\infty} a_n q^n \right\}$

- This space is $M_k(SL_2(\mathbb{Z}))$, "holo. m.f."

$\left(E_k \in M_k(SL_2(\mathbb{Z})) \right)$

- Since $X(1) \sim \mathbb{P}^1$:

Theorem $M_k(SL_2(\mathbb{Z}))$ is finite-dimensional

More precisely, $X(1) \to \mathbb{P}^1$ is a Deligne-Mumford Stack with coarse moduli space $\mathbb{P}^1$.

Theorem

$$\dim M_k(SL_2(\mathbb{Z})) = \begin{cases} 0 & \text{if } k \text{ odd} \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{if } k \not\equiv 2 \ (12) \\ \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 2 \ (12) \end{cases}$$

- Let $M(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} M_k(SL_2(\mathbb{Z}))$

be the $\mathbb{Z}$-graded ring of modular forms.

Then

Theorem $M(SL_2(\mathbb{Z})) \simeq \mathbb{C}[E_4, E_6]$

Proof: Use dimension formula above.

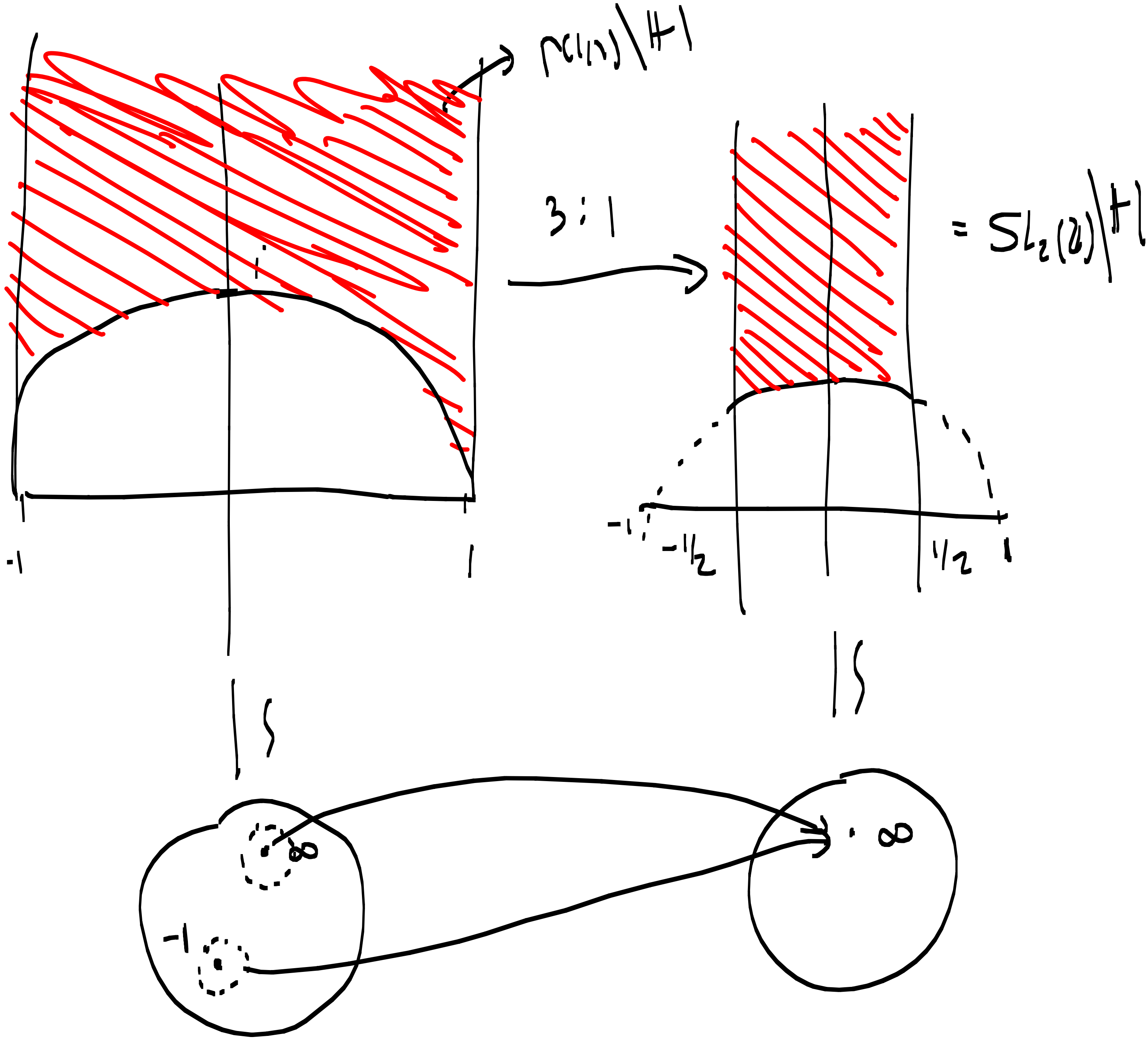
- This is an important fact that will be used later.

Geometric Picture for $\Gamma \supset \Gamma(1,2)$

$$\Gamma(1,2) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$$

$$\cap|$$

$$SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$



$X(\Gamma(1,2))$ = compactification
$Y(\Gamma(1,2)) \cup \underbrace{\{\infty, -1\}}_{\text{"cusps" of } \Gamma(1,2)}$

$\overline{\mathcal{L}_k}$ = line bundle of <u>holomorphic</u> m.f. on $\Gamma(1,2)$

$$M_k(\Gamma(1,2)) = \left\{ \begin{array}{l} \text{"holomorphic" mod. forms} \\ \text{of wt } k \text{ on } \Gamma(1,2) \end{array} \right\}$$

<u>E.g.</u> Let k be even. Then

$$\Theta^k = \sum_{m=1}^{\infty} r_k(n)\, q_2^n \qquad q_2 = e^{\pi i \tau}$$

is in $M_{k/2}(\Gamma(1,2))$.

<u>Theorem:</u> $\dim M_k(\Gamma(1,2)) = \left\lfloor \frac{k}{4} \right\rfloor + 1$

Sums-of-squares

Suppose we want to find a "formula" for

$$r_8(n) = \left\{ \begin{array}{l} \#\text{ ways to write } n \\ \text{as a sum of } 8 \text{ squares} \end{array} \right\}$$

- Note:

$$\Theta^8 = \sum_{n=0}^{\infty} r_8(n) q^{n/2} \qquad q = e^{2\pi i \tau}$$

$$\in M_4(\Gamma(1|2))$$

We also have:

Prop: 1) $E_4\left(\frac{\tau+1}{2}\right) \in M_4(\Gamma(1|2))$ (non-trivial)

2) $E_4(\tau) \in M_4(\Gamma(1|2))$ (trivial)

- Since $\dim M_4(\Gamma(1|2)) = \left\lfloor \frac{4}{4} \right\rfloor + 1 = 2$ we must have

$$\Theta^8 = a\, E_4(\tau) + b\, E_4\left(\frac{\tau+1}{2}\right)$$

Compare both sides:

$$\Theta^8 = \left(1 + 2q^{1/2} + \dots\right)^8 = 1 + 16q^{1/2} + 112q + \dots$$

$$= a\left(1 + 240q + 240\cdot 9\cdot q^2 + \dots\right) + b\left(1 - 240q^{1/2} + \dots\right)$$

$$\Longrightarrow a + b = 1, \quad -240b = 16$$

$$\Rightarrow \quad \Theta^8 = \frac{16}{15}E_4(\tau) - \frac{1}{15}E_4\left(\frac{\tau+1}{2}\right)$$

Compare Fourier coefficients:

$$r_8(n) = \begin{cases} 16\cdot\sigma_3(n) & n \text{ odd} \\ 16^2\cdot\sigma_3(n/2) - 16\,\sigma_3(n) & n \text{ even} \end{cases}$$

More concisely:

$$\boxed{r_8(n) = 16\sum_{d\mid n}(-1)^{n-d}\,d^3}$$