HOW MANY ADJUNCTIONS GIVE RISE TO THE SAME MONAD?

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Abstract. Given an adjoint pair of functors F, G, the composite GF naturally gets the structure of a monad. The same monad may arise from many such adjoint pairs of functors, however. Can one describe all of the adjunctions giving rise to a given monad? In this paper we single out a class of adjunctions with especially good properties, and we develop methods for computing all such adjunctions, up to natural equivalence, which give rise to a given monad. To demonstrate these methods, we explicitly compute the finitary homological presentations of the free A-module monad on the category of sets, for A a Dedekind domain. We also prove a criterion, reminiscent of Beck's monadicity theorem, for when there is essentially (in a precise sense) only a single adjunction that gives rise to a given monad.

Contents

1. Introduction.

If $\mathcal{C}, \mathcal{D}$ are categories and $G : \mathcal{D} \to \mathcal{C}$ a functor with a left adjoint F, then the composite GF gets the structure of a monad. However, given a monad $T : \mathcal{C} \to \mathcal{C}$, there may be *many* categories $\mathcal{D}$ and adjoint pairs F, G such that $GF = T$ as a monad. We will call such a choice of $\mathcal{D}, F$, and G a *presentation* for T.

It has long been known that, among all presentations of a given monad T, there is an *initial* presentation, the Kleisli category of T, and a *terminal* presentation, the Eilenberg-Moore category of T. Furthermore, Beck's monadicity theorem gives a necessary and sufficient condition on G for the presentation $(\mathcal{D}, F, G)$ to be equivalent to the Eilenberg-Moore category. (See [5] for a nice exposition of these ideas.) Beck's result has proven very useful, e.g. in algebraic geometry where, in its dual form for comonads, it is the foundation for the general theory of descent; see [2].

The applications of Beck's monadicity theorem have made it very clear that, given a presentation $(\mathcal{D}, F, G)$ of a monad T, it is very useful to be able to tell when $(\mathcal{D}, F, G)$ is the terminal presentation (i.e., the Eilenberg-Moore category) of T. However, the author has long wondered about what to do when one encounters a presentation of a monad which is *not* the terminal (Eilenberg-Moore) presentation, and not the initial (Kleisli) presentation. Can one describe the collection of all presentations of a monad? Even better, can one establish some kind of *coordinate system* on the collection of presentations of a monad, so that when one encounters a presentation of a monad (or a comonad, e.g. for applications in descent theory), one can give some kind of "coordinates" that describe where this presentation sits in relation to the initial and terminal presentations, and all the other presentations, of the same monad?

In this paper we study the collection of presentations of a given monad, but with a restriction on what presentations we are willing to consider. This is because, given a presentation $(\mathcal{D}, F, G)$, one can trivially produce many more presentations by taking the Cartesian product of $\mathcal{D}$ with any small category. We regard these presentations as degenerate, and we want to disregard presentations with this kind of redundant information in them. Consequently, in Definition 2.2, we make the definition that a presentation $(\mathcal{D}, F, G)$ is said to be *homological* if every object X of $\mathcal{D}$ can be recovered from the F, G-bar construction on X (see Definition 2.2 for the precise definition). This definition eliminates the "redundant" presentations we wanted to exclude, and has some other good properties, described in Remark 2.3. We also restrict our attention to what we call "coequalizable" monads, that is, those monads T for which the Eilenberg-Moore category has coequalizers; this property is satisfied in all cases of interest which we know of, and in Remark 2.9 we explain a bit about why that is.

Once these definitions are made, we can prove some nice theorems:

- In Theorem 2.13, we prove that, if T is coequalizable, then the category of natural equivalence classes of homological presentations of T is equivalent to the partially-ordered collection of reflective replete subcategories of the Eilenberg-Moore category $\mathcal{C}^T$ which contain the Kleisli category $\mathcal{C}_T$.

  This means the category of all natural equivalence classes of homological presentations of T is always well-behaved in at least one way: it can't be just any arbitrary category, rather, it is always partially-ordered (i.e., there is at most one morphism from any given object to any other given object).
- In Theorem 2.20, when T is coequalizable and $\mathcal{C}^T$ has a biproduct and is Krull-Schmidt, we actually construct a "coordinate system" on the natural equivalence classes of homological presentations of T! Any homological presentation is determined uniquely (up to natural equivalence) by specifying a suitable subcollection of the collection of isomorphism classes of indecomposable objects of $\mathcal{C}^T$. In some practical cases, $\mathcal{C}^T$ is the category of finitely generated modules over an algebra, and then the vertices of the Auslander-Reiten quiver of $\mathcal{C}^T$ act as "coordinates" for the collection of natural equivalence classes of homological presentations of T; see Remark 2.22.
- In Theorem 3.6, we give a simple and usable criterion for the *triviality* (up to natural equivalence) of the collection of homological presentations of T, i.e., a criterion for when, up to natural equivalence, there exists only one homological presentation of T (necessarily the Eilenberg-Moore category of T). As an example, in Corollary 3.8 we show that the base-change monad on module categories

associated to a field extension has this property of unique homological presentability.

- In Theorem 4.5, we use Theorems 2.13 and Theorem 2.20 to explicitly compute the collection of all (natural equivalence classes of) homological finitary presentations of each monad in one particular class of monads: namely, let A be a Dedekind domain, and let T denote the monad on the category of sets which sends a set to the underlying set of the free A-module it generates. We show that the partially-ordered set of natural equivalence classes of finitary homological presentations of T is equivalent to the set of functions
$$\mathrm{Max\,Spec}(A) \to \{0, 1, 2, \dots, \infty\}$$
from the set $\mathrm{Max\,Spec}(A)$ of maximal ideals of A to the set of extended natural numbers, under the partial ordering in which we let $f \leqslant g$ if and only if $f(\mathfrak{m}) \leqslant g(\mathfrak{m})$ for all $\mathfrak{m} \in \mathrm{Max\,Spec}(A)$. Since the ring of integers $\mathbb{Z}$ is a Dedekind domain, this is a very fundamental example. (Here a presentation for T is "finitary" if its right adjoint functor preserves filtered colimits; this is a condition that, roughly speaking, guarantees that the data of the presentation is determined by "finite input." See Remark 2.5.)

In many cases of interest (e.g. the base-change monads on module categories associated to maps of rings or maps of schemes), the Eilenberg-Moore category $\mathcal{C}^T$ is actually *abelian*, hence T is coequalizable and $\mathcal{C}^T$ has a biproduct automatically, and frequently $\mathcal{C}^T$ is actually quite computable and understandable. Under those circumstances our results seem to be fairly useful, and as we hope Theorem 4.5 demonstrates, they are actually applicable and give explicit nontrivial results in concrete situations of interest.

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2. Homological presentations of a monad are equivalent to replete reflective subcategories of its Eilenberg-Moore category.

2.1. **Preliminary definitions.** Throughout this paper, when T is a monad, when convenient we will sometimes also write T for the underlying functor of the monad.

Definition 2.1. *Let $\mathcal{C}$ be a category, T a monad on $\mathcal{C}$. If $\mathcal{D}$ is a category equipped with a functor $F' : \mathcal{C} \to \mathcal{D}$ and a right adjoint G' for F' such that the associated monad $G'F'$ is equal to T, we call the data $(\mathcal{D}, F', G')$ a* presentation of T. *Sometimes we shall just write $\mathcal{D}$ as shorthand for $(\mathcal{D}, F', G')$, when F', G' are clear from context.*

The collection of all presentations of T forms a large category, whose morphisms are morphisms of adjunctions (see IV.7 of [5] *for the definition of morphisms of adjunctions) which are the identity on $\mathcal{C}$. We call this large category* the category of presentations of T, *and for which we will write* $\mathrm{Pres}(T)$. *One also can consider natural transformations between morphisms of adjunctions, and we regard two morphisms as* homotopic *if there exists an invertible natural transformation between them; we will write* $\mathrm{Ho}(\mathrm{Pres}(T))$ *for the category of presentations of T but whose morphisms are homotopy classes of morphisms of adjunctions.*

We note that $\mathrm{Pres}(T)$ and $\mathrm{Ho}(\mathrm{Pres}(T))$ are not necessarily categories, but are *large categories*, because their hom-collections are not necessarily hom-sets. The notion of a category of presentations of a monad appears in VI.5 of [5], but was not there given a name.

For the purposes of this paper we will mostly be studying presentations of a monad which have the property that there is some minimal degree of compatibility between the category and the monad, enough compatibility to guarantee that e.g. some constructions in homological algebra can be made. Here is our definition:

Definition 2.2. *Let $\mathcal{C}$ be a category, T a monad, $(\mathcal{D}, F', G')$ a presentation of T. We will say that $\mathcal{D}$ is a* homological presentation of T *if, for every T-algebra $\rho : G'F'X \to X$, the coequalizer (in $\mathcal{D}$) of the two natural counit maps*

$$\epsilon_{F'X}, F'\rho : F'G'F'X \to F'X \tag{2.1.1}$$

exists, and for each object Y of $\mathcal{D}$, the canonical map

$$\mathrm{coeq}\{\epsilon_{F'G'Y}, F'G'\epsilon_Y\} \to Y \tag{2.1.2}$$

is an isomorphism. We will write $\mathrm{HPres}(T)$ *for the full large subcategory of* $\mathrm{Pres}(T)$ *generated by the homological presentations.*

We will write $\mathrm{Ho}(\mathrm{HPres}(T))$ *for the large category whose objects are homological presentations of T, and whose morphisms are homotopy classes of morphisms of adjunctions.*

Remark 2.3. The reason for the name "homological" for this kind of presentation is the following: if $(\mathcal{D}, F', G')$ is a homological presentation of a monad and $\mathcal{D}$ is abelian, then each object X in $\mathcal{D}$ admits a canonical resolution

$$0 \leftarrow X \leftarrow F'G'X \xleftarrow{d_0} F'G' \ker(\epsilon_{F'G'X} - F'G'\epsilon_X) \xleftarrow{d_1} \dots \tag{2.1.3}$$

obtained by repeatedly applying $F'G'$, forming the coequalizer 2.1.2, and taking the kernel of the coequalizer map. This resolution gives us a way to compute the left-derived functors of any functor on $\mathcal{D}$ which is acyclic on every object of the form $F'G'X$. If $(\mathcal{D}, F', G')$ fails to be homological, then at least for some objects X the chain complex 2.1.3 fails to be exact and hence cannot be used to compute derived functors in this way.

The resolution 2.1.3 is very familiar and commonplace in its various special cases. For example, when $\mathcal{C}$ is the category of sets and T the monad given on a set S by taking the underlying set of the free abelian group generated by S, then the category Ab is a presentation for T, and it is homological (because it is the terminal presentation, i.e., the Eilenberg-Moore category of T, which in Corollary 2.14 we prove is always homological for any coequalizable monad T). The resolution 2.1.3 is the elementary resolution one uses in a first course in homological algebra to prove that free resolutions exist in the category of abelian groups: given an abelian group X, one can form the direct sum $\oplus_{x\in X}\mathbb{Z}$, one can let X_0 be the kernel of the obvious surjection $\oplus_{x\in X}\mathbb{Z} \to X$, then iterate to form a free resolution of X.

When $\mathcal{D}$ fails to be abelian, instead of 2.1.3 one forms the simplicial resolution

$$\dots \longrightarrow F'G'F'G'F'G'X \longrightarrow F'G'F'G'X \longrightarrow F'G'X$$

of X, and one can use this resolution to compute more general kinds of derived functors (e.g. if $\mathcal{D}$ has the structure of a model category). In every case the condition that $(\mathcal{D}, F', G')$ is homological is really the condition that F', G' gives us a way to form a canonical resolution of any object in $\mathcal{D}$. In some sense one should think of a homological presentation for a monad T as *a category equipped with a way of forming a canonical resolution of any object by T-free objects,* and that means that this paper is in some sense really about classifying various ways of forming canonical resolutions.

Finally, one more note about the map 2.1.2: after applying G', the map always becomes an isomorphism, because the fork

$$G'F'G'F'G'X \underset{G'F'G'\epsilon_X}{\overset{G'\epsilon_{F'G'X}}{\rightrightarrows}} G'F'G'X \longrightarrow G'X$$

is always split by the unit map $\eta_{G'X} : G'X \to G'F'G'X$, hence the fork is a split coequalizer. But the map 2.1.2 can fail to be an isomorphism *before* applying G'.

Informally, the general trend is that monads tend to either have a single homological presentation up to equivalence, or a truly enormous collection of homological presentations, big enough to make it very difficult to explicitly classify them. However, even within the homological presentations, there is an even more restricted class of presentations of a monad which we can reasonably restrict our attention to, namely, the *finitary* homological presentations:

Definition 2.4. *Let $\mathcal{C}$ be a category, T a monad, $(\mathcal{D}, F', G')$ a presentation of T. We will say that $\mathcal{D}$ is a* finitary *presentation of T if G' preserves all filtered colimits which exist in $\mathcal{D}$.*

We will write $\mathrm{Fin\,Pres}(T)$ *for the large category of finitary presentations of T,* $\mathrm{Fin\,HPres}(T)$ *for the large category of finitary homological presentations, and* $\mathrm{Ho}(\mathrm{Fin\,HPres}(T))$ *for the large category of finitary homological presentations up to natural equivalence.*

Remark 2.5. In the category of modules over a ring, every object is a filtered colimit of finitely generated modules. Consequently, when $(\mathcal{D}, F', G')$ is a finitary presentation and $\mathcal{D}$ a category of modules over a ring, then G' can be computed on any object if one knows how to compute G' on finitely generated modules. This is actually quite useful; see the proof of Theorem 4.5, for example.

Remark 2.6. It is *not* in general true that every presentation of a monad T is finitary, even if T itself preserves filtered colimits; for example, let $\mathcal{C}$ be the category of sets, let T be the monad which sends a set S to the underlying set of the free abelian group generated by S, and let $(\mathcal{D}, F', G')$ be the presentation for T in which $\mathcal{D}$ is the category of *reduced* abelian groups, $F' : \mathcal{C} \to \mathcal{D}$ is the free abelian group functor, and $G' : \mathcal{D} \to \mathcal{D}$ is the forgetful functor. (An abelian group is said to be "reduced" when it has no nontrivial divisible subgroup.) Then, for any prime number p, the colimit of the filtered diagram

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \dots$$

in $\mathcal{D}$ is zero, but the colimit of

$$G'(\mathbb{Z}) \xrightarrow{G'(p)} G'(\mathbb{Z}) \xrightarrow{G'(p)} \dots$$

in $\mathcal{C}$ is the underlying set of $\mathbb{Z}[\frac{1}{p}]$.

Recall that a subcategory is said to be *replete* if it contains every object isomorphic to one of its own objects, and *reflective* if it a full subcategory and the inclusion of the subcategory admits a left adjoint. We include fullness as part of the definition of a reflective subcategory; this seems to be relatively standard, although not universal (see e.g. [5]), in the literature.

It is standard that a reflective subcategory is closed under limits computed in the ambient category; see e.g. the beginning of section 2 in [7].

Definition 2.7. *Let $\mathcal{C}$ be a category, T a monad, $\mathcal{C}^T$ the Eilenberg-Moore category of T-algebras. If $\mathcal{D}$ is a replete reflective subcategory of $\mathcal{C}^T$, we will say that $\mathcal{D}$* presents *T if $\mathcal{D}$ contains all the free T-algebras, i.e., if $\mathcal{D}$ contains the T-algebra $TTX \xrightarrow{\mu_X} TX$ for every object X of $\mathcal{C}$.*

We will write $\mathrm{Loc}(T)$ for the partially-ordered collection of all replete reflective subcategories $\mathcal{C}^T$ which present T.

Finally, we will say that an element $\mathcal{D}$ of $\mathrm{Loc}(T)$ is finitary *if the forgetful functor from $\mathcal{D}$ to $\mathcal{C}^T$ preserves all filtered colimits which exist in $\mathcal{D}$. We will write $\mathrm{Fin}\,\mathrm{Loc}(T)$ for the subcollection of $\mathrm{Loc}(T)$ consisting of the finitary elements.*

The notation "$\mathrm{Loc}(T)$" is motivated by the notion that a replete reflective subcategory of a category is, speaking intuitively and roughly, a kind of "localization" of that category.

We note that $\mathrm{Loc}(T)$ is not necessarily a set, *nor even a class* (we are grateful to Mike Shulman for pointing out to us that the collection of subcategories of a category is not necessarily a class!). Sometimes the term "conglomerate" is used for a collection too large to form a class. In other words, if one wants to use Grothendieck universes, one must expand the universe *twice* to go from sets to conglomerates. In practical algebraic, geometric, and topological situations, however, it seems likely that $\mathrm{Loc}(T)$ will form a set. For example, see Corollary 2.16, where we show that mild conditions (a biproduct condition, a Krull-Schmidt condition, and a smallness condition) imply that $\mathrm{Loc}(T)$ is a set.

Finally, it will sometimes be convenient to have coequalizers in Eilenberg-Moore categories. We introduce a definition which describes monads which have this agreeable property:

Definition 2.8. *Let $\mathcal{C}$ be a category, T a monad, $\mathcal{C}^T$ the Eilenberg-Moore category of T-algebras. We will say that T is* coequalizable *if $\mathcal{C}^T$ has coequalizers.*

Remark 2.9. There are many known conditions on T which guarantee that T is coequalizable; for example, in Lemma II.6.6 in [3] it is shown that, if T preserves reflexive coequalizers, then T is coequalizable. Consequently, many interesting examples of monads T are coequalizable.

For example, suppose $R \to S$ is a map of commutative rings. Then the base-change monad $T : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$, i.e., the composite of the extension of scalars functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ with the restriction of scalars functor $\mathrm{Mod}(S) \to \mathrm{Mod}(R)$, is coequalizable, since extension of scalars and restriction of scalars are both right exact, preserving all coequalizers. If S is finitely generated as an R-module then the base-change monad $\mathrm{fgMod}(R) \to \mathrm{fgMod}(R)$ on the finitely generated module category is also coequalizable, for the same reason. Then the Eilenberg-Moore category $\mathrm{Mod}(R)^T$ is equivalent to $\mathrm{Mod}(S)$.

More generally, if $f : Y \to X$ is a map of schemes and $\mathrm{QC}\,\mathrm{Mod}(\mathcal{O}_X)$ the category of quasicoherent $\mathcal{O}_X$-modules, then the base-change monad f_*f^* is coequalizable if f is an affine morphism, since in that case f_* is right exact (and f^* is always right exact, regardless of whether f is affine). Then the Eilenberg-Moore category $\mathrm{QC}\,\mathrm{Mod}(\mathcal{O}_X)^{f_*f^*}$ is equivalent to $\mathrm{QC}\,\mathrm{Mod}(\mathcal{O}_Y)$, by the results of EGA II.1.4, [4].

Usually (e.g. in the examples above, and in our Theorem 4.5) we will have an explicit description of the category $\mathcal{C}^T$ and we will know that it has coequalizers; what will be interesting and new will be the description of the *rest* of $\mathrm{HPres}(T)$.

Definition 2.10. *Suppose $\mathcal{C}$ is a category with coproduct $\oplus$. We say that an object X of $\mathcal{C}$ is* indecomposable *if $X \cong Y \oplus Z$ implies either $Y \cong 0$ or $Z \cong 0$. We say that*

$\mathcal{C}$ is weakly Krull-Schmidt *if every object X of $\mathcal{C}$ admits a decomposition into a finite coproduct of indecomposable objects, and that decomposition is unique up to permutation and isomorphism of the summands.*

Suppose furthermore that $\mathcal{A}$, $\mathcal{G}$ are two collections of objects of $\mathcal{C}$. We say that $\mathcal{C}$ is weakly Krull-Schmidt relative to $\mathcal{A}$ and generated by $\mathcal{G}$ *if every object X of $\mathcal{C}$ admits a decomposition into a finite coproduct of the form $X_0 \oplus X_1 \oplus \cdots \oplus X_n$ where X_0 is in $\mathcal{A}$, each of the objects $X_1, \dots, X_n$ are indecomposable and contained in $\mathcal{G}$, and that decomposition is unique up to isomorphism of the summand X_0 and permutation and isomorphism of the summands $X_1, \dots, X_n$.*

Our definition of weakly Krull-Schmidt differs from the usual definition of a Krull-Schmidt category in that we do not require the indecomposable objects to have local endomorphism rings. See Remark 4.2 for some discussion of the purpose of the definition of "weakly Krull-Schmidt relative to $\mathcal{A}$ and generated by $\mathcal{G}$."

2.2. **Replete reflective subcategories presenting a monad are equivalent to its homological presentations.**

Lemma 2.11. *Let $\mathcal{C}$ be a category, T a monad on $\mathcal{C}$, $(\mathcal{D}, F', G')$ a presentation of T. If $\mathcal{D}$ has coequalizers of all pairs of maps of the form 2.1.1, then the canonical comparison functor $K : \mathcal{D} \to \mathcal{C}^T$ has a left adjoint. Conversely, if T is coequalizable and K is full and faithful and has a left adjoint, then $\mathcal{D}$ has all coequalizers (and in particular, all pairs of maps of the form 2.1.1).*

Proof. We will write $F : \mathcal{C} \to \mathcal{C}^T$ for the canonical functor and G for its right adjoint. When $\mathcal{D}$ has coequalizers of all parallel pairs of the form 2.1.1, the comparison functor K admits a left adjoint V, defined on objects as follows: if $TX \xrightarrow{\rho_X} X$ is the structure map of a T-algebra, then V applied to that T-algebra is the coequalizer of the maps

$$F'\rho_X, \epsilon_{F'X} : F'G'F'X \to F'X,$$

using the fact that $G'F' = GF = T$. (The result that K has a left adjoint if $\mathcal{D}$ has coequalizers is an old one: it appears in Beck's thesis [1], and even appears as an exercise in VI.7 of [5]. But the only coequalizers one actually needs are the ones used in the construction of the left adjoint, i.e., those of the form 2.1.1.)

For the converse: suppose T is coequalizable and K is full and faithful and has a left adjoint V. Since left adjoints preserve colimits and since fullness and faithfulness of K is equivalent to $VK \cong \mathrm{id}_{\mathcal{D}}$, we can compute the coequalizer of any pair $f, g : X \to Y$ in $\mathcal{D}$ by computing the coequalizer of Kf, Kg in $\mathcal{C}^T$ (which exists since T is coequalizable) and then applying V. □

Lemma 2.12. *Let $\mathcal{C}$ be a category, T a monad on $\mathcal{C}$, $(\mathcal{D}, F', G')$ a presentation of T. Suppose $\mathcal{D}$ has coequalizers of all pairs of maps of the form 2.1.1. Then the comparison functor $K : \mathcal{D} \to \mathcal{C}^T$ is full and faithful if and only if $(\mathcal{D}, F', G')$ is homological.*

Proof. We use the same notation as in the proof of Lemma 2.11. That K is full and faithful is equivalent to the counit map $VK \to \mathrm{id}_{\mathcal{D}}$ of the adjunction being an isomorphism. We recall that K is defined on objects by letting KX be the T-algebra with structure map $G'F'G'X = TG'X \to G'X$ given by the counit natural transformation $F'G' \to \mathrm{id}_{\mathcal{D}}$. Now VKX is precisely the coequalizer of the two maps

$$\epsilon_{F'G'X}, F'G'\epsilon_X : F'G'F'G'X \to F'G'X,$$

and the map $VKX \to X$ is precisely the map 2.1.2. So the condition that $(\mathcal{D}, F', G')$ be homological is equivalent to the condition that $VK \to \mathrm{id}_{\mathcal{D}}$ be an isomorphism of functors, i.e., the condition that K be full and faithful. $\square$

Theorem 2.13. *Let $\mathcal{C}$ be a category, T a coequalizable monad on $\mathcal{C}$. Then the large homotopy category* $\mathrm{Ho}(\mathrm{HPres}(T))$ *of homological presentations of T is equivalent to the partially-ordered collection* $\mathrm{Loc}(T)$ *of replete reflective subcategories of $\mathcal{C}^T$ which present T.*

Furthermore, if the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ preserves filtered colimits, then this equivalence restricts to an equivalence between the subcollection $\mathrm{Ho}(\mathrm{Fin}\,\mathrm{HPres}(T))$ *of* $\mathrm{Ho}(\mathrm{HPres}(T))$ *and the subcollection* $\mathrm{Fin}\,\mathrm{Loc}(T)$ *of* $\mathrm{Loc}(T)$.

Proof. We write $\mathrm{Kl}(T)$ for the Kleisli category of T, we write $F : \mathcal{C} \to \mathcal{C}^T$ for the canonical functor and G for its right adjoint, and we write $F'' : \mathcal{C} \to \mathrm{Kl}(T)$ for the canonical functor and G'' for its right adjoint. The theorem follows almost immediately from Lemma 2.12, which gives us that every homological presentation $(\mathcal{D}, F', G')$ of T has the property that K is faithful and full, hence $\mathcal{D}$ is canonically equivalent to a full replete subcategory of $\mathcal{C}^T$, and Lemma 2.11, which gives us that that full replete subcategory is reflective. That reflective replete subcategory contains the free T-algebras, i.e., the Kleisli category of T, since the Kleisli category is initial among presentations of T. So that reflective replete subcategory of $\mathcal{C}^T$ is an element of $\mathrm{Loc}(T)$. If we furthermore assume that G preserves filtered colimits and $(\mathcal{D}, F', G')$ is finitary, then since $G' = G \circ K$, for any filtered diagram $\mathcal{X}$ in $\mathcal{D}$, we have the natural commutative diagram

$$\begin{tikzcd}
\mathrm{colim}\, G'(\mathcal{X}) \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & G'(\mathrm{colim}\, \mathcal{X}) \\
& G(\mathrm{colim}\, K(\mathcal{X})), \arrow[ur] &
\end{tikzcd}$$

so the map $G(\mathrm{colim}\, K(\mathcal{X})) \to G'(\mathrm{colim}\, \mathcal{X}) = G(K(\mathrm{colim}\, \mathcal{X}))$ is an isomorphism, and since G reflects isomorphisms, K preserves filtered colimits.

Conversely, if $\mathcal{D}$ is a reflective replete subcategory of $\mathcal{C}^T$ with inclusion $K : \mathcal{D} \to \mathcal{C}^T$ having left adjoint V, suppose we write $S : \mathrm{Kl}(T) \to \mathcal{D}$ for the inclusion of the free T-algebras. We claim that the composite $S \circ F'' : \mathcal{C} \to \mathcal{D}$ has right adjoint $G \circ K : \mathcal{D} \to \mathcal{C}$, and that the composite monad $G \circ K \circ S \circ F''$ is equal to the monad T. The second claim is very easy: the composite $G \circ K \circ S$ is equal to G'', so

$$G \circ K \circ S \circ F'' = G'' \circ F'' = T.$$

The first claim is also not difficult: since $F = K \circ S \circ F''$ and $V \circ K \simeq \mathrm{id}_{\mathcal{D}}$, we have

$$V \circ F \simeq V \circ K \circ S \circ F'' \simeq S \circ F''.$$

Now V is left adjoint to K and F left adjoint to G, so $S \circ F'' \simeq V \circ F$ is left adjoint to $G \circ K$, proving our first claim. It follows that $(\mathcal{D}, S \circ F'', G \circ K)$ is a presentation of T.

All that remains to be proven is that $(\mathcal{D}, S \circ F'', G \circ K)$ is a *homological* presentation of T. By construction, K is full and faithful, so by Lemma 2.11, $\mathcal{D}$ has coequalizers of all parallel pairs of the form 2.1.1. So by Lemma 2.12, $(\mathcal{D}, S \circ F'', G \circ K)$ is homological. If we furthermore assume that G preserves filtered colimits and that $\mathcal{D}$ is finitary, then of course the composite $G \circ K$ preserves filtered colimits, and consequently $(\mathcal{D}, S \circ F'', G \circ K)$ is also a finitary homological presentation. $\square$

Corollary 2.14. *If T is coequalizable, the Eilenberg-Moore adjunction $(\mathcal{C}^T, F, G)$ of T is a homological presentation of T. (And, consequently, the terminal homological presentation of T.)*

Corollary 2.15. *If T is coequalizable, the large homotopy category* $\mathrm{Ho}(\mathrm{HPres}(T))$ *of homological presentations of T is partially-ordered, i.e., for any objects $\mathcal{A}, \mathcal{B}$ of* $\mathrm{Ho}(\mathrm{HPres}(T))$*, there is at most one morphism $\mathcal{A} \to \mathcal{B}$. If we furthermore assume that the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ preserves filtered colimits, then large homotopy category* $\mathrm{Ho}(\mathrm{Fin}\,\mathrm{HPres}(T))$ *of finitary homological presentations of T is also partially-ordered.*

Corollary 2.16. *Suppose T is coequalizable and $\mathcal{C}^T$ is weakly Krull-Schmidt. Suppose the collection of isomorphism classes of indecomposable objects forms a* set *(not a proper class!), and suppose that set has cardinality κ. Then* $\mathrm{Ho}(\mathrm{HPres}(T))$ *is equivalent to a partially-ordered* set *of cardinality no greater than $2^{\aleph_0^\kappa}$.*

Proof. The partially-ordered collection $\mathrm{Loc}(T)$, which by Theorem 2.13 is equivalent to $\mathrm{Ho}(\mathrm{HPres}(T))$, is contained in the collection of subcollections of the collection of finite formal sums of indecomposable objects. This collection is, in turn, contained in the collection of subcollections of the collection of not-necessarily-finite formal sums of indecomposable objects in which each indecomposable object appears only finitely many times. This last collection has cardinality $2^{\aleph_0^\kappa}$. □

We greatly improve this cardinality bound in Corollary 2.21 under the assumption that $\mathcal{C}^T$ has a biproduct.

2.3. Coordinatization of the collection of homological presentations of a monad.

Definition 2.17. *Recall that a category $\mathcal{C}$ is said to* have a biproduct *if it a zero object, finite products, and finite coproducts, and, for each finite family $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$, the canonical map $\coprod_{i\in I} X_i \to \prod_{i\in I} X_i$ is an isomorphism.*

Lemmas 2.18 and 2.19 are easy and must be well-known, but we do not know where they already appear in the literature.

Lemma 2.18. *Suppose $\mathcal{A}$ is a replete reflective subcategory of a category with biproduct. Then $\mathcal{A}$ has a biproduct.*

Proof. Reflective subcategories are closed under limits, so the biproduct computed in the ambient category is also contained in $\mathcal{A}$. It is an easy exercise to verify that this biproduct also has the universal properties of the product and coproduct in $\mathcal{A}$. □

Lemma 2.19. *Suppose $\mathcal{A}$ is a replete reflective subcategory of a category $\mathcal{C}$ with biproduct $\oplus$. If $X \cong Y \oplus Z$ in $\mathcal{C}$, then X is in $\mathcal{A}$ if and only if both Y and Z are in $\mathcal{A}$.*

Proof. We write $L : \mathcal{C} \to \mathcal{C}$ for the composite of the reflector functor $\mathcal{C} \to \mathcal{A}$ with the inclusion $\mathcal{A} \to \mathcal{C}$. By Lemma 2.18, $\mathcal{A}$ has a biproduct. Since L is a composite of a left adjoint (the reflector functor) with a right adjoint (the inclusion functor), it preserves biproducts, since the biproduct is both the finite coproduct and the finite product. So $LX \cong LY \oplus LZ$, and if Y, Z are in $\mathcal{A}$, then the unit maps $Y \to LY$ and $Z \to LZ$ are both isomorphisms. So $X \cong Y \oplus Z \to LY \oplus LZ \cong LX$ is an isomorphism. So X is in $\mathcal{A}$.

For the converse: if X and Y are objects in $\mathcal{C}$ and $X \oplus Y$ is in $\mathcal{A}$, write $i : X \to X \oplus Y$ and $\pi : X \oplus Y \to X$ for the summand inclusion and projection maps. Then X is the equalizer of the two maps

$$\mathrm{id}, i \circ \pi : X \oplus Y \to X \oplus Y$$

(this is easy to prove, by "splitting the cofork" in the sense of section VI.6 of [5]). So X is a limit of a diagram in $\mathcal{A}$, so X is in $\mathcal{A}$ since reflective subcategories are closed under limits. $\square$

Theorem 2.20. *(***Coordinatization.***) Let $\mathcal{C}$ be a category, T a coequalizable monad on $\mathcal{C}$. Suppose the Eilenberg-Moore category $\mathcal{C}^T$ has a biproduct and is weakly Krull-Schmidt. Write $\Gamma(\mathcal{C}^T)$ for the collection of isomorphism classes of indecomposable objects in $\mathcal{C}^T$. Then* $\mathrm{Ho}(\mathrm{HPres}(T))$ *embeds by an order-preserving map into the collection of subcollections of $\Gamma(\mathcal{C}^T)$.*

Proof. By Theorem 2.13, specifying an element of $\mathrm{Ho}(\mathrm{HPres}(T))$ is equivalent to specifying a replete reflective subcategory of $\mathcal{C}^T$ which contains $\mathcal{C}_T$, hence is determined uniquely by which isomorphism classes of objects in $\mathcal{C}^T$ are contained in the replete reflective subcategory. But by Lemma 2.19, a replete reflective subcategory of a weakly Krull-Schmidt category with biproduct is determined uniquely by which indecomposables are contained in it. $\square$

In other words: under the conditions of Theorem 2.20, a homological presentation of T can be specified by specifying a suitable subcollection of $\Gamma(\mathcal{C}^T)$ (which, as we describe in the last section of this paper, is actually computable in cases of interest). Since $\mathcal{C}^T$ is often computable and understandable, Theorem 2.20—when it applies—gives a coordinatization of the collection of homological presentations of T, as desired.

Corollary 2.21. *Suppose T is coequalizable and $\mathcal{C}^T$ is weakly Krull-Schmidt and has a biproduct. Suppose the collection of isomorphism classes of indecomposable objects forms a* set *(not a proper class!), and suppose that set has cardinality κ. Then* $\mathrm{Ho}(\mathrm{HPres}(T))$ *is equivalent to a partially-ordered* set *of cardinality no greater than* 2^κ.

Remark 2.22. Suppose that k is a field and A is a k-algebra. Let $\mathcal{C}$ be any category and T any monad on $\mathcal{C}$ whose Eilenberg-Moore category $\mathcal{C}^T$ is equivalent to the category $\mathrm{fgMod}(A)$ of finitely generated A-modules; for example, we could let $\mathcal{C}$ be the category of B-modules, for some reasonable subalgebra B of A, and we could let T be the monad associated to the free-forgetful adjunction between B-modules and A-modules. In Auslander-Reiten theory, the set of isomorphism classes of indecomposable finitely generated A-modules is exactly the set $\Gamma(\mathrm{fgMod}(A))$ of vertices in the Auslander-Reiten quiver of $\mathcal{C}^T$. So one can regard the vertices of the Auslander-Reiten quiver of $\mathcal{C}^T$ as "coordinates" for the collection of natural equivalence classes of homological presentations of T: Theorem 2.20 gives an embedding of $\mathrm{Ho}(\mathrm{HPres}(T))$ into the partially-ordered set of subsets of $\Gamma(\mathrm{fgMod}(A))$.

3. A criterion for unique homological presentability of a monad.

3.1. **Preliminary definitions.** Some monads can (up to natural equivalence) only be homologically presented in a single way, i.e., $\mathrm{Ho}(\mathrm{HPres}(T))$ is equivalent to a one-object category. Here is the relevant definition:

Definition 3.1. *Suppose T is a monad. If* $\mathrm{Ho}(\mathrm{HPres}(T))$ *has only a single element up to isomorphism, then we say that T is* uniquely homologically presentable.

We give a concrete algebraic class of examples (base-change monads associated to field extensions) of uniquely homologically presentable monads in Corollary 3.8.

Because we will need to make use of it, we state Beck's monadicity theorem (see e.g. VI.7 of [5]):

Theorem 3.2. (Beck.) *Suppose $\mathcal{C}, \mathcal{D}$ are categories, $G : \mathcal{D} \to \mathcal{C}$ a functor with a left adjoint F. Then the comparison functor $\mathcal{D} \to \mathcal{C}^{GF}$ is an equivalence of categories if and only if, whenever a parallel pair $f, g : X \to Y$ in $\mathcal{D}$ is such that Gf, Gg has a split coequalizer in $\mathcal{C}$, each of the following conditions hold:*

- *f, g has a coequalizer* $\mathrm{coeq}\{f, g\}$ *in $\mathcal{D}$,*
- *G preserves the coequalizer of f, g, i.e., the natural map* $\mathrm{coeq}\{Gf, Gg\} \to G\,\mathrm{coeq}\{f, g\}$ *is an isomorphism,*
- *and G reflects the coequalizer of f, g, i.e., if Z is a cocone over the diagram $f, g : X \to Y$ such that GZ is a coequalizer of Gf, Gg, then Z is a coequalizer of f, g.*

Here is a very classical definition:

Definition 3.3. *When G is a functor with left adjoint, we say that G is* monadic *if G satisfies the equivalent conditions of Theorem 3.2.*

We offer a (to our knowledge, new) variant on this definition which will be essential to our criterion for unique homological presentability of a monad.

Definition 3.4. *Suppose $\mathcal{C}, \mathcal{D}$ are categories, $G : \mathcal{D} \to \mathcal{C}$ a functor. We say that G is* absolutely monadic *if G has a left adjoint and, whenever a parallel pair $f, g : X \to Y$ in $\mathcal{D}$ is such that Gf, Gg has a split coequalizer in $\mathcal{C}$, then:*

- *f, g has a* split *coequalizer* $\mathrm{coeq}\{f, g\}$ *in $\mathcal{D}$,*
- *G preserves the coequalizer of f, g, i.e., the natural map* $\mathrm{coeq}\{Gf, Gg\} \to G\,\mathrm{coeq}\{f, g\}$ *is an isomorphism,*
- *and G reflects the coequalizer of f, g, i.e., if Z is a cocone over the diagram $f, g : X \to Y$ such that GZ is a coequalizer of Gf, Gg, then Z is a coequalizer of f, g.*

Note that a functor that is absolutely monadic is also monadic, but the converse does not always hold.

3.2. **A criterion for unique homological presentability.** Now we present and prove the main result of this section.

First we will need a lemma. We suspect that this lemma is already well-known, but we do not know an already-existing reference in the literature.

Lemma 3.5. *Suppose $\mathcal{D}, \mathcal{E}$ are categories, and $\mathcal{D} \xrightarrow{S} \mathcal{E}$ is a full, faithful functor with a left adjoint. Then S is monadic.*

Proof. Since S is full and faithful, we regard it as inclusion of a subcategory $\mathcal{D}$ of $\mathcal{E}$. Then since S has a left adjoint, $\mathcal{D}$ is a reflective subcategory of $\mathcal{E}$. We write V for the left adjoint of S. Let $f, g : X \to Y$ be a pair of maps in $\mathcal{D}$ such that Sf, Sg has a split coequalizer Z Then we can apply V together with the natural equivalence $VS \simeq \mathrm{id}_{\mathcal{D}}$ to get that VZ is a split coequalizer of f, g. Hence S sends a cofork in $\mathcal{D}$ to a split coequalizer in $\mathcal{E}$ if and only if the cofork was already a split coequalizer in $\mathcal{D}$. So S preserves coequalizers of all pairs in $\mathcal{D}$ with a S-split coequalizer, and since S is faithful and injective on objects, it reflects isomorphisms; so S is monadic. □

Theorem 3.6. *Suppose $\mathcal{C}$ is a category, T a coequalizable monad on $\mathcal{C}$. We write F for the canonical functor $\mathcal{C} \to \mathcal{C}^T$ and G for its right adjoint. If G is absolutely monadic, then T is uniquely homologically presentable.*

Proof. Suppose G is absolutely monadic, and suppose that $(\mathcal{D}, F', G')$ is a presentation of T. We have the comparison functor $K : \mathcal{D} \to \mathcal{C}^T$, and we have that $G' = G \circ K$. We are going to show that, if $(\mathcal{D}, F', G')$ is homological, then K is an equivalence.

Suppose f, g is a parallel pair in $\mathcal{D}$ such that $G'f, G'g$ has a split coequalizer in $\mathcal{C}$. Then Kf, Kg is a parallel pair in $\mathcal{C}^T$ such that $G(Kf), G(Kg)$ has a split coequalizer in $\mathcal{C}$, and since G is absolutely monadic, Kf, Kg has a split coequalizer Z such that GZ is the given split coequalizer for $G'f, G'g$. But, by Lemma 3.5, K is monadic, hence, by Theorem 3.2, f, g has a coequalizer W such that KW is Z. Hence f, g has a coequalizer in $\mathcal{D}$ and G' preserves that coequalizer.

Now we check that G' reflects appropriate coequalizers. Suppose

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \longrightarrow Z \tag{3.2.1}$$

is a cofork in $\mathcal{D}$ such that the cofork

$$G'X \underset{G'g}{\overset{G'f}{\rightrightarrows}} G'Y \longrightarrow G'Z \tag{3.2.2}$$

is a split coequalizer sequence in $\mathcal{C}$. Again using the fact that $G' = G \circ K$ and using that G is absolutely monadic, we have that the cofork

$$KX \underset{Kg}{\overset{Kf}{\rightrightarrows}} KY \longrightarrow KZ \tag{3.2.3}$$

in $\mathcal{C}^T$ is a split coequalizer sequence; finally, by Lemma 3.5 and Theorem 3.2, K reflects such coequalizers, so cofork 3.2.1 is a coequalizer sequence in $\mathcal{D}$.

Hence G' preserves and reflects coequalizers of all parallel pairs f, g such that Gf, Gg has a split coequalizer. Hence, by Theorem 3.2, G' is monadic, and the comparison map $\mathcal{D} \to \mathcal{C}^{G'F'} = \mathcal{C}^T$ is an equivalence of categories.

Hence every homological presentation $\mathcal{D}$ of T is equivalent to the entire Eilenberg-Moore category of T. Hence every element of $\mathrm{Ho}(\mathrm{HPres}(T))$ is isomorphic to the Eilenberg-Moore presentation. □

Corollary 3.7. *Suppose $\mathcal{C}$ is an abelian category and T a monad on $\mathcal{C}$ such that $\mathcal{C}^T$ is abelian and the canonical functor $G : \mathcal{C}^T \to \mathcal{C}$ is additive. Suppose that, if*

$$X \to Y \to Z$$

is a pair of maps in $\mathcal{C}^T$ such that

$$GX \to GY \to GZ \to 0$$

is split exact in $\mathcal{C}$, then

$$X \to Y \to Z \to 0$$

is split exact in $\mathcal{C}^T$. Then T is uniquely homologically presentable.

Proof. The assumed condition on G is precisely what absolute monadicity of G means in the abelian setting. □

Corollary 3.8. *Suppose L/K is a field extension and $T : \mathrm{Mod}(K) \to \mathrm{Mod}(K)$ the associated base change monad, i.e., TM is the underlying K-module of $L \otimes_K M$. Then T is uniquely homologically presentable.*

4. Explicit examples: Dedekind domains.

First, recall the well-known classification of finitely generated modules over a Dedekind domain, which we will use throughout this section:

Theorem 4.1. *Let A be a Dedekind domain, and let M be a finitely generated A-module. Then M is isomorphic to a direct sum of a finitely generated projective A-module and finitely many A-modules of the form $A/\mathfrak{m}^n$ for various maximal ideals $\mathfrak{m}$ in A and various positive integers n. This decomposition is unique up to isomorphism and up to permutation and isomorphism of the indecomposable factors $A/\mathfrak{m}^n$.*

Remark 4.2. In Theorem 4.1, the projective summand also decomposes into a direct sum of indecomposables, but the decomposition of the projective summand does *not* have to be unique up to permutation and isomorphism of the summands unless the class group of the Dedekind domain vanishes. So the finitely generated modules over a Dedekind domain are weakly Krull-Schmidt relative to the projectives and generated by the indecomposable torsion modules, but not weakly Krull-Schmidt unless the class group vanishes (see Definition 2.10 for these definitions). (We are grateful to the anonymous referee for pointing out that we had been careless about this effect of the class group in an earlier version of this paper.)

Now here is a result which we will use in the proof of Theorem 4.5:

Proposition 4.3. *Let A be a Dedekind domain and let* $\mathrm{fgMod}(A)$ *denote the category of finitely generated A-modules. Then the partially-ordered set of reflective replete subcategories of* $\mathrm{fgMod}(A)$ *which contain the free A-modules is is isomorphic to the set of functions*

$$\mathrm{Max\,Spec}(A) \to \{0, 1, 2, \dots, \infty\}$$

from the set $\mathrm{Max\,Spec}(A)$ *of maximal ideals of A to the set of extended natural numbers, under the partial ordering in which we let $f \leqslant g$ if and only if $f(\mathfrak{m}) \leqslant g(\mathfrak{m})$ for all* $\mathfrak{m} \in \mathrm{Max\,Spec}(A)$.

Proof. Suppose that $\mathcal{D}$ is a replete reflective subcategory of $\mathrm{fgMod}(A)$ which contains the free A-modules. Lemma 2.19 implies that $\mathcal{D}$ contains all the summands of free T-algebras, and that $\mathcal{D}$ is closed under the biproduct. Let $\mathcal{G}$ denote the collection of indecomposable torsion finitely generated A-modules. By Theorem 4.1, $\mathrm{fgMod}(A)$ is weakly Krull-Schmidt relative to the projective A-modules and generated by $\mathcal{G}$, so given an object X of $\mathcal{D}$, we can decompose X as $X_0 \oplus X_1 \oplus \cdots \oplus X_n$ with X_0 a summand of a free T-algebra and $X_1, \dots, X_n$ in $\mathcal{G}$, and Lemma 2.19 now implies that $X_1, \dots, X_n$ are all in $\mathcal{D}$ as well. We conclude that $\mathcal{D}$ is determined uniquely by which indecomposable objects of $\mathcal{G}$ are contained in $\mathcal{D}$, i.e., by a subcollection of $\Gamma(\mathcal{G})$.

Now suppose that $\mathfrak{m}$ is a maximal ideal of A and that $A/\mathfrak{m}^n$ is in $\mathcal{A}$, and let $i \leqslant n$. Then $A/\mathfrak{m}^i$ is the kernel of a map $A/\mathfrak{m}^n \to A/\mathfrak{m}^n$, and consequently $A/\mathfrak{m}^i$ is also in $\mathcal{A}$, since a reflective category is closed under limits. Consequently, $\mathcal{A}$ is completely determined by a single function $f_{\mathcal{A}} : \mathrm{Max\,Spec}(A) \to \{0, 1, 2, \dots, \infty\}$ from the set of maximal ideals of A to the set of extended natural numbers; namely, $f_{\mathcal{A}}$ is the function sending a maximal ideal $\mathfrak{m}$ of A to the largest integer n such that $A/\mathfrak{m}^n$ is in $\mathcal{A}$, and by letting $f_{\mathcal{A}}(\mathfrak{m}) = \infty$ if $A/\mathfrak{m}^n$ is in $\mathcal{A}$ for all n.

We claim that, for each function $f : \mathrm{Max\,Spec}(A) \to \{0, 1, 2, \dots, \infty\}$, there does indeed exist a replete reflective subcategory $\mathcal{A}$ of $\mathrm{fgMod}(A)$ such that $f_{\mathcal{A}} = f$. Let $\mathcal{A}_f$ denote the full subcategory of $\mathrm{fgMod}(A)$ generated by the projective A-modules and all the A-modules M with the property that, for each maximal ideal $\mathfrak{m}$ of A, if there exists a monomorphism

$A/\mathfrak{m}^n \to M$ of A-modules, then $n \leqslant f(\mathfrak{m})$. Clearly $\mathcal{A}_f$ contains A as well as $A/\mathfrak{m}^n$ for all $n \leqslant f(\mathfrak{m})$, and $\mathcal{A}_f$ does *not* contain $A/\mathfrak{m}^n$ if $n > f(\mathfrak{m})$. Hence (using Theorem 4.1) $f_{\mathcal{A}_f} = f$.

Clearly $\mathcal{A}_f$ is full and replete in $\mathrm{fgMod}(A)$, so the only remaining question is whether $\mathcal{A}_f$ is reflective. We now construct an explicit left adjoint for the inclusion functor $\mathcal{A}_f \hookrightarrow \mathrm{fgMod}(A)$. Given an A-module M, let $t_f(M)$ denote the subset of M consisting of all elements $x \in M$ such that, for some maximal ideal $\mathfrak{m}$ of A,

- $x \in \mathfrak{m}^{f(\mathfrak{m})}M$, and
- there exists some positive integer n such that $ax = 0$ for all $a \in \mathfrak{m}^n$.

Let $u_f(M)$ denote the sub-A-module of M generated by the subset $t_f(M)$. Clearly, if $g : M' \to M$ is an A-module homomorphism, then $g(t_f(M')) \subseteq t_f(M)$. Consequently u_f is a functor from $\mathrm{fgMod}(A)$ to $\mathrm{fgMod}(A)$, and u_f is equipped with a natural transformation $i_f : u_f \to \mathrm{id}$, namely, the natural inclusion of $u_f(M)$ into M. Let $v_f : \mathrm{fgMod}(A) \to \mathrm{fgMod}(A)$ be the functor given by $v_f(M) = \operatorname{coker} i_f(M)$. Clearly v_f is equipped with a natural transformation $\eta_f : \mathrm{id} \to v_f$, namely, the natural projection of M onto $M/u_f(M)$.

By Theorem 4.1, every finitely generated A-module is isomorphic to $P \oplus \coprod_{i=1}^m A/\mathfrak{m}_i^{\epsilon_i}$ for some projective A-module P, some nonnegative integers m, n, some sequence of maximal ideals $(\mathfrak{m}_1, \dots, \mathfrak{m}_m)$ of A, and some sequence of positive integers $(\epsilon_1, \dots, \epsilon_m)$. Clearly,

$$v_f\left(P \oplus \coprod_{i=1}^m A/\mathfrak{m}_i^{\epsilon_i}\right) = P \oplus \coprod_{i=1}^m A/\mathfrak{m}_i^{\min\{\epsilon_i, f(\mathfrak{m})\}}$$

and $\eta_f(P \oplus \coprod_{i=1}^m A/\mathfrak{m}_i^{\epsilon_i})$ is the obvious projection map. Consequently, for every finitely generated A-module M, $v_f(M)$ is in $\mathcal{A}_f$, and since every map from $P \oplus \coprod_{i=1}^m A/\mathfrak{m}_i^{\epsilon_i}$ to an object of $\mathcal{A}_f$ factors uniquely through the projection map η_f, the functor v_f, with its codomain restricted to $\mathcal{A}_f$, is left adjoint to the inclusion $\mathcal{A}_f \hookrightarrow \mathrm{fgMod}(A)$. Consequently $\mathcal{A}_f$ is reflective. $\square$

The anonymous referee pointed out to us that results similar to Proposition 4.3 are also obtained in [6]; while Proposition 4.3 does not appear in [6], it is also easy to prove Proposition 4.3 using the results of section 5 of [6] as a starting point.

Corollary 4.4. *Let K, L be number fields, that is, finite extensions of $\mathbb{Q}$, with rings of integers A and B, respectively. Let L/K be a field extension. Let T be the monad associated to the induction-restriction adjunction between* $\mathrm{fgMod}(A)$ *and* $\mathrm{fgMod}(B)$*, i.e., $T(M)$ is the underlying A-module of $B \otimes_A M$. Then* $\mathrm{Ho}(\mathrm{HPres}(T))$ *has only a single element up to isomorphism. That is, there exists (up to natural equivalence) only one homological presentation of T.*

Proof. The Kleisli category $\mathrm{fgMod}(B)_T$ contains all the B-modules of the form $B \otimes_A M$ for M a finitely generated A-module. For any maximal ideal $\mathfrak{m}$ of B, and any positive integer n, $\mathrm{fgMod}(B)_T$ contains a module $B \otimes_A M$ with $B/\mathfrak{m}^i$ as a summand for some $i \geqslant n$, namely, let $\mathfrak{p}$ be the (unique) prime of A under $\mathfrak{m}$, and let $M = A/\mathfrak{p}^n$. Consequently the only reflective replete subcategory of $\mathrm{fgMod}(B)$ which contains $\mathrm{fgMod}(B)_T$ is the one which, in the language of Proposition 4.3, corresponds to the function $\mathrm{Max\,Spec}(B) \to \{0, 1, \dots, \infty\}$ sending every maximal ideal to ∞, i.e., $\mathrm{fgMod}(B)$ itself. $\square$

Theorem 4.5. *Let A be a Dedekind domain, let* Sets *denote the category of sets, and let T denote the monad on* Sets *associated to the free-forgetful adjunction between* $\mathrm{Mod}(A)$ *and* Sets*, i.e., $T(S)$ is the underlying set of the free A-module generated by S. Then the* *partially-ordered collection* $\mathrm{Ho}(\mathrm{Fin\,HPres}(T))$ *of natural equivalence classes of finitary homological presentations of T is equivalent to the set of functions*

$$\mathrm{Max\,Spec}(A) \to \{0, 1, 2, \ldots, \infty\}$$

from the set $\mathrm{Max\,Spec}(A)$ *of maximal ideals of A to the set of extended natural numbers, under the partial ordering in which we let* $f \leqslant g$ *if and only if* $f(\mathfrak{m}) \leqslant g(\mathfrak{m})$ *for all* $\mathfrak{m} \in \mathrm{Max\,Spec}(A)$.

Proof. Since every A-module is the colimit of its finitely generated sub-A-modules, and since the partially-ordered set of sub-A-modules of a given A-module is filtered, we know that every A-module is a filtered colimit of finitely generated A-modules. Hence a finitary element of $\mathrm{Loc}(\mathrm{Sets}^T) \cong \mathrm{Loc}(\mathrm{Mod}(A))$ is determined by which finitely generated A-modules it contains.

From here, the proof resembles that of Proposition 4.3: given a finitary element $\mathcal{A}$ of $\mathrm{Loc}(\mathrm{Sets}^T)$, since $\mathcal{A}$ is reflective, it is closed under limits computed in $\mathrm{Mod}(A)$. Consequently, if $A/\mathfrak{m}^n$ is in $\mathcal{A}$, then so is $A/\mathfrak{m}^i$ for all $i \leqslant n$. Of course A is also in $\mathcal{A}$. Hence we can specify which finitely generated A-modules are contained in $\mathcal{A}$, and consequently all of $\mathcal{A}$, by specifying (as in the proof of Proposition 4.3) a function $f_{\mathcal{A}} : \mathrm{Max\,Spec}(A) \to \{0, 1, 2, \ldots, \infty\}$ from the set of maximal ideals of A to the set of extended natural numbers; namely, $f_{\mathcal{A}}$ is the function sending a maximal ideal $\mathfrak{m}$ of A to the largest integer n such that $A/\mathfrak{m}^n$ is in $\mathcal{A}$, and by letting $f_{\mathcal{A}}(\mathfrak{m}) = \infty$ if $A/\mathfrak{m}^n$ is in $\mathcal{A}$ for all n.

Now we still need to know that, for each $f : \mathrm{Max\,Spec}(A) \to \{0, 1, 2, \ldots, \infty\}$, there does indeed exist a replete reflective subcategory $\mathcal{A}$ of $\mathrm{Mod}(A)$ such that $f_{\mathcal{A}} = f$. The argument is as follows: let $\mathcal{A}_f$ be the replete full subcategory of $\mathrm{Mod}(A)$ generated by the projective A-modules and all the A-modules M with the property that, for each maximal ideal $\mathfrak{m}$ of A, if $A/\mathfrak{m}^n \to M$ is a monomorphism of A-modules, then $n \leqslant f(\mathfrak{m})$. Clearly $f_{\mathcal{A}_f} = f$ and $\mathcal{A}_f$ is replete, full, and contains the projective A-modules. We need to know that $\mathcal{A}_f$ is reflective and finitary. Let $L : \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ be the functor given by letting $L(M)$ be the colimit $\mathrm{colim}_{X \in fg(M)}\, \nu_f(X)$, where $fg(M)$ is the (filtered) category of finitely generated sub-A-modules of M, and ν_f is the functor defined on $\mathrm{fgMod}(A)$ in the proof of Proposition 4.3. Let $\eta M : M \to L(M)$ be the natural map $M \cong \mathrm{colim}_{X \in fg(M)}\, X \to \mathrm{colim}_{X \in fg(M)}\, \nu_f(X)$ given by the natural transformation $\eta_f : \mathrm{id} \to \nu_f$. Then $L(M)$ is in $\mathcal{A}_f$, since $A/\mathfrak{m}^n$ is finitely generated and hence every monomorphism from $A/\mathfrak{m}^n$ to the filtered colimit $\mathrm{colim}_{X \in fg(M)}\, \nu_f(X)$ factors through a monomorphism $A/\mathfrak{m}^n \to \nu_f(X)$ for some $X \in fg(M)$. Furthermore, if T is an object of $\mathcal{A}_f$, then

$$\begin{aligned}
\hom_{\mathrm{Mod}(A)}(M, T) &\cong \lim_{X \in fg(M)} \hom_{\mathrm{Mod}(A)}(X, T) \\
&\cong \lim_{X \in fg(M)} \hom_{\mathrm{Mod}(A)}(\nu_f(X), T) \\
&\cong \hom_{\mathrm{Mod}(A)}(\mathrm{colim}_{X \in fg(M)}\, \nu_f(X), T) \\
&\cong \hom_{\mathrm{Mod}(A)}(L(M), T)
\end{aligned}$$

so L is indeed left adjoint to the inclusion $\mathcal{A}_f \hookrightarrow \mathrm{Mod}(A)$. (More carefully: $L = GF$, where G is the inclusion $\mathcal{A}_f \hookrightarrow \mathrm{Mod}(A)$, and F is L with its codomain restricted to $\mathcal{A}_f$.) So $\mathcal{A}_f$ is reflective. The functor $L = GF$ commutes with filtered colimits by construction, and F is a left adjoint and hence commutes with all colimits, and G is full and faithful and hence reflects colimits; so G commutes with filtered colimits, and hence $\mathcal{A}_f$ is finitary. □

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