New Symmetries of Stable Homotopy Groups

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Abstract

Work of Bousfield, Ravenel, Morava, Miller, Wilson, and others during the 1970s and 1980s established that the stable homotopy groups of spheres can be decomposed into many periodic families, each of which repeats every $2(p^n$ -1) dimensions, where p is a prime and n is a non-negative integer; and furthermore, if one fixes p and n, then there are spectral sequences that one can, in principle, use to calculate the $2(p^n$ -1) - periodic families in the stable homotopy groups of spheres. These spectral sequence calculations are extremely difficult, however, and complete calculations have only been made for $n < 3$. When $p > n + 1$, these spectral sequence calculations begin with the cohomology of a certain differential graded algebra defined by Ravenel. In this paper, we give a very explicit description of this surprisingly simple differential graded algebra, and we explore the problem of finding symmetries (i.e., automorphisms) of this differential graded algebra for large n. We show how we reduce the problem of finding such symmetries to a very explicit and elementary problem in number theory, and we demonstrate some new symmetries - which give rise to new operations on the $2(p^n$ 1)-periodic stable homotopy groups of certain CW-complexes called Smith-Toda complexes - which our elementary, explicit approach produces. We then attempt to prove a folklore conjecture by Hopkins regarding the cohomology of the "Novikov diagonal", the diagonal line with slope -1 passing through bidegree $(0, 0)$ in the $K(n)$-local E_n Adams spectral sequence for Smith Toda $V(n-1)$, using the same structures used to produce our automorphisms. We also mention possible future directions for research, including pursuing an original conjecture made by Salch about the interaction between a certain class of denominators of the Riemann zeta function and Euler's totient function. This is joint work with Prof. Andrew Salch (Wayne State University) and John Diehr (Wayne State University).

1 Background

We make the following definition first to motivate our work:

Definition 1.1. Given a commutative ring R, by the bigraded differential algebra over R, we mean a bigraded R- algebra A equipped with an R-linear map $d : A \longrightarrow A$ that preserves internal degree and increases cohomological degree by 1, and which satisfies the Leibniz rule $d(ab) = d(a)b + (-1)^{|a||b|}ad(b)$ for all elements $a, b \in A$ which are homogeneous with respect to cohomological degree, and where $|a|$ denotes the cohomological degree of a.

- A differential graded algebra is a ring with an added chain complex structure - that is, it's equipped with a sequence of homomorphisms of abelian groups $... \longleftarrow C_{-1} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow C_2 \longleftarrow ...$ such that $d_i \circ d_{i+1} = 0$ for all i (i.e., im $d_{i+1} \subseteq$ ker d_i, and where $H_i(C_i) := \frac{kerd_i}{imd_{i+1}}$). By graded algebra (or graded ring, here), we mean a ring R equipped with an isomorphism of abelian groups $R \cong ... \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus R_2...$ such that if $r \in R_i, r' \in R_j$, then $r \cdot r' \in R_{i+j}$. An element $r \in R_i$ is homogeneous of degree r. In our specific setting, we will be working with a cochain complex, where $d_i \circ d_{i-1} = 0$ for all i, as d increases cohomological degree by 1.

We begin with an exterior algebra on generators $h_{i,j}$ defined by Ravenel as $\{h_{i,j} : 1 \leq i \leq \lfloor \frac{pn}{p-1} \rfloor, 0 \leq j < n\}$ for each nonnegative integer n and prime number p, where $\lfloor \frac{pn}{p-1} \rfloor$ is the greatest integer less than or equal to $\frac{pn}{p-1}$. This is the beginning of Ravenel's construction of a sequence of spectral sequences used as a tool for computation of the cohomology of a Morava stabilizer group. (In particular, Ravenel's filtration on the Morava stabilizer algebras gives rise to a graded Lie algebra. One has such a graded Lie algebra for each prime p and nonnegative integer n, and these Lie algebra cohomology calculations are the input for a sequence of spectral sequences, which compute the group cohomology of a Morava stabilizer group. This group cohomology is, in turn, the input for the chromatic spectral sequence, whose output is the input for the Adams-Novikov spectral sequence, whose output is the stable homotopy groups of spheres. This is the computational approach to the stable homotopy groups of spheres which turns out to cleave more strongly to number theory and arithmetic geometry. One can find Ravenel's constructions and a careful introduction to this subject in [1].) Throughout this paper, we will be working with the Adams-Novikov spectral sequence.

For $p > n + 1$, we can make the following argument regarding the number $\lfloor \frac{pn}{p-1} \rfloor$:

$$n = \frac{(p-1)n}{p-1}$$

$$= \frac{pn}{p-1} - \frac{n}{p-1}$$

$$< \frac{pn}{p-1}$$

$$< \frac{pn}{p-1} + \frac{p-(n+1)}{p-1}$$

$$= \frac{pn-n+p-1}{p-1}$$

$$= \frac{(p-1)(n+1)}{p-1}$$

$$= n + 1$$

Then, the inequalities $n < \lfloor \frac{pn}{p-1} \rfloor < n + 1$ give us that $\lfloor \frac{pn}{p-1} \rfloor = n$. Thus, for $p > n + 1$, we make the following definition:

Definition 1.2. We define $\Lambda_n = \Lambda(h_{i,j} : 1 \leq i \leq n, 0 \leq j < n)$ as the differential graded algebra (and bigraded exterior algebra over $\mathbb{Z}$ on generators $h_{i,j}$) for each nonnegative integer n, where the second index in the notation $h_{i,j}$ is regarded as being defined modulo n (so that $h_{i,n} = h_{i,0}$).

- Each generator of Λ_n is assigned a cohomological degree s, internal degree t, and Adams degree = internal degree - cohomological degree = $t - s$. An element $h_{i,j}$ is given cohomological degree 1, internal degree $2p^j(p^i - 1)$, and the resulting Adams degree is $2p^j(p^i - 1) - 1$, while an element $h_{i,j}h_{k,l}$ is given cohomological degree 2 (a multiplication of n generators yields a cohomological degree of n), internal degree $2p^j(p^i - 1) + 2p^l(p^k - 1)$ (the internal degree of a product of generators is equal to the sum of the internal degrees of each generator), and the resulting Adams degree is $2p^j(p^i - 1) - 1 + 2p^l(p^k - 1) - 1$. Note that, when discussing the product of generators $h_{i,j}h_{k,l}$, for example, one can use the notation $h_{i,j} \wedge h_{k,l}$ since we're an exterior algebra setting where the exterior product, or wedge product, is defined.

- Λ_n is also equipped with the differential d such that

$$d(h_{i,j}) = \sum_{k=1}^{i-1} h_{k,j}h_{i-k,j+k} \tag{1.0.1}$$

  (and we can extend the differential from generators $\{h_{i,j}\}$ to all of Λ_n by use of the Leibniz rule and linearity).

Work of Bousfield, Ravenel, Morava, Miller, Wilson, et al. established that the stable homotopy groups of spheres can be decomposed into periodic families, each repeating every $2(p^n - 1)$ dimensions for a prime p and nonnegative integer n, and for fixed p and n, one can use the machinery of spectral sequences to calculate the $2(p^n - 1)$ periodic families in the stable homotopy groups of spheres. This periodicity gives rise to chromatic homotopy theory. These spectral sequence calculations have only been made for very low values of n ($n < 3$).

Due to Ravenel, Hopkins, Devinatz, Adams, and Novikov, if $p > n + 1$, there exists a spectral sequence which takes as input the E_2 term,

$$E_2^{s,t} \cong H^{s,t}(\Lambda(h_{i,j} : 1 \leq i \leq n, 0 \leq j < n)) \tag{1.0.2}$$

, the $(s,t)^{th}$ cohomology of our differential graded algebra Λ_n. By the $K(n)$ - local Adams Novikov spectral sequence,

$$E_2^{s,t} \Rightarrow \pi_{t-s}(L_{K(n)}V(n-1)) \tag{1.0.3}$$

, i.e. $E_2^{s,t}$ converges to $\pi_{t-s}(L_{K(n)}V(n-1))$, where π_{t-s} is the $(t-s)$ stable homotopy group, and $L_{K(n)}V(n-1)$ is the Bousfield localization of the nth Morava K-theory, with $V(n-1)$ the Smith-Toda complex. The Smith-Toda complex is a CW complex, a type of topological space introduced by Whitehead to generalize simplicial complexes to homotopy theory, where one has a Hausdorff space together with a partition of the space into certain open cells, where an n-dimensional open cell is a topological space homeomorphic to the n-dimensional open ball.

Definition 1.3. An element $E_r^{s,t}$ is said to have internal degree t, filtration s, and total degree $t-s$ (so that the set of Adams degrees of generators of the exterior algebra in Definition 1.2 equivalent to the set of total degrees of terms $E_r^{s,t}$ of the Adams spectral sequence).

- The Adams spectral sequence has differential d_r which maps left 1 and up r, so that $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$. [2]

Due to Ravenel, Miller, and Wilson, there exists a sequence of spectral sequences

$$\pi_* L_{K(n)} V(n-1) \Rightarrow ... \Rightarrow \pi_* L_{K(n)} S$$

,called the Bockstein spectral sequences, where $\pi_* L_{K(n)} S$ denotes the $K(n)$-local stable homotopy groups of spheres, and $L_{K(n)}$ denotes the Bousfield localization at $K(n)$. Then, there exists an infinite sequence of abelian groups that acts as the glue to stitch together these $K(n)$-local stable homotopy groups, giving the Bousfield localization of the nth Johnson-Wilson theory. Finally, due to Ravenel and Hopkins, the chromatic convergence theorem gives

$$\pi_* S \cong \lim_n (\pi_* L_{E(n)} S) \tag{1.0.4}$$

That is, the stable homotopy groups of spheres is just the limit over n of the stable homotopy groups of the Johnson-Wilson localizations,i.e., the limit over all n of the $K(n)-local$ Adams-Novikov spectral sequence.

We can discuss the results of complete spectral sequence calculations that have been made for $n < 3$. For $n = 0$, Serre's computations showed that

$$(\pi_i S) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i = 0 \\ 0 & i \neq 0 \end{cases} \cong \pi_i(L_{E(0)} S)$$

,where the last isomorphism is due to Sullivan, Bousfield and Quillen. This gives rational stable homotopy theory. When $n = 1$, $\pi_* L_{E(1)} S$ is essentially the same as "im j", where j is the whitehead j-homomorphism. In fact, by Adams, the number of $\pi_{2n-1}(imj)$ is equal to the denominators of $\zeta(-i)$ (up to sign and a power of 2), where ζ denotes the Riemann-zeta function (in other words, for a positive integer n, $\zeta(1-2n)$ is equal to the number of $\pi_{4n-1}(L_{K(n)} S)$). In fact,

$$\zeta(1-2n) = \frac{\# K_{4n-2}(\mathbb{Z})}{\# K_{4n-1}(\mathbb{Z})}$$

up to sign and a power of 2 , where K denotes a K-group, giving a result in algebraic K-theory. The significance of this last result yields the study of the denominators of the Riemann-zeta function at special values $\zeta(1-2n)$ for a positive integer n, which we mention later in this paper as a future direction for research. For $n = 2$, $\pi_* L_{E(2)} S$ is very close to the ring of topological modular forms $\pi_* tmf$.

Now, we return to our differential graded algebra Λ_n and perform sample computations for $n = 2$. The dimension of Λ_n as an $\mathbb{F}_p$ vector space, $\dim_{\mathbb{F}_p} \Lambda_n$, is equal to 2^{n^2}, while by explicit computation, we can compute the dimension of the cohomology of Λ_n as an $\mathbb{F}_p$ vector space, $\dim_{\mathbb{F}_p} H^*(\Lambda_n)$, for $p > n+1$. A short table is provided below containing these figures for the first few values of n.

n	$\dim_{\mathbb{F}_p}\Lambda_n$	$\dim_{\mathbb{F}_p}H^*(\Lambda_n), p > n+1$
1	2	2
2	16	12
3	512	152
4	65536	3440

Table 1.4: $\mathbb{F}_p$ vector space dimensions of Λ_n and its cohomology

For $n = 2$ and $p > n + 1 = 3$, we compute the $K(2)$-local Adams Novikov spectral sequence. $\Lambda_2 = \Lambda(h_{10}, h_{11}, h_{20}, h_{21})$, so we compute the differential d on elements of Λ_n in the table below. In the below computations, we also make use of the Leibniz rule, and keep in mind that, for reasons of grading, when dealing with a product of two or more generators, swapping two neighboring generators yields a negative sign (and, because of this, the computations below are only done on one possible permutation of a product of generators).

element in Λ_n	differential d on element
h_{10}	$d(h_{10}) = 0$
h_{11}	$d(h_{11}) = 0$
h_{20}	$d(h_{20}) = h_{10}h_{11}$
h_{21}	$d(h_{21}) = h_{11}h_{12} = h_{11}h_{10} = -h_{10}h_{11}$
$h_{10}h_{11}$	$d(h_{10}h_{11}) = 0$
$h_{10}h_{20}$	$d(h_{10})h_{20} - h_{10}d(h_{20}) = 0$
$h_{10}h_{21}$	$d(h_{10})h_{21} - h_{10}d(h_{21}) = 0$
$h_{11}h_{20}$	$d(h_{11})h_{20} - h_{11}d(h_{20}) = 0$
$h_{11}h_{21}$	$d(h_{11})h_{21} - h_{11}d(h_{21}) = 0$
$h_{20}h_{21}$	$h_{10}h_{11}h_{21} - h_{20}h_{11}h_{10} = h_{10}h_{11}h_{21} + h_{10}h_{11}h_{20}$
$h_{10}h_{11}h_{20}$	$d(h_{10}h_{11})h_{20} + h_{10}h_{11}d(h_{20}) = 0$
$h_{10}h_{11}h_{21}$	$d(h_{10})h_{11}h_{21} + h_{10}h_{11}d(h_{21}) = 0$
$h_{10}h_{20}h_{21}$	$d(h_{10})h_{20}h_{21} + h_{10}h_{20}d(h_{21}) = 0$
$h_{11}h_{20}h_{21}$	$d(h_{11})h_{20}h_{21} + h_{11}h_{20}d(h_{21}) = 0$
$h_{10}h_{11}h_{20}h_{21}$	$d(h_{10})h_{11}h_{20}h_{21} - h_{10}h_{11}h_{20}d(h_{21}) = 0$

Table 1.5: Computation of the differential on elements in the Λ_2 algebra

We also keep in mind here that d on the sum of two generators is simply equal to the sum of d on each individual generator. Then, we can compute the cohomology of Λ_2 as an $\mathbb{F}_p$ vector space in the usual way, kernel of d modulo the image of d, so that

$$H^*(\Lambda_2) = \ker(d)/\mathrm{im}(d)$$
$$= \mathbb{F}_p\{1, h_{10}, h_{11}, \zeta_2 = h_{20} + h_{21}, h_{10}h_{20}, h_{11}h_{20}, h_{10}h_{21}, h_{11}h_{21}, h_{10}h_{11}h_{20} \sim -h_{10}h_{11}h_{21},$$
$$\mathrm{h}_{10}h_{20}h_{21}, h_{11}h_{20}h_{21}, h_{10}h_{11}h_{20}h_{21}\}$$

(Note that, since we'd like to make computations for the $K(2)$-local Adams-Novikov spectral sequence for $V(1)$, we compute the cohomology of Λ_n where $n = 2$). Now, for each element in $H^*(\Lambda_2)$, we assign a cohomological degree s, internal degree t, and Adams degree $t - s$, where degrees are defined modulo $2(p^2 - 1)$. A table below shows these degrees.

element in $H^*(\Lambda_2)$	cohomological degree	internal degree	Adams degree
1	0	0	0
h_{10}	1	2(p-1)	2p-3
h_{11}	1	2p(p-1)	$2p^2 - 2p - 1$
$\zeta_2 = h_{20} + h_{21}$	1	0	-1
$h_{10}h_{20}$	2	2(p-1)	2p-4
$h_{11}h_{20}$	2	2p(p-1)	$2p^2 - 2p - 2$
$h_{10}h_{21}$	2	2(p-1)	2p-4
$h_{11}h_{21}$	2	2p(p-1)	$2p^2 - 2p - 2$
$h_{10}h_{11}h_{20} \sim -h_{10}h_{11}h_{21}$	3	2(p-1) + 2p(p-1)	$2p^2 - 5$
$h_{10}h_{20}h_{21}$	3	2(p-1)	2p-5
$h_{11}h_{20}h_{21}$	3	2p(p-1)	$2p^2 - 2p - 3$
$h_{10}h_{11}h_{20}h_{21}$	4	2(p-1) + 2p(p-1)	$2p^2 - 6$

Table 1.6: Degrees of elements in the cohomology of Λ_2

Note 1.7. It is traditional to display the Adams spectral sequence with total degree (Adams degree) $t - s$ increasing from left to right and the filtration (cohomological degree) s increasing vertically. Here, we take the chromatic approach, so that the decomposition of stable homotopy groups of spheres into $2(p^n - 1)$ -periodic families yields Adams degrees congruent to $2(p^n - 1)$; and, as a result, the entire Adams spectral sequence plot under the chromatic approach repeats every $2(p^n - 1)$ Adams degree (the non-chromatic approach gives no upper bound on i or j in the Λ_n DGA, giving a differential graded algebra called the May E_1 term, giving the classical Adams spectral sequence rather than the Adams-Novikov spectral sequence).

The cocycles given in the $\mathbb{F}_p$ vector space cohomology of Λ_n (and, in this particular example, Λ_2) give candidates for elements of the exterior algebra that survive the spectral sequence calculations. Then, we consider the differential d_r (in the current example, d_2) defined earlier: recalling that the d_r differential maps left 1 and up r, in the Adams spectral sequence plot, if applying the d_r operator to a nonzero cocycle α does not yield another nonzero cocycle, then α survives the spectral sequence calculations. If it does, α does not survive.

In fact, for $p > \frac{n^2+1}{2}$, the columns in the Adams-Novikov spectral sequence plot representing the various Adams degrees are sufficiently far apart, so the differentials d_r for any value of r on a nonzero cocycle will never yield another nonzero cocycle, giving that every

element survives the spectral sequence calculations. Thus, for $p > \frac{n^2+1}{2}$, the E_2 page of the Adams-Novikov spectral sequence is isomorphic to the E_r page of the spectral sequence. (One calls the E_r term the E_r page of the Adams-Novikov spectral sequence, since the input for a page of the Adams spectral sequence is the result of the taking the cohomology on the previous page in the spectral sequence, and so by taking cohomology, one is "turning the page").

We continue with our example for $n = 2$, let $p = 7$ (which, note, satisfies $p > n + 1$). Then, creating our plot for the E_2 term of the $K(2)$-local Adams-Novikov spectral sequence (the E_2 term is traditionally called the E_2 page of the Adams-Novikov spectral sequence), applying the d_2 operator to any cocycle does not yield another nonzero cocycle for reasons of bidegree - thus, every element survives on the E_2 page for the $K(2)$-local Adams-Novikov spectral sequence for Smith-Toda $V(1)$ by the d_2 differential. Since $p > \frac{n^2+1}{2}$, as well, we can conclude that this also happens on any page of the Adams-Novikov spectral sequence.

Calculations for n=3 of the cohomology of the differential graded algebra Λ_n have been made by Ravenel for n and $p > n+1$ in [1], and for $n = 4$, these calculations have been made by Salch (although, never printed). Notice that, in the above considerations, we fixed our value of p such that $p > n + 1$ - one can leave p as a variable and ask, does the d_r operator on a cocycle α yield another nonzero cocycle, giving that α does not survive the $K(n)$-local Adams-Novikov spectral sequence for $V(n-1)$?

Note that, regardless of the values of n and p such that $p > n+1$, the cohomology of Λ_n as an $\mathbb{F}_p$ vector space $H^*(\Lambda_n)$ always contains the element 1 - and this element has bidegree given by cohomological degree 0 and Adams degree 0 (giving an element in (0,0)-bidegree in the Adams spectral sequence plot). This yields the following question:

Question 1.8. For any value of n and $p > n + 1$, does the element 1 survive the $K(n)$-local Adams-Novikov spectral sequence for Smith-Toda $V(n-1)$ (on any page of the Adams spectral sequence) ? That is, fixing n and choosing $p > n + 1$, is there no nonzero cocycle in $H^*(\Lambda_n)$ such that $d_r(1)$ could be that element?

For reasons of bidegree, if the answer to Question 1.8 is yes, then the cohomological degree of that element must be greater than or equal to 2, and the Adams degree must be congruent to -1 modulo $2(p^n - 1)$. That element is a sum of products of generators $h_{i,j}$, where each $h_{i,j}$ gives an element $2(p^i - 1)p^j - 1$ corresponding to its Adams degree. The set of $h_{i,j}$'s in that product gives a subset of the set of Adams degrees of generators of Λ_n $\{2(p^i-1)p^j-1 : 1 \leq i \leq n, 0 \leq j < n\}$ with at least two elements and whose sum is congruent to -1 modulo $2(p^n - 1)$. This conclusion regarding degrees will allow us to formulate our question in a number theoretic setting in the next section, using modular arithmetic to give results.

If the answer to Question 1.8 is no, then we know that the first family of obstructions to the existence of the next Smith-Toda complex $V(n)$ is automatically zero. This first family of

obstructions is the family of $E(n)$ - Adams differentials on an element v_n in an Ext functor involving Smith Toda $V(n-1)$.This is because if v_n survives the $E(n)$ - Adams spectral sequence, then it represents a class in $\pi_{2(p^n-1)}(L_{E(n)}V(n-1))$, which is a homotopy class f of continuous maps $S^{2(p^n-1)} \longrightarrow L_{E(n)}V(n-1)$. That is, if v_n survives the spectral sequence, it gives us the attaching map for the next Smith-Toda complex $V(n)$ to exist.

2 A Number Theoretic Formulation

Let $n \in \mathbb{N}$ and p a prime such that $p > n+1$. Recall that if there is an element in $H^*(\Lambda_n)$ such that $d_r(1)$ is that element, where $1 \in H^*(\Lambda_n)$ is in bidegree $(0,0)$ in the Adams-Novikov spectral sequence plot, then that element has bidegree $(s, t-s)$ such that $s \geq 2$ and $t-s$ is congruent to -1 modulo $2(p^n-1)$ (these elements make up a vertical line in the Adams spectral sequence plot with Adams degree -1, a vertical line which repeats every $2(p^n-1)$ dimensions on a given page of the Adams spectral sequence); that is, the element is a sum of products of generators $h_{i,j}$, where a generator $h_{i,j}$ is in bidegree $(1, 2p^j(p^i-1)-1)$. (If there's nothing the element 1 can hit by the differential, then there's nothing that v_n can hit by that same differential). Thus, the answer as to whether or not such an element exists is equivalent to answering the following question:

Question 2.1. For each prime number p and positive integer n such that $p > n+1$, does there exist a subset Y of $X_{p,n} = \{2(p^i-1)p^j - 1 : 1 \leq i \leq n, 0 \leq j < n\}$ containing at least two elements such that the sum of numbers in Y is congruent to -1 modulo $2(p^n-1)$? For what values of n and p does $X_{p,n}$ contain such a subset? Is there an infinite family of sets $X_{p,n}$ such that there is no such subset?

Proposition 2.2. $X_{p,n}$ does not contain such a subset Y if Y has fewer than $2p-1$ elements.

Proof. Suppose that $X_{p,n}$ contains a subset Y such that the sum of elements in Y is congruent to -1 modulo $2(p^n-1) = 2(p-1)(1+p+p^2+...+p^{n-1})$.

1) Since the sum of elements in Y is congruent to -1 modulo 2, and each individual element in $X_{p,n}$ is congruent to -1 modulo 2, the number of elements in Y must be congruent to 1 modulo 2.

2) Furthermore, since each element of $X_{p,n}$ is congruent to -1 modulo $p-1$, in order for the sum of elements in Y to be congruent to -1 modulo $p-1$ as desired, we need that the number of elements in Y is congruent to 1 modulo $p-1$.

Therefore, with 1) and 2) together, if $X_{p,n}$ contains the desired subset Y, then the number of elements in Y must be congruent to 1 modulo $2(p-1)$. Thus, $X_{p,n}$ does not contain such a subset Y if Y has fewer than $2p-1$ elements (i.e., if $n^2 < 2p-1$, i.e., if $p > \frac{n^2+1}{2}$). $\square$

Due to Proposition 2.2, since summing up $2k(p-1)+1$ elements in $X_{p,n}$, where k is a positive integer, yields a sum congruent to -1 modulo $2(p-1)$, and $2(p^n-1) = 2(p-1)(1+p+p^2+...+p^{n-1})$, in investigating if there is an infinite family of sets $X_{p,n}$ such that there is no such subset Y of $X_{p,n}$, we consider all sums of subsets with $2k(p-1)+1$ elements modulo $(1+p+p^2+...+p^{n-1})$.

Proposition 2.3. Suppose that $X_{p,n}$ is built from $n \in \mathbb{N}$ and prime p such that $p > n+1$, $n^2 = 2k(p-1)+1$, and our subset W of $X_{p,n}$ contains all n^2 elements. Then W gives a subset of $X_{p,n}$ containing $2k(p-1)+1$ elements and with sum of elements in W not congruent to -1 modulo $(1+p+p^2+...+p^{n-1})$.

Proof. The sum of elements in W is equal to

$$\sum_{i=1}^{n}\sum_{j=0}^{n-1} 2(p^i-1)p^j - 1 \equiv -n^2 \text{ modulo } (1+p+p^2+...+p^{n-1})$$

Suppose this sum is congruent to -1 modulo $(1+p+p^2+...+p^{n-1})$,giving us a subset Y described in Question 2.1. Then

$$n^2-1 \equiv 0 \text{ modulo } (1+p+p^2+...+p^{n-1})$$
$$\Rightarrow \tfrac{2k(p-1)}{(1+p+p^2+...+p^{n-1})} \text{ is an integer.}$$

But the denominator of this fraction is greater than the numerator: letting $A = 2k(p-1)+1$, we have

$$A < \tfrac{p^n-1}{p-1} + 1 \Rightarrow p^{\sqrt{A}} + p > 2 + (p-1)A$$

, which is a true inequality since $p > 2$ and $p^{\sqrt{A}} > (p-1)A$ for $1 < n < p-1$. Thus, this gives a contradiction since $\frac{2k(p-1)}{(1+p+p^2+...+p^{n-1})}$ is not an integer $\Rightarrow$ subsets of $X_{p,n}$ containing all n^2 elements of $X_{p,n}$ never satisfy the conditions for a subset Y described in Question 2.1, giving an infinite family of sets $X_{p,n}$ such that there is no such subset Y. □

Proposition 2.4. Denote $n^2-(2p-1)$ by g, where p is a prime and $n \in \mathbb{N}$ such that $p > n+1$ (i.e. $2n+1 < 2p-1$). If $n \geq g-1$ such that $2n+1 < 2p-1 < n^2$, there does exist such a subset Y of $X_{p,n}$ as described in Question 2.1.

Proof. Suppose that we sum up $2p-1$ elements in $X_{p,n}$ so that $2p-1 < n^2$ and $p > n+1$. If all elements in $X_{p,n}$ are summed up, we found earlier that the sum is congruent to $-n^2$ modulo $(1+p+p^2+...+p^{n-1}) \Rightarrow$ the sum is congruent to $-(2p-1+g)$ modulo $(1+p+p^2+...+p^{n-1})$. Subtract out the element in $X_{p,n}$ with $i = n-1, j = 1$ (which is congruent to $-2p+1$ modulo $(1+p+p^2+...+p^{n-1})$) from this sum. Then, our sum is now congruent to $-g$ modulo $(1+p+p^2+...+p^{n-1})$. If $n \geq g-1$, then since there are n elements in $X_{p,n}$ congruent to -1 modulo $(1+p+p^2+...+p^{n-1})$ (all elements with $i = n$), we can achieve a sum congruent to -1 modulo $(1+p+p^2+...+p^{n-1})$ by subtracting out elements in $X_{p,n}$ with $i = n$ until we achieve such a sum. Therefore, if $n \geq g-1$ for a prime p and $n \in \mathbb{N}$ such that $2n+1 < 2p-1 < n^2$, there does exist such a subset Y of $X_{p,n}$, giving an infinite family of sets $X_{p,n}$ yielding this subset. □

We next investigate the following question, which ultimately leads us to our three new symmetries of stable homotopy groups.

Question 2.5. For a given prime p and nonnegative integer n such that $1 < n < p - 1$, how many m-element subsets congruent to $-m$ modulo $(1 + p + p^2 + ... + p^{n-1})$ are there in $X_{p,n}$?

We can use the concept of a combinatorial radon transform found in the literature to help answer Question 2.5. Since Question 2.5 is very combinatoric in nature, we use the coding language SageMath, an open source mathematical software, to assist us. SageMath is especially useful for number theoretic and combinatorial questions. The source code below in Figure 2.6 was run using the Wayne State University Grid in SageMath (specifically, using Putty as an SSH client and running an interactive session). It aims to count the number of subsets described in Question 2.5 and attempt to see a pattern in these counts for certain values of n and p such that $p > n + 1$.

Since the computations here can be very large, the code employs a variable easiness_cutoff, which lets you skip the combinatorial Radon transforms which have more elements than the number easiness_cutoff. Also, in the output of the code, you'll notice that, for a given value of p and n, the number of m-element subsets with sum of elements congruent to $-m$ modulo $\frac{p^n-1}{p-1}$ in $X_{p,n}$ is equal to the number of (n^2-m)-element subsets with sum of elements congruent to $-(n^2-m)$ modulo $\frac{p^n-1}{p-1}$ in $X_{p,n}$. This is due to symmetry about $m = \frac{n^2}{2}$. There are certainly variations on the code below to employ different easiness_cutoffs. We can also employ a short code in SageMath to output the $n \times n$ matrix of elements in $X_{p,n}$ for a given value of p and n, where all elements are modulo $\frac{p^n-1}{p-1}$, and where i increases in the output as one moves right and j increases as one moves down. An example of such a code is shown below, which outputs the matrix of elements in $X_{7,3}$ modulo $\frac{p^n-1}{p-1}$:

```
p = 7
n = 3
m = matrix(QQ,n,n,lambda j,i: 2*(p^(i+1)-1)*p^j-1) % ((p^n-1)/(p-1))
```

Our discovery of new symmetries on $X_{p,n}$, which we discuss more closely in section 3, came about by studying these matrices of low values of n. It's certainly worth it to ask about other possible symmetries that arise in these matrices for higher values of n, which is why this code can be useful for those matrices for which it becomes tedious to compute the specific elements by hand.

```
# This variant counts subsets which sum to RESIDUE, not to -1.

# gapsize = 4
starting_n_value = 6
ending_n_value = 6
easiness_cutoff = 10000000
# RESIDUE = 0

for n in range(starting_n_value,1 + ending_n_value):
      for p in range(n+2,floor((n^2+1)/2)+1):
           for RESIDUE in range(-2*p+1,2*p-2):
         #  for p in range(11,12):
         #   for p in range(3,floor((n^2+1)/2)+1):
              if is_prime(p):
                  list1 = []

                  for j in range(0,n):
                      for i in range(1,n+1):
                          list1.append(2*(p^i-1)*p^j - 1)

                  # mset= set(list1)
                  for l in range(ceil(RESIDUE/(2*p-2)),floor((n^2 + RESIDUE)/(2*p-2))+1):
                       # if (n^2 - (l*(2*p - 2) + 1)) == gapsize:
                       if binomial(n^2, 2*(p-1)*l - RESIDUE) < easiness_cutoff:
                          subs = Subsets(list1,(2*p-2)*l - RESIDUE, submultiset = True)
                          radon = map(sum,subs)

                          #Here's what the various data structures are, at this point:
                          # list is an ordered list of the integers in X_{p,n},
                          # set is the same as list, but forgetting the ordering, i.e., set = X_{p,n},
                          # subs is the set of c-element subsets of set,
                          # radon is the set of sums of c-element subsets of set.

                          bigresidues = []
                          for m in range(0,len(radon)):
                              bigresidues.append(radon[m] % (2*p^n - 2))

                          # bigresidues is the set of residues of the elements of radon, modulo 2p^n - 2.
                          # print radon
                          # print bigresidues
                          totalcount = bigresidues.count(RESIDUE % (2*p^n - 2))
                          if totalcount == 0:
                              print "(p,n,c) = ",p,",",n,",",l,"*(2p-2) -",RESIDUE, totalcount
                          else:
                              print "(p,n,c) = ",p,",",n,",",l,"*(2p-2) -",RESIDUE, totalcount, "=",factor(totalcount)
                       else:
                          print "(p,n,c) = ",p,",",n,",",l,"*(2p-2) -",RESIDUE,"was skipped
```

Figure 2.6: SageMath code to find specific types of subsets of $X_{p,n}$

Shown below is some sample output from the above code for low values of n.

```
(p,n,c) =  5 , 3 , -1 *(2p-2) - -9 1 = 1
(p,n,c) =  5 , 3 , 0 *(2p-2) - -9 1 = 1
(p,n,c) =  5 , 3 , -1 *(2p-2) - -8 0
(p,n,c) =  5 , 3 , 0 *(2p-2) - -8 3 = 3
(p,n,c) =  5 , 3 , 0 *(2p-2) - -7 6 = 2 * 3
(p,n,c) =  5 , 3 , 0 *(2p-2) - -6 12 = 2^2 * 3
(p,n,c) =  5 , 3 , 0 *(2p-2) - -5 18 = 2 * 3^2
(p,n,c) =  5 , 3 , 0 *(2p-2) - -4 18 = 2 * 3^2
(p,n,c) =  5 , 3 , 0 *(2p-2) - -3 12 = 2^2 * 3
(p,n,c) =  5 , 3 , 0 *(2p-2) - -2 6 = 2 * 3
(p,n,c) =  5 , 3 , 0 *(2p-2) - -1 3 = 3
(p,n,c) =  5 , 3 , 1 *(2p-2) - -1 0
(p,n,c) =  5 , 3 , 0 *(2p-2) - 0 1 = 1
(p,n,c) =  5 , 3 , 1 *(2p-2) - 0 1 = 1
(p,n,c) =  5 , 3 , 1 *(2p-2) - 1 4 = 2^2
(p,n,c) =  5 , 3 , 1 *(2p-2) - 2 8 = 2^3
(p,n,c) =  5 , 3 , 1 *(2p-2) - 3 11 = 11
(p,n,c) =  5 , 3 , 1 *(2p-2) - 4 11 = 11
(p,n,c) =  5 , 3 , 1 *(2p-2) - 5 8 = 2^3
(p,n,c) =  5 , 3 , 1 *(2p-2) - 6 4 = 2^2
(p,n,c) =  5 , 3 , 1 *(2p-2) - 7 1 = 1
(p,n,c) =  5 , 3 , 2 *(2p-2) - 7 0
(p,n,c) =  7 , 4 , -1 *(2p-2) - -13 1 = 1
(p,n,c) =  7 , 4 , 0 *(2p-2) - -13 36 = 2^2 * 3^2
(p,n,c) =  7 , 4 , -1 *(2p-2) - -12 0
(p,n,c) =  7 , 4 , 0 *(2p-2) - -12 90 = 2 * 3^2 * 5
(p,n,c) =  7 , 4 , 0 *(2p-2) - -11 180 = 2^2 * 3^2 * 5
(p,n,c) =  7 , 4 , 0 *(2p-2) - -10 292 = 2^2 * 73
(p,n,c) =  7 , 4 , 0 *(2p-2) - -9 388 = 2^2 * 97
(p,n,c) =  7 , 4 , 0 *(2p-2) - -8 426 = 2 * 3 * 71
(p,n,c) =  7 , 4 , 0 *(2p-2) - -7 388 = 2^2 * 97
(p,n,c) =  7 , 4 , 0 *(2p-2) - -6 292 = 2^2 * 73
(p,n,c) =  7 , 4 , 0 *(2p-2) - -5 180 = 2^2 * 3^2 * 5
(p,n,c) =  7 , 4 , 0 *(2p-2) - -4 90 = 2 * 3^2 * 5
(p,n,c) =  7 , 4 , 1 *(2p-2) - -4 0
(p,n,c) =  7 , 4 , 0 *(2p-2) - -3 36 = 2^2 * 3^2
(p,n,c) =  7 , 4 , 1 *(2p-2) - -3 1 = 1
(p,n,c) =  7 , 4 , 0 *(2p-2) - -2 12 = 2^2 * 3
(p,n,c) =  7 , 4 , 1 *(2p-2) - -2 6 = 2 * 3
(p,n,c) =  7 , 4 , 0 *(2p-2) - -1 4 = 2^2
(p,n,c) =  7 , 4 , 1 *(2p-2) - -1 21 = 3 * 7
(p,n,c) =  7 , 4 , 0 *(2p-2) - 0 1 = 1
(p,n,c) =  7 , 4 , 1 *(2p-2) - 0 56 = 2^3 * 7
(p,n,c) =  7 , 4 , 1 *(2p-2) - 1 118 = 2 * 59
(p,n,c) =  7 , 4 , 1 *(2p-2) - 2 198 = 2 * 3^2 * 11
(p,n,c) =  7 , 4 , 1 *(2p-2) - 3 268 = 2^2 * 67
(p,n,c) =  7 , 4 , 1 *(2p-2) - 4 296 = 2^3 * 37
(p,n,c) =  7 , 4 , 1 *(2p-2) - 5 268 = 2^2 * 67
(p,n,c) =  7 , 4 , 1 *(2p-2) - 6 198 = 2 * 3^2 * 11
(p,n,c) =  7 , 4 , 1 *(2p-2) - 7 118 = 2 * 59
(p,n,c) =  7 , 4 , 1 *(2p-2) - 8 56 = 2^3 * 7
(p,n,c) =  7 , 4 , 2 *(2p-2) - 8 0
(p,n,c) =  7 , 4 , 1 *(2p-2) - 9 21 = 3 * 7
(p,n,c) =  7 , 4 , 2 *(2p-2) - 9 0
(p,n,c) =  7 , 4 , 1 *(2p-2) - 10 6 = 2 * 3
(p,n,c) =  7 , 4 , 2 *(2p-2) - 10 0
(p,n,c) =  7 , 4 , 1 *(2p-2) - 11 1 = 1
(p,n,c) =  7 , 4 , 2 *(2p-2) - 11 2 = 2
(p,n,c) =  7 , 5 , -1 *(2p-2) - -13 1 = 1
(p,n,c) =  7 , 5 , 0 *(2p-2) - -13 35160 = 2^3 * 3 * 5 * 293
(p,n,c) =  7 , 5 , 1 *(2p-2) - -13 0
(p,n,c) =  7 , 5 , -1 *(2p-2) - -12 0
(p,n,c) =  7 , 5 , 0 *(2p-2) - -12 35160 = 2^3 * 3 * 5 * 293
(p,n,c) =  7 , 5 , 1 *(2p-2) - -12 1 = 1
(p,n,c) =  7 , 5 , 0 *(2p-2) - -11 30860 = 2^2 * 5 * 1543
(p,n,c) =  7 , 5 , 1 *(2p-2) - -11 8 = 2^3
(p,n,c) =  7 , 5 , 0 *(2p-2) - -10 23800 = 2^3 * 5^2 * 7 * 17
(p,n,c) =  7 , 5 , 1 *(2p-2) - -10 40 = 2^3 * 5
```

Figure 2.7: Sample output of SageMath code to find specific subsets of $X_{p,n}$

The output in Figure 2.7 is written such that (p, n, c) is equal to the value of p, followed by the value of n, then $l*(2p-2)$ minus the residue $-m$ modulo $(1+p+p^2+...+p^{n-1})$ (altogether giving the number of elements in our subset of $X_{p,n}$ whose elements we are summing up), and finally, the number of subsets congruent to this residue modulo $(1+p+p^2+...+p^{n-1})$. For example, for the row that reads $(p, n, c) = 5, 3, 0 * (2p-2) - -9\ 1 = 1$, we interpret this as saying that there is only 1 subset of $X_{5,3}$ containing 9 elements, and such that the sum of elements in this subset is congruent to -1 modulo $(1+p+p^2+...+p^{n-1})$ (we saw

earlier that this subset containing every element of $X_{5,3}$ has the property that the sum of elements in this subset is congruent to $-n^2$ modulo $(1+p+p^2+...+p^{n-1}) = -9$ modulo $(1+p+p^2+...+p^{n-1}))$. Also, for the row that reads $(p,n,c) = 5,3,0*(2p-2)--3\ 12 = 12$, we interpret this as saying that there are 12 subsets of $X_{5,3}$ containing 3 elements, and such that the sum of elements in this subset is congruent to -3 modulo $(1+p+p^2+...+p^{n-1})$ (you can explicitly write out all of the elements in $X_{5,3}$ and each of their residues modulo $(1+p+p^2+...+p^{n-1})$ to see this).

We notice that the code suggests that, for fixed n and choice of p such that $p > n+1$, the number of m-element subsets of $X_{p,n}$ congruent to $-m$ modulo $(1+p+p^2+...+p^{n-1})$ is independent of the value of p; that is, the number of m-element subsets congruent to $-m$ modulo $(1+p+p^2+...+p^{n-1})$ appears equal for the sets $X_{p,n}$ and $X_{p',n}$, $p \neq p'$. We can explicitly see this for low values of n by seeing precisely what subsets these are; in fact, in order to see what elements $a_{i,j}$ of $X_{p,n}$ make up these subsets, recall that this amounts to computing the Adams degrees of corresponding generators $h_{i,j}$ of the bigraded exterior differential algebra Λ_n and products of generators (whose Adams degree is given by the sums of degrees of individual generators) modulo $2(p^n-1)$. From now, we make use of the following definition:

Definition 2.8. The critical complex of $X_{p,n}$ consists of the m-element subsets of $X_{p,n}$ with the property that the sum of elements in the subset is congruent to $-m$ modulo $(1+p+p^2+...+p^{n-1})$.

- In the exterior differential graded algebra Λ_n setting, the critical complex of $X_{p,n}$ would be precisely those generators and product of generators with cohomological degree m and whose Adams degree is congruent to $-m$ modulo $2(p^n-1)$
- This gives a diagonal on the Adams spectral sequence plot, the diagonal with bidegree $(m,-m)$ modulo $2(p^n-1)$.

In the next section, we construct three new symmetries on this diagonal and explore their properties.

3 New Symmetries on the Critical Complex of $X_{p,n}$

Recall that there are n^2 elements in $X_{p,n} = \{2p^j(p^i-1)-1 | 1 \leq i \leq n, 0 \leq j < n\}$, where $1 < n+1 < p$. Thus, we begin by constructing an $n \times n$ matrix of the elements in $X_{p,n}$, where i increases as the column number increases and j increases as the column number increases. Again, we can identify m-element subsets within the matrix with sum of elements congruent to $-m$ modulo $(1+p+p^2+...+p^{n-1})$. We will write $a_{i,j}$ as shorthand for $2(p^i-1)p^j-1$, where the second subscript of an element is defined modulo n. We also make use of the definition below:

Definition 3.1. Given a positive integer m, let $S^m_{p,n} = \{S \subseteq X_{p,n} : \#(S) = m, \sum_{s\in S} \equiv -m \mod \frac{p^n-1}{p-1}\}$ - $S^m_{p,n}$ is exactly the m-element subsets of $X_{p,n}$ with sum of elements congruent to $-m$ modulo $(1 + p + p^2 + ... + p^{n-1})$.

We arrive at three new automorphisms of $X_{p,n}$, and give a geometric definition (in the matrix setting) and algebraic definition of each below.

Definition 3.2.

- Let σ be the automorphism of $X_{p,n}$ given by $\sigma(a_{i,j}) = a_{i,j+1}$; σ is the operator that shifts an element down one row.
- Let τ be the automorphism of $X_{p,n}$ given by letting $\tau(a_{i,j}) = a_{i,j}$ if $i \neq n$ and by letting $\tau(a_{n,j}) = a_{n,j+1}$; τ is the operator that shifts an element in the righthand column down one row.
- Let β be the automorphism of $X_{p,n}$ given by letting $\beta(a_{i,j}) = a_{n-i,n-j-1}$; β is the operator that reflects an element about the first $n-1$ columns, and reflects the element about the n rows.
- For a subset S of $X_{p,n}$, we denote σS by the subset $\{\sigma x : x \in S\}$ of $X_{p,n}$, τS by the subset $\{\tau x : x \in S\}$ of $X_{p,n}$, and βS by the subset $\{\beta x : x \in S\}$ of $X_{p,n}$.

We now explore properties of our three automorphisms σ, τ and β of $X_{p,n}$.

Claim 3.3. σ has order n, τ has order n, and β has order 2. Also, σ and τ commute, $\sigma\beta = \beta\sigma^{-1}$, and $\tau\beta = \beta\tau^{-1}$.

Proof. Geometrically, in the context of our $n \times n$ matrix, moving an element $a_{i,j}$ down one row to $a_{i,j+1}$ and applying this n times, where the second subscript is always defined modulo n, gets us back to our original element. Algebraically, $\sigma^n(a_{i,j}) = a_{i,j}$, and similarly for τ. Thus, σ and τ both have order n.

Algebraically, since $\beta(\beta(a_{i,j})) = \beta(a_{n-i,n-j-1}) = a_{i,j}$, we see that β has order 2, since applying β to an element 2 times gets us back to the original element. We can also see this geometrically in our $n \times n$ matrix - reflecting an element about the first $n-1$ columns, then about the n rows, and repeating this process again, we get back to our original element.

Algebraically, since, if $i \neq n$, $\sigma\tau(a_{i,j}) = \sigma(a_{i,j}) = \tau\sigma(a_{i,j})$, and if $i = n$, $\sigma\tau(a_{i,j}) = \sigma(a_{i,j+1}) = \tau\sigma(a_{i,j})$, σ and τ commute. Furthermore, since $\sigma\beta(a_{i,j}) = \sigma(a_{n-i,n-j-1}) = a_{n-i,n-j} = \beta\sigma^{-1}(a_{i,j}) = \beta(a_{i,j-1})$, $\sigma\beta = \beta\sigma^{-1}$. Lastly, we verify that $\tau\beta = \beta\tau^{-1}$ in the same way that we verified $\sigma\beta = \beta\sigma^{-1}$, since τ is simply σ restricted to a certain set of elements in $X_{p,n}$.

□

By the above observations, we can verify that σ, τ and β generate a group, the details of which we give below.

Definition 3.4. We denote $G_{p,n}$ by the group generated by our three automorphisms of $X_{p,n}$, σ, τ and β. $G_{p,n}$ is a subgroup of the automorphism group of $X_{p,n}$, $\mathrm{Aut}(X_{p,n})$. A group presentation of $G_{p,n}$ is given by

$$G_{p,n} =< \sigma, \tau, \beta : \sigma^n, \tau^n, \beta^2, \sigma\tau = \tau\sigma, \sigma\beta = \beta\sigma^{-1}, \tau\beta = \beta\tau^{-1} >$$

Also, $G_{p,n} \subseteq D_{2n} \times D_{2n}$, where D_{2n} denotes the dihedral group of $2n$ elements.

Definition 3.5. By the Novikov diagonal, we mean the diagonal line with slope -1 passing through bidegree $(0,0)$ in the $K(n)$-local E_n Adams spectral sequence, or the $K(n)$-local Adams-Novikov spectral sequence, for Smith Toda $V(n-1)$.

We notice that in the context of our $n \times n$ matrix containing elements of $X_{p,n}$, applying any one of the group actions (σ, τ, or β) to a set in $S^m_{p,n}$ for a given value of m gives another set in $S^m_{p,n}$. This leads us to Theorem 3.6 below. The result of Theorem 3.6 gives new symmetries on the critical complex of $X_{p,n}$, which, in turn, give automorphisms of the differential graded algebra Λ_n from Definition 1.2 and, in particular, new symmetries on the Novikov diagonal defined in Definition 3.5. When we produce such symmetries on the Novikov diagonal, we are really making computations for varying j in our DGA Λ_j regarding the E_2 page of the Novikov diagonal.

Theorem 3.6. Suppose $S \in S^m_{p,n}$. Then the subsets σS and τS and βS of $X_{p,n}$ are each elements of $S^m_{p,n}$.

Proof. First, we show that for a set $S \in S^m_{p,n}$, $\tau S \in S^m_{p,n}$. Since τ is defined as $\tau(a_{i,j}) = a_{i,j}$ if $i \neq n$ and $\tau(a_{n,j}) = a_{n,j+1}$, the only elements in S that τ shifts are those of the form $a_{n,j} \in X_{p,n}$; but, we saw earlier that every element of this form is congruent to -1 modulo $\frac{p^n-1}{p-1}$, so τ will still preserve the same residue modulo $\frac{p^n-1}{p-1} \Rightarrow$ for a set $S \in S^m_{p,n}$, $\tau S \in S^m_{p,n}$.

Next, we show that for a set $S \in S^m_{p,n}$, $\sigma S \in S^m_{p,n}$. Let $\sum(\sigma S)$ be the sum of elements in the set σS, and $\sum S$ be the sum of elements in S, where $S \in S^m_{p,n}$. Then,

$$\sum(\sigma S) - \sum S$$

$$= \textstyle\sum_{a_{i,j} \in S}(\sigma a_{i,j} - a_{i,j})$$

$$= \textstyle\sum_{a_{i,j} \in S}[2(p^i - 1)p^{j+1} - 1 - 2(p^i - 1)p^j + 1]$$

$$= \textstyle\sum_{a_{i,j} \in S} 2(p^i - 1)p^j(p-1).$$

Now, note that

$$\textstyle\sum S = \sum_{a_{i,j} \in S}(2p^j(p^i - 1) - 1) \equiv -m \bmod \frac{p^n-1}{p-1}$$

$$= -m + \textstyle\sum_{a_{i,j} \in S} 2p^j(p^i - 1)$$

, where the last equality is due to the fact that S contains exactly m elements. For $\sum S$ to be congruent to $-m \bmod \frac{p^n-1}{p-1}$, then, we have that

$$\sum_{a_{i,j}\in S} 2p^j(p^i-1) \equiv 0 \bmod \tfrac{p^n-1}{p-1}$$

$$\Rightarrow \sum(\sigma S) - \sum S = \sum_{a_{i,j}\in S} 2(p^i-1)p^j(p-1)$$

$$= 0 \bmod \tfrac{p^n-1}{p-1}$$

$$\Rightarrow \sum S \equiv \sum(\sigma S) \bmod \tfrac{p^n-1}{p-1}.$$

This gives that for a $S \in S^m_{p,n}$, $\sigma S \in S^m_{p,n}$.

Finally, we show that for a set $S \in S^m_{p,n}$, $\beta S \in S^m_{p,n}$. Let $\sum(\beta S)$ be the sum of elements in the set βS, and $\sum S$ be as above. Then

$$\sum(\beta S) - \sum S$$

$$= \sum_{a_{i,j}\in S}[\beta(a_{i,j} - a_{i,j})]$$

$$= \sum_{a_{i,j}\in S}[2(p^{n-i}-1)p^{n-j-1} - 1 - 2(p^i-1)p^j - 1]$$

$$= 2\sum_{a_{i,j}\in S}(p^{2n-i-j-1} - p^{n-j-1} - p^{j+i} + p^j)$$

$$= 2\sum_{a_{i,j}\in S}(-p^j(p^i-1) + p^{2n-i-j-1} - p^{n-j-1})$$

, where the last equality is due to the fact that $-1(p^{2n-i-j-1} - p^{n-j-1} - p^{j+i} + p^j) + p^{n-j-1}(p^{n-i}-1) = p^j(p^i-1)$. We need to show that $\sum(\beta S) - \sum S \equiv 0 \bmod \frac{p^n-1}{p-1}$. Since, from the previous proof regarding σS, $2(p^i-1)p^j \equiv 0 \bmod \frac{p^n-1}{p-1}$, it remains to show that

$$2\sum_{a_{i,j}\in S}(p^{2n-i-j-1} - p^{n-j-1}) \equiv 0 \bmod \tfrac{p^n-1}{p-1}$$

Note that, since $p^{2n} - p^n = p^n(p^n-1) \equiv 0 \bmod \frac{p^n-1}{p-1}$, $p^{2n} \equiv p^n \bmod \frac{p^n-1}{p-1}$. Thus, it suffices to show that

$$2\sum_{a_{i,j}\in S}(p^{n-i-j-1} - p^{n-j-1}) \equiv 0 \bmod \tfrac{p^n-1}{p-1}$$

This is equivalent to showing that $2\sum_{a_{i,j}\in S} p^{n-i-j-1}(1-p^i) \equiv 0 \bmod \frac{p^n-1}{p-1}$. But

$$2\sum_{a_{i,j}\in S} p^{n-i-j-1}(1-p^i) = 2\sum_{a_{i,j}\in S} -p^j(p^{n-i-2j-1})(p^i-1)$$

This is congruent to 0 mod $\frac{p^n-1}{p-1}$ since we know that $2(p^i-1)p^j \equiv 0 \bmod \frac{p^n-1}{p-1}$. Thus, $\sum S \equiv \sum(\beta S) \bmod \frac{p^n-1}{p-1}$. This gives that for a $S \in S^m_{p,n}$, $\beta S \in S^m_{p,n}$. □

As a consequence of Theorem 3.6, $S^m_{p,n}$ is a $G_{p,n}$-set.

We return to our exterior algebra over $\mathbb{Z}$ on the set of generators $\{h_{i,j} : i \in \{1, ..., n\}, j \in \{0, ..., n-1\}\}$, Λ_n, which we defined in Definition 1.2. We let $G_{p,n}$ act on Λ_n by $\mathbb{Z}$-algebra automorphisms in the evident way, i.e.,

- $\sigma(h_{i,j}) = h_{i,j+1}$
- $\tau(h_{i,j}) = h_{i,j}$ if $i \neq n$, and $\tau(h_{n,j}) = h_{n,j+1}$

- and $\beta(h_{i,j}) = h_{n-i,n-j-1}$

The action of $G_{p,n}$ on Λ_n preserves cohomological degree but not internal degree. The subject of Proposition 3.7 below is whether the action of $G_{p,n}$ commutes with the differential d on Λ_n; that is, Proposition 3.7 addresses the relations between the differential d on Λ_n and each of the three $\mathbb{Z}$-algebra automorphisms defined above.

Proposition 3.7 Given $h_{i,j} \in \Lambda_n$, we have the following relations:

- $d(\sigma(h_{i,j})) = \sigma(d(h_{i,j}))$
- $\tau(d(h_{i,j}) = d(h_{i,j})$ and $d(\tau(h_{i,j})) = \begin{cases} \tau(d(h_{i,j})) = d(h_{i,j}) & i \neq n \\ \tau(d(h_{i,j+1})) & i = n \end{cases}$. Denoting a cocycle by a sum of products of generators θ such that $d(\theta) = 0$, we see that τ sends cocycles to cocycles.

Proof. We can verify the relations above algebraically. First,

$$d(\sigma(h_{i,j})) = d(h_{i,j+1})$$

$$= \textstyle\sum_{k=1}^{i-1} h_{k,j+1} h_{i-k,j+k+1}$$

$$= \sigma(d(h_{i,j})) = \sigma(\textstyle\sum_{k=1}^{i-1} h_{k,j} h_{i-k,j+k})$$

, so the first relation in the proposition is clear.

Next, since the automorphism τ only acts nontrivially on elements of the form $h_{n,j}$, it's clear that $\tau(d(h_{i,j})) = d(h_{i,j})$ for $i \neq n$. For $i = n$,

$$\tau(d(h_{n,j})) = \tau(\textstyle\sum_{k=1}^{n-1} h_{k,j} h_{n-k,j+k})$$

But, note that there will be no elements of the form $h_{n,j}$ in the sum, and so again, τ will not act on any elements in this sum nontrivially $\Rightarrow \tau(d(h_{i,j})) = d(h_{i,j})$ for $i = n$ as well.

To verify the last relation in the proposition, we can easily verify that $d(\tau(h_{i,j})) = \tau(d(h_{i,j})) = d(h_{i,j})$ if $i \neq n$ (where the last equality is due to the last relation we verified), since τ acts nontrivially only on elements $h_{n,j}$. For $i = n$, we have that $d(\tau(h_{i,j})) = d(h_{i,j+1}) = \tau(d(h_{i,j+1}))$, where, again, the last equality is due to the relation $\tau(d(h_{i,j}) = d(h_{i,j})$. □

Note that relations between the $\mathbb{Z}$-algebra automorphism β and the differential d on Λ_n were not mentioned in Proposition 3.7. β and the differential do not commute - the interaction between β and d is the subject of Proposition 3.8, since it yields a new differential graded algebra on which we define a new differential map. β and d also do not commute on elements in $X_{p,n}$ that define a cocycle, where a cocycle is defined as an element such that the differential d computed on the element is 0. For example, in $X_{p,3}$, one can quickly show by computation that β and d do not commute on the cocycle $h_{1,0}h_{1,1}h_{1.2}$. For this same reason,

since $h_{1,0}h_{1,1}h_{1,2}$ defines an m-element subset of $X_{p,3}$ with sum congruent to -3 modulo $\frac{p^3-1}{p-1}$, β and d do not commute on cocycles in the critical complex for $X_{p,n}$.

One can also note that $\beta(d(h_{n,j})) \neq d(\beta(h_{n,j}))$, where $h_{n,j}$ lies in the critical complex for $X_{p,n}$ as a 1-element subset of $X_{p,n}$ congruent to -1 modulo $\frac{p^n-1}{p-1}$, so that β and d do not commute on the critical complex in general; however, upon noticing this, we get an additional relation not provided in Claim 3.3, which is that $\beta(d(h_{n,j})) = d(\sigma^{-2j-1}(h_{n,j}))$, which can also be seen geometrically in the context of our $n \times n$ matrix.

Remark 3.81 One can make an attempt to develop an abstract theory of the Hodge Laplacian to define this operator on our differential graded algebra, and this requires a few quirks to the classical theory. In this case, to define a theory of the Hodge Laplacian suitable for an application to the cohomology of equivariant Lie algebras which occurs as input for the spectral sequences used to compute the cohomology of Morava stabilizer groups and, from there, the $K(n)$-local stable homotopy groups of finite CW-complexes, one must

- work over an arbitrary field (not necessarily the real numbers or complex numbers, since we'd need to work over a finite field eventually),
- develop theory to suit motivating examples arising naturally as Lie algebras rather than Lie groups (as our DGA arises from a Lie algebra, which we discuss in the next section), and
- develop the theory for Lie algebras equipped with additional structure of a group action (to support our automorphisms on the critical complex).

To build foundation on how a Hodge Laplacian-like operator may act on elements in our DGA, the absence of β being commutative with the differential d, or having any nice relation with d that we can see, for that matter, causes us to ask what arises from having β behave like the hodge star operator * in our DGA - once we develop the theory for Lie algebras above equipped with a choice of symmetric nondegenerate bilinear form on their linear duals, the hodge star * is defined on our exterior algebra. Then, with $* = \beta$, we form the co-differential $\delta = *d* = \beta d \beta$, and finally, the Hodge Laplacian operator $\Delta = d\delta + \delta d = d\beta d\beta + \beta d\beta d$; the absence of the differential d being commutative with the co-differential δ gives rise to the Hodge Laplacian. With this, we see that Δ doesn't send cocycles to cocycles, and Δ doesn't send elements in the critical complex to elements in the critical complex.

Proposition 3.82 Define

$$d'(h_{i,j}) = \sum_{k=1}^{n} h_{k,j} h_{i-k,j+k} \tag{3.0.1}$$

,defining a new differential graded algebra $\Lambda_n^{'}$ (equipped with the same structure as Λ_n, except with the differential d' on $\Lambda_n^{'}$, differing from (1.0.1)). Then β and d' commute.

Proof. Using the definition of d' above and the definition of β as $\beta(h_{i,j}) = h_{n-i,n-j-1}$, we have that

$$\beta(d'(h_{i,j})) = \beta(\textstyle\sum_{k=1}^{n} h_{k,j}h_{i-k,j+k})$$

$$= \textstyle\sum_{k=1}^{n} h_{n-k,n-j-1}h_{n-i+k,n-j-k-1} \ (*)$$

, while

$$d'(\beta(h_{i,j})) = d'(h_{n-i,n-j-1})$$

$$= \textstyle\sum_{k=1}^{n} h_{k,n-j-1}h_{n-i-k,n-j+k-1} \ (**)$$

Now, by explicitly writing out terms, we can see that the sum (*) and the sum (**) are actually equal; the two sums share the same terms, with the only difference being the summands in (*) being permuted to achieve (**).

□

Note 3.9. The change in the definition of the differential d to d' is a very natural change based on the relation of our work to the Lie algebra setting, which we discuss in the next section. Define the filtration of an element $h_{i,j}$ as the first subscript i. Then, to go from the computation of d' on an element to the computation of d on that element, keep only the terms in the resulting computation of d' with total filtration equal to the filtration of the element that the differential d' is mapping. For a product of generators, the total filtration is just the sum of filtrations of the individual generators in the product.
We can push this further by constructing another differential graded algebra $\Lambda_n^{''}$, equipped with the differential d'' (which differs from both 3.0.1 and 1.0.1). Now, define the filtration of an element $h_{i,j}$ as the second subscript j, where $j \in \{0, 1, ..., n-1\}$. Then, to go from the computation of d' on an element to the computation of d'' on that element, keep only the terms in the resulting computation of d' with total filtration equal to the filtration of the element that the differential d'' is mapping, or of filtration one lower. Again, when discussing a product of generators, this total filtration is the sum of filtrations of individual generators involved in the product.

In the Lie algebra setting, we will discuss both algebras achieved through filtration $\Lambda_n^{'}$ and $\Lambda_n^{''}$ described in Note 3.9 in a more structured context. For this, we view each of our three differential graded algebras as arising from Lie algebras (which we will describe in the next section), and construct the Chevellay-Eilenberg DGAs of these Lie algebras. This will assist us in our attempt to prove a conjecture of ours regarding the cohomology of the critical complex of $X_{p,n}$ agreeing with the cohomology of the $\Lambda_n^{'}$ DGA. This conjecture is also a crucial step in proving a larger statement conjectured by Hopkins, which we also discuss in the next section.

4 A Lie algebra Formulation

Definition 4.1. If L is a Lie algebra over a field K, its Chevalley-Eilenberg differential graded algebra is given by $CE(L) = \Lambda(\iota)$, where ι is a k-linear basis for L^*, the dual of L, and Λ denotes an exterior algebra. The elements of $CE(L)$ are those dual to L, and the Chevallay-Eilenberg differential encodes the Lie bracket on L; we have a map $L \times L \longrightarrow L : (x, y) \longmapsto [x, y]$, and L^* = cohomological degree 1 part of $CE(L) \longrightarrow (L \times L)^* \cong L^* \times L^*$ = cohomological degree 2 part of $CE(L)$, and we achieve a map d from the cohomological degree 1 part of $CE(L)$ to its cohomological degree 2 part.

The cochain cohomology of the cochain complex of $CE(L)$ is the Lie algebra cohomology of L. This differential on L^* is the dual of the Lie bracket extended in a unique way to be a graded derivation. Note that, in particular, the fact that the differential d has that $d \circ d = 0$ tells us precisely that the Lie bracket satisfies the Jacobi identity.

From Definition 4.1, conversely, every differential graded algebra structure on an exterior algebra arises in this way from a Lie algebra. Thus, in this section, we view the differential graded algebras discussed earlier in this paper (such as Λ_n and Λ'_n) as Chevalley-Eilenberg bigraded DGAs.

Recall that our story began with Ravenel's filtration on the Morava stabilizer algebra, which gives rise to a graded Lie algebra, and that one has such a graded Lie algebra for each prime p and nonnegative integer n. It is well known in the field that, if one fixes n, the cohomology of these graded Lie algebras does not depend on p as long as p is sufficiently large; by this, we don't mean that the cohomology for one value of p can be isomorphic to the cohomology for any other value of p, since for $p \neq l$, an $\mathbb{F}_p$ vector space cannot be isomorphic to an $\mathbb{F}_l$ vector space, but rather that one can give a uniform description (with the same generators and relations) of the cohomology of this graded Lie algebra for a fixed value of n as long as $p > n + 1$. In order to make large-primary cohomological calculations for fixed n, we introduce a common deformation, defined over $\mathbb{Z}$, of all the graded Lie algebras for the various choice of prime p. Since the grading degree of a standard basis element $x_{i,j}$ of one of these graded Lie algebras of primitives depends on p (specifically, it is $2(p^i - 1)p^j$), the grading degree we have on our common deformation of these Lie algebras is not a $\mathbb{Z}$-grading, but rather, it will be a $\mathbb{Z}[\mathbf{p}]$-grading, where $\mathbf{p}$ is a polynomial indeterminate; by choosing a prime p and setting $p = \mathbf{p}$, one recovers the grading on the original graded Lie algebra for given n and p.

Naturally, given the terminology in previous sections, for a bigraded algebra, we refer to the first grading degree as the cohomological degree, and second grading degree (taken to be a $\mathbb{Z}[\mathbf{p}]$-grading rather than a $\mathbb{Z}$-grading) as the internal degree. With this, we can amend Definition 1.2 to replace p with $\mathbf{p}$ in defining Λ_n, but then, given a prime p, we let $\Lambda_{p,n}$ be the reduction modulo p of Λ_n, with internal grading degrees reduced modulo the relation $\mathbf{p} = p$.

We now define the Lie algebra $L(n, n)$, which agrees with the Lie algebra of primitives constructed by Ravenel (by Theorem 6.3.2 of [1]) that gives rise to the Chevalley-Eilenberg bigraded DGA $\Lambda_{p,n}$.

Definition 4.2. Given integers m, n, we let $\overline{L}(n,m)$ denote the Lie algebra over $\mathbb{Z}$ with $\mathbb{Z}$-linear basis given by the set of symbols $\{x_{i,j} : 1 \leq i \leq m, 0 \leq j \leq n-1\}$ and with Lie bracket given by

$$[x_{i,j}, x_{k,l}] = \begin{cases} \delta^l_{i+j} x_{i+k,j} - \delta^j_{k+l} x_{i+k,l} & if \;\; i+k \leq m \\ 0 & if \;\; i+k > m \end{cases} \tag{4.0.1}$$

where δ is the Kronecker delta modulo n, i.e., $\delta^a_a = 1$ and $\delta^a_b = 0$ if $a \neq b$ modulo n. We equip $\overline{L}(n,m)$ with the $\mathbb{Z}[\mathbf{p}]$-grading in which $x_{i,j}$ is in grading degree $2(\mathbf{p}^i - 1)\mathbf{p}^j$. Given a prime number p, we write $L(n,m)$ for the reduction of $\overline{L}(n,m)$, with the $\mathbb{Z}[\mathbf{p}]$-grading reduced to a $\mathbb{Z}$-grading by reducing modulo the relation $\mathbf{p} = p$.

Remark 4.3 Throughout this section, when we write "graded Lie algebra", we shall always mean one in with the $[x,y]$ is equal to the sum of grading degrees of x and y; this differs from the convention sometime used, in which the grading degree of the Lie bracket of two elements is one less than the sum of the grading degrees of those elements. With our convention, the grading on $\overline{L}(n,m)$ defined in Definition 4.2 makes $\overline{L}(n,m)$ into a $\mathbb{Z}[\mathbf{p}]$-graded Lie algebra over $\mathbb{Z}$, while $L(n,m)$ is a $\mathbb{Z}[\mathbf{p}]$-graded Lie algebra over $\mathbb{F}_p$.

Remark 4.4 It is easy to check that the Chevalley-Eilenberg bigraded DGA of $\overline{L}(n,n)$ is Λ_n, while the Chevalley-Eilenberg bigraded DGA of $L(n,n)$ is $\Lambda_{p,n}$. In other words, the Chevalley-Eilenberg bigraded DGA of $\overline{L}(n,n)$ is exactly the original differential graded algebra that we discussed in previous sections (with the amended definition discussed before Definition 4.2), on which $G_{p,n}$ acts by the three $\mathbb{Z}$-algebra automorphisms that we constructed.

We now define a graded Lie algebra closely and naturally related to $\overline{L}(n,n)$.

Definition 4.5 Let n be a positive integer. By $\overline{M}(n)$ we will mean the Lie algebra over $\mathbb{Z}$ with $\mathbb{Z}$-linear basis the set of symbols $\{x_{i,j} : 1 \leq i \leq n, 0 \leq j \leq n-1\}$, and Lie bracket given by

$$[x_{i,j} x_{k,l}] = \delta^l_{i+j} x_{i+k,j} - \delta^j_{k+l} x_{i+k,l} \tag{4.0.2}$$

We equip $\overline{M}(n)$ with the $\mathbb{Z}[\mathbf{p}]$-grading in which $x_{i,j}$ is in grading degree $2(\mathbf{p}^i - 1)\mathbf{p}^j$. Given a prime p, we let $M(n)$ denote the reduction modulo p of $\overline{M}(n)$, with the $\mathbb{Z}[\mathbf{p}]$-grading reduced to a $\mathbb{Z}$-grading by reducing modulo the relation $\mathbf{p} = p$.

Remark 4.6 It is easy to check that the Chevalley-Eilenberg bigraded DGA of $\overline{M}(n)$ is Λ'_n, the DGA closely related to Λ_n discussed in previous sections, and also with the same amendment to its definition as the amendment to Λ_n discussed before Definition 4.2. Furthermore, it is an easy yet tedious to exercise to verify that the Chevalley-Eilenberg DGAs $C^\bullet(\overline{M}(n))$ and $C^\bullet(M(n))$ share the same distinguished basis $\{h_{i,j} : 1 \leq i \leq n, j \in \mathbb{Z}/n\mathbb{Z}\}$ as Λ_n and $\Lambda_{p,n}$, but with differential

$$d(h_{i,j}) = \sum_{k=1}^{n} h_{k,j} h_{i-k,j+k}$$

Note that this is the same as the formula for d' presented in (3.0.1).

Definition 4.7 Let n be a nonnegative integer. We will write $\overline{N}(n)$ for the $\mathbb{Z}[\mathbf{p}]$-graded Lie algebra over $\mathbb{Z}$ constructed as follows: a $\mathbb{Z}$-linear basis for $\overline{N}(n)$ is given by the set of symbols $\{x_{i,j} : 1 \leq i \leq n, 0 \leq j \leq n-1\}$, with Lie bracket given by

$$[x_{i,j}, x_{k,l}] = \epsilon_l \delta^l_{i+j} x_{i+k,j} - \epsilon_j \delta^j_{k+l} x_{i+k,l} \tag{4.0.3}$$

where ϵ_m is defined to be 1 if $m = 0$ or $m = 1$, and defined to be zero otherwise. The $\mathbb{Z}[\mathbf{p}]$-grading on $\overline{N}(n)$ is the same one defined in Definition 4.2, i.e., $x_{i,j}$ is in internal grading degree $2(\mathbf{p}^i - 1)\mathbf{p}^j$.
Given a prime number p, we will write $N(n)$ for the reduction of $\overline{N}(n)$ modulo p, with the $\mathbb{Z}[\mathbf{p}]$-grading reduced to a $\mathbb{Z}$ grading by reducing modulo the relation $\mathbf{p} = p$.

Remark 4.8 We can check that the Chevalley-Eilenberg bigraded DGA of $\overline{N}(n)$ is Λ''_n, the DGA closely related to Λ_n discussed in previous sections, and also with the same amendment to its definition as the amendment to Λ_n discussed before Definition 4.2.

As we did in previous sections, one can make use of code in SageMath (and, in particular, run code using the WSU Grid) to assist with calculations involving the three Chevellay-Eilenberg bigraded DGAs of the three Lie algebras defined above. One does this by defining the DGAs in SageMath - i.e., defining the generators of cohomological degree 1, and defining how the differential maps these generators. SageMath is then equipped with the appropriate foundation to say where products and sums of generators get sent to by the differential, and defining differential graded algebras with these basic specifications. A sample code that defines these DGAs for $n = 2$ and $n = 3$ in SageMath is shown below.

```
#L(2,2)
A.<h10,h11,h20,h21> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1)) B = A.cdg_algebra({h10: 0, h11: 0, h20: h10*h11, h21: h11*h10})
#M(2)
A.<h10,h11,h20,h21> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1)) B = A.cdg_algebra({h10: h10*h21 - h10*h20, h11: h11*h20 - h11*h21, h20: h10*h11, h21: h11*h10})
#N(2) (zero filts. up)
A.<h10,h11,h20,h21> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1)) B = A.cdg_algebra({h10: -h10*h20, h11: h11*h20, h20: 0, h21: h11*h10})
#L(3,3)
A.<h10,h11,h12,h20,h21,h22,h30,h31,h32> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1,1,1,1,1,1))
B = A.cdg_algebra({h10: 0, h11: 0, h12:0, h20: h10*h11, h21: h11*h12, h22: h12*h10, h30: h10*h21 + h20*h12, h31: h11*h22 + h21*h10, h32: h12*h20 + h22*h11})
```

```
#M(3)
A.<h10,h11,h12,h20,h21,h22,h30,h31,h32> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1,1,1,1,1,
1))
B = A.cdg_algebra({h10: -h10*h30 + h20*h22 + h10*h31, h11: -h11*h31 - h20*h21 +
h11*h32, h12: -h12*h32 - h21*h22 + h12*h30, h20: h10*h11 - h20*h30 + h20*h32, h21:
h11*h12 - h21*h31 + h21*h30, h22: h12*h10 - h22*h32 + h22*h31, h30: h10*h21 + h20*h12,
h31: h11*h22 + h21*h10, h32: h12*h20 + h22*h11})
#modified N(3) (one filt. up)
A.<h10,h11,h12,h20,h21,h22,h30,h31,h32> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1,1,1,1,1,
1))
B = A.cdg_algebra({h10: -h10*h30 + h10*h31, h11: -h11*h31 - h20*h21 , h12: - h21*h22
+ h12*h30, h20: h10*h11 - h20*h30 , h21: -h21*h31 + h21*h30, h22: h12*h10 + h22*h31,
h30: h10*h21 , h31: h21*h10, h32: h12*h20 + h22*h11})
```

Note that the above computations of the differential d will agree with the three respective definitions of the differential on each of our three naturally and closely related DGAs.

Observation 4.9 If we filter $\overline{M(n)}$ by first subscript (so that the filtration of $h_{i,j}$ is defined to be i), this filtration respect the Lie bracket on $\overline{M}(n)$, and the associated graded Lie algebra of this filtration is $\overline{L}(n,n)$. The filtration by first subscript on $\overline{M}(n)$ is compatible with the actions of σ and τ, as well. The operation β is a Lie algebra automorphism of $M(n)$, although it doesn't respect the filtration, and isn't a Lie algebra automorphism of $L(n,n)$.
The derivation of $\overline{N}(n)$ from $\overline{M}(n)$ is more subtle: one filters $\overline{M}(n)$ by the second subscript, where the second subscript is an integer in $\{0,1,...,n-1\}$ (using the ordering on that set, and not as an abstract element of the cyclic group of order n) and one defines a Lie bracket on the same distinguished basis $\{x_{i,j}\}$ as used to define $\overline{M}(n)$, but in the new Lie bracket, we define $[x_{i,j}, x_{k,l}]$ to be all the terms in the $\overline{M}(n)$ Lie bracket of those elements which are of the same filtration as the sum of filtrations of $x_{i,j}$ and $x_{k,l}$ or one filtration lower (If we did not include the phrase "or one filtration lower" here, we would just be giving an alternative description of the associated graded Lie algebra of the filtration of $\overline{M}(n)$ by second subscript). Consequently, the Chevalley-Eilenberg DGA $C^\bullet(\overline{N}(n))$ has the same distinguised basis $\{h_{i,j}\}$ as $C^\bullet(\overline{M}(n))$, but with differential given by letting $d(h_{i,j})$ in $C^\bullet(\overline{N}(n))$ be the sum of the terms in $d(h_{i,j})$ computed in $C^\bullet(\overline{M}(n))$ which are of the same filtration as $h_{i,j}$ - i.e., j - or one higher.
Note that, the descriptions of these two filtrations is equivalent to the description provided in Note 3.9, where we discussed the computations of d, d', and d'', and how these computations are related - this formulation of descriptions in the Lie algebra setting is another way to say how our three Chevalley-Eilenberg DGAs are related.
The grading on $L(n,n)$ and $M(n)$ induces a grading (in the obvious way, by total degree) on the Chevellay-Eilenberg DGAs of $L(n,n)$ and $M(n)$. σ and τ preserve grading degree on $L(n,n)$ and $M(n)$, hence also on their Chevellay-Eilenberg DGAs. β does not preserve grading degree on $L(n,n)$ and $M(n)$, unless $n=1$. The second filtration is not compatible with the actions of σ and τ on $\overline{M}(n)$, so $\overline{N}(n)$ does not inherit an action of σ or τ.

Peter May's thesis [3] discusses the cohomology $H*$ of the associated graded Lie algebra of a Lie algebra g, and a corresponding spectral sequence, titled the May spectral sequence. Thus, there is a May spectral sequence $H^*(L(n,n)) \longrightarrow H^*(M(n))$. We can use the May spectral sequence to give an isomorphism between the two cohomologies - in particular, it encodes how to take the cohomology of an associated graded algebra and compute the cohomology of the original.

Example 4.10 For the sake of concreteness and developing intuition, we write out the Lie brackets, and their dual Chevalley-Eilenberg differentials, in $\overline{M}(n)$, in $\overline{N}(n)$, and in $\overline{L}(n,n)$ for low values of n. We skip the trivial case $n = 1$.

$n = 2$:

Lie bracket	in $\overline{M}(2)$	in $\overline{L}(2,2)$	in $\overline{N}(2)$
$[x_{1,0}, x_{1,1}]$	$x_{2,0} - x_{2,1}$	$x_{2,0} - x_{2,1}$	$x_{2,0} - x_{2,1}$
$[x_{1,0}, x_{2,0}]$	$-x_{1,0}$	0	$-x_{1,0}$
$[x_{1,0}, x_{2,1}]$	$x_{1,0}$	0	0
$[x_{1,1}, x_{2,0}]$	$x_{1,1}$	0	$x_{1,1}$
$[x_{1,1}, x_{2,1}]$	$-x_{1,1}$	0	0
$[x_{2,0}, x_{2,1}]$	0	0	0
Chevalley-Eilenberg differential	in $C^\bullet(\overline{M}(2))$	in $C^\bullet(\overline{L}(2,2))$	in $C^\bullet(\overline{N}(2))$
$d(h_{1,0})$	$h_{1,0}h_{2,1} - h_{1,0}h_{2,0}$	0	$-h_{1,0}h_{2,0}$
$d(h_{1,1})$	$h_{1,1}h_{2,0} - h_{1,1}h_{2,1}$	0	$h_{1,1}h_{2,0}$
$d(h_{2,0})$	$h_{1,0}h_{1,1}$	$h_{1,0}h_{1,1}$	$h_{1,0}h_{1,1}$
$d(h_{2,1})$	$-h_{1,0}h_{1,1}$	$-h_{1,0}h_{1,1}$	$-h_{1,0}h_{1,1}$

$n = 3$:

Lie bracket	in $\overline{M}(3)$	in $\overline{L}(3,3)$	in $\overline{N}(3)$
$[x_{1,0}, x_{1,1}]$	$x_{2,0}$	$x_{2,0}$	$x_{2,0}$
$[x_{1,0}, x_{1,2}]$	$-x_{2,2}$	$-x_{2,2}$	$-x_{2,2}$
$[x_{1,1}, x_{1,2}]$	$x_{2,1}$	$x_{2,1}$	0
$[x_{1,0}, x_{2,0}]$	0	0	0
$[x_{1,1}, x_{2,1}]$	0	0	0
$[x_{1,0}, x_{2,1}]$	$x_{3,0} - x_{3,1}$	$x_{3,0} - x_{3,1}$	$-x_{3,1}$
$[x_{1,1}, x_{2,2}]$	$x_{3,1} - x_{3,2}$	$x_{3,1} - x_{3,2}$	$-x_{3,2}$
$[x_{1,2}, x_{2,0}]$	$x_{3,2} - x_{3,0}$	$x_{3,2} - x_{3,0}$	$x_{3,2}$
$[x_{1,0}, x_{2,2}]$	0	0	0
$[x_{1,1}, x_{2,0}]$	0	0	0
$[x_{1,2}, x_{2,1}]$	0	0	0
$[x_{2,0}, x_{2,1}]$	$-x_{1,1}$	0	$-x_{1,1}$
$[x_{2,1}, x_{2,2}]$	$-x_{1,2}$	0	$-x_{1,2}$
$[x_{2,2}, x_{2,0}]$	$-x1,0$	0	0
$[x_{1,0}, x_{3,0}]$	$-x_{1,0}$	0	$-x_{1,0}$
$[x_{1,1}, x_{3,1}]$	$-x_{1,1}$	0	$-x_{1,1}$

$n = 3$ (continued) :

$[x_{1,2}, x_{3,2}]$	$-x_{1,2}$	0	0
$[x_{1,0}, x_{3,1}]$	$x_{1,0}$	0	$x_{1,0}$
$[x_{1,1}, x_{3,2}]$	$x_{1,1}$	0	0
$[x_{1,2}, x_{3,0}]$	$x_{1,1}$	0	$x_{1,2}$
$[x_{1,0}, x_{3,2}]$	0	0	0
$[x_{1,1}, x_{3,0}]$	0	0	0
$[x_{1,2}, x_{3,1}]$	0	0	0
$[x_{2,0}, x_{3,0}]$	$-x_{2,0}$	0	$-x_{2,0}$
$[x_{2,1}, x_{3,1}]$	$-x_{2,1}$	0	$-x_{2,1}$
$[x_{2,2}, x_{3,2}]$	$-x_{2,2}$	0	0
$[x_{2,0}, x_{3,1}]$	0	0	0
$[x_{2,1}, x_{3,2}]$	0	0	0
$[x_{2,2}, x_{3,0}]$	0	0	0
$[x_{2,0}, x_{3,2}]$	$x_{2,0}$	0	0
$[x_{2,1}, x_{3,0}]$	$x_{2,1}$	0	$x_{2,1}$
$[x_{2,2}, x_{3,1}]$	$x_{2,2}$	0	$x_{2,2}$
$[x_{3,0}, x_{3,1}]$	0	0	0
$[x_{3,1}, x_{3,2}]$	0	0	0
$[x_{3,2}, x_{3,0}]$	0	0	0
Chevalley-Eilenberg differential	in $C^\bullet(\overline{M}(3))$	in $C^\bullet(\overline{L}(3,3))$	in $C^\bullet(\overline{N}(3))$
$d(h_{1,0})$	$-h_{1,0}h_{3,0} + h_{2,0}h_{2,2} + h_{1,0}h_{3,1}$	0	$-h_{1,0}h_{3,0} + h_{1,0}h_{3,1}$
$d(h_{1,1})$	$-h_{1,1}h_{3,1} - h_{2,0}h_{2,1} + h_{1,1}h_{3,2}$	0	$-h_{2,0}h_{2,1} - h_{1,1}h_{3,1}$
$d(h_{1,2})$	$-h_{1,2}h_{3,2} - h_{2,1}h_{2,2} + h_{1,2}h_{3,0}$	0	$h_{1,2}h_{3,0} - h_{2,1}h_{2,2}$
$d(h_{2,0})$	$h_{1,0}h_{1,1} - h_{2,0}h_{3,0} + h_{2,0}h_{3,2}$	$h_{1,0}h_{1,1}$	$-h_{2,0}h_{3,0} + h_{1,0}h_{1,1}$
$d(h_{2,1})$	$h_{1,1}h_{1,2} - h_{2,1}h_{3,1} + h_{2,1}h_{3,0}$	$h_{1,1}h_{1,2}$	$h_{2,1}h_{3,0} - h_{2,1}h_{3,1}$
$d(h_{2,2})$	$-h_{1,0}h_{1,2} - h_{2,2}h_{3,2} + h_{2,2}h_{3,1}$	$-h_{1,0}h_{1,2}$	$-h_{1,0}h_{1,2} + h_{2,2}h_{3,1}$
$d(h_{3,0})$	$h_{1,0}h_{2,1} - h_{1,2}h_{2,0}$	$h_{1,0}h_{2,1} - h_{1,2}h_{2,0}$	$h_{1,0}h_{2,1}$
$d(h_{3,1})$	$h_{1,1}h_{2,2} - h_{1,0}h_{2,1}$	$h_{1,1}h_{2,2} - h_{1,0}h_{2,1}$	$-h_{1,0}h_{2,1}$
$d(h_{3,2})$	$h_{1,2}h_{2,0} - h_{1,1}h_{2,2}$	$h_{1,2}h_{2,0} - h_{1,1}h_{2,2}$	$h_{1,2}h_{2,0} - h_{1,1}h_{2,2}$

Note that the Chevellay-Eilenberg differential computations in the three DGAs above are consistent with the descriptions of filtration that we gave in Observation 4.9.

Definition 4.11 Let σ, τ and β be the permutations of the set of symbols $\{x_{i,j} : 1 \leq i \leq n, 0 \leq j \leq n-1\}$ given as follows:

- $\sigma(x_{i,j}) = x_{i,j+1}$
- $\tau(x_{i,j}) = x_{i,j}$ if $i \neq n$ and $\tau(x_{n,j}) = x_{n,j+1}$
- $\beta(x_{i,j}) = x_{n-i,n-j-1}$

Here, we regard the second subscript in the symbol $x_{i,j}$ as being defined modulo n, so that, for example, $x_{i,n} = x_{i,0}$.
By $\mathbb{F}_p$-linearity, we extend the actions of σ, τ and β on $\{x_{i,j} : 1 \leq i \leq n, 0 \leq j \leq n-1\}$ to an action of σ, τ and β on the Lie algebras $\overline{L}(n,n)$, $\overline{M}(n)$ and $\overline{N}(n)$.
By taking Chevalley-Eilenberg DGAs, we pass from the G_n-action on the Lie algebras, as defined in above, to the $G_{p,n}$-action on the DGAs, as defined in Definition 3.4.
We write $h_{i,j}$ for the dual basis element to $x_{i,j}$.

Definition 4.12 Suppose g is one of the $\mathbb{Z}[\mathbf{p}]$-graded Lie algebras $\overline{L}(n,m)$ or $\overline{M}(n)$ or $\overline{N}(n)$. By the critical complex of g we will mean the sub-cochain-complex $cc^\bullet(g)$ of the Chevalley-Eilenberg DGA $C^\bullet(g)$ consisting of the elements of $C^\bullet(g)$ which are in internal grading degree divisible by $2(\mathbf{p}^n - 1)$.
Similarly, if g is one of the $\mathbb{Z}$-graded Lie algebras $L(n,n)$ or $M(n)$ or $N(n)$, then by the critical complex of g we mean the sub-cochain-complex $cc^\bullet(g)$ of the Chevalley-Eilenberg DGA $C^\bullet(g)$ consisting of the elements of $C^\bullet(g)$ which are in internal grading degree divisible by $2(p^n - 1)$.

Note that the above is consistent with how the critical complex is defined earlier in this paper, and in particular, the description given in Definition 2.8.

Conjecture 4.13 The cohomology of the critical complex of $L(n,n)$ $H^*(cc^\bullet(L(n,n))$ (or the cohomology of the sub-cochain-complex of the Chevellay-Eilenberg DGA $C^\bullet(L(n,n))$ $= \Lambda_{p,n}$ defined above) is isomorphic to the cohomology of $\Lambda'_{p,n}$ (which, from Remark 4.6, is the Chevalley-Eilenberg bigraded DGA of $M(n)$).

We quickly remark that we can show the above conjecture is true by using $\mathbb{Z}[\mathbf{p}]$-grading instead of $\mathbb{Z}[p]$-grading.

We now discuss some approaches to proving Conjecture 4.13 that we've worked on:

1) One approach is to recall that, since $\overline{L}(n,n)$ is the associated graded Lie algebra of a filtration of $\overline{M}(n)$ by Observation 4.9, there exists a May spectral sequence $H^*(L(n,n)) \longrightarrow H^*(M(n))$. In other words, since d is the associated graded of a filtration of d', there exists a May spectral sequence $H^*(\Lambda_{p,n}) \longrightarrow H^*(\Lambda'_{p,n})$.

To do calculations for the May spectral sequence for $n = 2$, let $\zeta_2 = h_{2,0} + h_{2,1}$, and let $\eta_2 = h_{2,0} - h_{2,1}$ (and, in general, we will use the notation $\zeta_n = h_{n,0} + h_{n,1}$ and $\eta_n = h_{n,0} - h_{n,1}$). Recall from Table 1.4 that $dim_{\mathbb{F}_p} H^*(\Lambda_n) = 12$ when $n = 2$ and for $p > n+1$. To compute the May differential on an element x in $H^*(\Lambda_n)$, we compute $d'(x)$. The computations are as follows:

$d'(1) = 0$	$d'(h_{1,0}) = -h_{1,0}\eta_2$	$d'(h_{1,1}) = h_{1,1}\eta_2$,
$d'(h_{1,0}\eta_2) = 0$	$d'(h_{1,1}\eta_2) = 0$	$d'(h_{1,0}h_{1,1}\eta_2) = 0$
$d'(\zeta_2) = 0$	$d'(\zeta_2 h_{1,0}) = -\zeta_2 h_{1,0}\eta_2$	$d'(\zeta_2 h_{1,1}) = \zeta_2 h_{1,1}\eta_2$
$d'(\zeta_2 h_{1,0}\eta_2) = 0$	$d'(\zeta_2 h_{1,1}\eta_2) = 0$	$d'(\zeta_2 h_{1,0}h_{1,1}\eta_2) = 0.$

(These differentials obey a Leibniz rule, so you really only have to compute $d'(h_{1,0}), d'(h_{1,1})$ and $d'(\zeta_2)$; all the other differentials follow from these ones.)

So the input for the May spectral sequence is a 12-dimensional (as a vector space over $\mathbb{F}_p$) DGA, whose cohomology is exterior on ζ_2 and $h_{1,0}h_{1,1}\eta_2$, i.e. it's the cohomology of $M(2)$ and isomorphic to the cohomology of $U(2)$ (which we will see in a proof later on in this section). If you do the same thing for $n = 3$, the input for the May spectral sequence is a 152-dimensional DGA, which presumably has a pattern of 72 nonzero differentials that wipes out all but the 8 classes in the exterior algebra on 3 elements that gives you the cohomology $M(3)$, i.e., it's the cohomology of $U(3)$ (as we will see from a proof of a theorem later on). There may be some insight that lets you conclude that this has to happen for all n, but this seems like a very difficult approach to the problem. May differentials would have to wipe out everything except for the Novikov diagonal. Instead, we use the May spectral sequence to verify our conjecture computationally for low values of n.

2) Another approach to proving Conjecture 4.13 is to show that the critical complex of the Λ_n DGA is chain-homotopy equivalent to the Λ'_n DGA. Let f be a chain map from Λ_n to Λ'_n (a map between the two co-chain complexes), and g a chain map from Λ'_n to Λ_n. A chain homotopy h from f to g is such that $d_n h_n + h_{n+1} d_{n+1} =$ identity $-(f_{n+1} \circ g_{n+1})$. f and g are then chain-homotopy-inverse to one another so that $f \circ g$ is chain-homotopic to the identity map on the critical complex of the Λ_n DGA (or chain-homotopic to it) and $g \circ f$ is chain-homotopic to the identity map on the Λ'_n DGA (or chain-homotopic to it).

In other words, if $\phi : C^\bullet \longrightarrow D^\bullet, \psi : C^\bullet \longrightarrow D^\bullet$ are chain maps, a chain homotopy from ϕ to ψ is a map $h : C^\bullet \longrightarrow D^{\bullet-1}$ such that $dh + hd = \phi - \psi$ (ignoring subscripts for a moment); in our case, $g \circ f = \psi$ and $\psi = id_{\Lambda'_n}$. Since chain-homotopic maps induce the same map on homology, you immediately get that if our two co-chain complexes are chain-homotopy equivalent, then they have isomorphic cohomology groups, even if the two co-chain complexes aren't actually isomorphic to one another.

In particular, we have $H^*(\Lambda_n) \xrightarrow{H^*(f)} H^*(\Lambda'_n) \xrightarrow{H^*(g)} H^*(\Lambda_n)$ such that $H^*(g \circ f) : H^*(\Lambda_n) \longrightarrow H^*(\Lambda_n)$, in which case, $g \circ f$ can't be the identity map or chain-homotopic to the identity map unless we restrict f to the critical complex of Λ_n. To focus in on Conjecture 4.13, this means that the chain map f is from the critical complex in Λ_n to Λ'_n, and g is the chain map from Λ'_n to the critical complex of Λ_n.

To this point, since f is a chain map from Λ_n to Λ'_n, we know that the equation $f(d(x)) = d'(f(x))$ must be satisfied for an element $x \in \Lambda_n$, since d is the differential on Λ_n and d' is the differential on Λ'_n. Using this equation, we see that f sends everything in the critical complex of the Λ_n DGA to itself,i.e., f is the identity map on the critical complex of Λ_n. Thus, we define the chain map $f = \iota_n$, where ι_n is the inclusion map on the critical complex of Λ_n.

We will let g be the chain map such that, for $x \notin imf$, $g(x) = 0$, and for $x \in imf$, g is the identity map. In other words, we will define $g = pr_n$, where g is the projection map to the critical complex of Λ_n. (We note that, from the equations $d(g(x)) = g(d'(x))$ for $x \in \Lambda'_n$, $d(g(x)) = d(x)$ for $x \in imf$ and $g(d'(x)) = d'(x)$ for $x \in imf$, an important requirement for g to be a chain map is that $d = d'$ on the critical complex, or that, the critical complex of $M(n)$ is exactly equal to the critical complex of $L(n, n)$ - this is a fact we'd have to prove

later on in our proof of Conjecture 4.13). We get the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^0 \arrow[r, "d_0"] & \Lambda^1 \arrow[r, "d_1"] & \Lambda^2 \arrow[r, "d_2"] & \dots \\
0 \arrow[r] & CC^0 \arrow[r, "d_0"] \arrow[u, hook, "\iota_0"] & CC^1 \arrow[r, "d_1"] \arrow[u, hook, "\iota_1"] & CC^2 \arrow[r, "d_2"] \arrow[u, hook, "\iota_2"] & \dots \\
0 \arrow[r] & \Lambda^0 \arrow[r, "d_0"] \arrow[u, "pr_0"] \arrow[uur, dotted, bend right, "h_0"'] & \Lambda^1 \arrow[r, "d_1"] \arrow[u, "pr_1"] \arrow[uur, dotted, bend right, "h_1"'] & \Lambda^2 \arrow[r, "d_2"] \arrow[u, "pr_2"] \arrow[uur, dotted, bend right, "h_2"'] & \dots
\end{tikzcd}$$

Figure 4.14 - our chain maps and chain-homotopy equivalence constructed in an attempt to prove Conjecture 4.13, assuming such a chain-homotopy exists

Remark 4.15 By the above diagram, note that while the differential d gives the sequence of Λ^n's the structure of a co-chain complex, the sequence is also made into a chain complex by the k-linear map h, provided that h exists. Assuming our chain homotopy h exists, we will also see a compatibility condition between the maps d and h later on. One can try to ask, because the underlying complex of the Λ_n DGA is given the structure of both a chain complex and co-chain complex, if we can build a simplicial set (with face and degeneracy maps acting like our d and h maps) out of this sequence - however, we've checked that this is not the case for our sequence.

Remark 4.16 Since the critical complex is closed under the composition $d_n \circ \iota_n : CC^n \longrightarrow \Lambda^{n+1}$ in the above diagram, the sequence of CC^n's, the critical complex, is given the structure of a co-chain complex (which follows from our grading degrees). Even more, the critical complex is actually a sub-DGA of Λ_n.

Now, using the fact that our chain homotopy h (a k-linear map) must satisfy $d_n h_n + h_{n+1} d_{n+1} = id_{\Lambda_{n+1}} - (\iota_{n+1} \circ pr_{n+1})$, we arrive at 14 equations for $n = 2$ that must be satisfied:

(1) $h(1) = 0$
(2) $h(h_{1,0}h_{1,1}) = 0$
(3) $h(h_{1,0}(h_{2,1} - h_{2,0}) = h_{1,0}$
(4) $h(h_{1,1}(h_{2,1} - h_{2,0}) = -h1,1$
(5) $d'(h(h_{1,0}h_{2,0})) - h(h_{1,0}h_{2,0}h_{2,1}) = h_{1,0}h_{2,0}$
(6) $d'(h(h_{1,0}h_{2,1})) + h(h_{1,0}(h_{2,1} - h_{2,0})h_{2,1})$
=h$_{1,0}h_{2,1}$
(7) $d'(h(h_{1,1}h_{2,0})) + h(h_{1,1}h_{2,0}h_{2,1}) = h_{1,1}h_{2,0}$
(8) $d'(h(h_{1,1}h_{2,1})) + h(h_{1,1}h_{2,0}h_{2,1}) = h_{1,1}h_{2,1}$
(9) $d'(h(h_{2,0}h_{2,1})) = 0$
(10) $d'(h(h_{1,0}h_{1,1}h_{2,0})) = 0$

(11) $d'(h(h_{1,0}h_{1,1}h_{2,1})) = 0$
(12) $d'(h(h_{1,0}h_{2,0}h_{2,1})) = h_{1,0}h_{2,0}h_{2,1}$
(13) $d'(h(h_{1,1}h_{2,0}h_{2,1})) = h_{1,1}h_{2,0}h_{2,1}$
(14) $d'(h(h_{1,0}h_{1,1}h_{2,0}h_{2,1})) = 0$

This then yields the following chain homotopy h for $n = 2$:

$h(1) = 0$
$\mathrm{h}(\mathrm{h}_{1,0}) = h_{1,0}$
$\mathrm{h}(\mathrm{h}_{1,1}) = h_{1,1}$
$\mathrm{h}(\mathrm{h}_{2,0}) = h_{2,0}$
$\mathrm{h}(\mathrm{h}_{2,1}) = h_{2,1}$
$\mathrm{h}(\mathrm{h}_{1,0}h_{1,1}) = 0$
$\mathrm{h}(\mathrm{h}_{1,0}h_{2,0}) = -h_{1,0}$
$\mathrm{h}(\mathrm{h}_{1,0}h_{2,1}) = 0$
$\mathrm{h}(\mathrm{h}_{1,1}h_{2,0}) = h_{1,1}$
$\mathrm{h}(\mathrm{h}_{1,1}h_{2,1}) = h(h_{2,0}h_{2,1}) = h(h_{1,0}h_{1,1}h_{2,0}) = h(h_{1,0}h_{1,1}h_{2,1}) = 0$
$\mathrm{h}(\mathrm{h}_{1,0}h_{2,0}h_{2,1}) = h_{2,1}h_{1,0}$
$\mathrm{h}(\mathrm{h}_{1,1}h_{2,0}h_{2,1}) = h_{1,1}h_{2,1}$
$\mathrm{h}(\mathrm{h}_{1,0}h_{1,1}h_{2,0}h_{2,1}) = 0$

We quickly remark that this chain homotopy h on the 16 elements in the Λ' DGA can also be produced on a basis for Λ' consisting of $\eta_2 = h_{2,0} - h_{2,1}$ and $\zeta_2 = h_{2,0} + h_{2,1}$. Also, in the above mapping for $n = 2$, the system of equations we arrived at actually turned out to be overdetermined; one realizes that h on the 1-cochains is actually free! In fact, we can choose h to be 0 on the 1-cochains.

(3) Another approach to finding our chain homotopy h is to produce an explicit formula for h for all values of n based on what it does to subscripts of generators $h_{i,j}$. One can try to see if h obeys some sort of "twisty" Leibniz rules and generalized derivations, for example. One can also try to exploit as many symmetries as possible, such as checking that our chain homotopy h preserves sigma-equivariance, i.e., that $\sigma h = h\sigma$ or that $\sigma h = -h\sigma$, for example.

Another remark helps us in our search for a chain homotopy h:

Remark 4.17 Recall that a Gerstenhaber algebra over a field k is a graded commutative k-algebra A equipped with a Lie bracket $[,]$ of degree -1 satisfying the Poisson identity

$$[x, yz] = [x, y]z + (-1)^{(|x|-1)|y|}y[x, z] \tag{4.0.4}$$

for all homogeneous $x, y, z \in A$. This bracket is called the Gerstenhaber bracket.
For $n = 2$, our chain homotopy h produced satisfies being a Gerstenhaber bracket; in fact, the only property of being a Gerstenhaber bracket that we can't check directly is that h satisfies the Poisson identity, so we will assume this holds for now. Once we identify h on 2-cochains, we can use the Poisson identity to extend the Gerstenhaber bracket to 3-cochains and higher (higher cohomological degrees), although it's not clear to us yet how to do this. (In this extension to higher cochains, one would also use the Jacobi identity, and anti-symmetry). For $n = 2$, using the basis for Λ_n involving $\zeta_1, \eta_1, \zeta_2, \eta_2$, we get that on 2-cochains,

$$h(\zeta_1, \eta_2) = -\eta_1 \qquad h(\eta_1, \eta_2) = -\zeta_1, \qquad h(\zeta_1, \eta_1) = 0.$$

h does not satisfy being a differential Gerstenhaber bracket - you would think h satisfies this since there's the extra structure that the differential d provides on our DGA, but this is not the case.

(4) Let g be a Lie algebra, and $CE^\bullet(g) = \Lambda^\bullet(g^*)$ be the Chevalley-Eilenberg DGA of g. Suppose we want a chain homotopy retraction h of $CE^\bullet(g)$ onto some sub-complex of $CE^\bullet(g)$, so that $h : CE^\bullet(g) \longrightarrow CE^{\bullet-1}(g)$.

For a moment, we write pr to mean the composition $\iota \circ pr$, projection to that sub-complex of $CE^\bullet(g)$. We know that we'd like the equation $dh + hd = id - pr$ to be satisfied. Solving for pr, we get $pr = id - dh - hd$. To be a projection to a subcomplex, pr must be idempotent, i.e., $pr \circ pr = pr$. If $pr \circ pr = pr$, then, we get that $id - pr$ is also idempotent, so that pr is a projection to a subcomplex if and only if $id - pr$ is idempotent, i.e., if and only if $dh + hd$ is idempotent. So the question becomes, when is $dh + hd$ idempotent?

In solving for this, we get the equation $0 = (dh + hd) \circ (dh + hd) - (dh + hd) = dhdh + dhhd + hdhd - dh - hd$. Thus, one approach to solving for h is to choose h so that $h_{n-1} \circ h_n = 0$ and dh and hd are idempotent. In particular, since we saw that h satisfies a Gerstenhaber bracket for $n = 2$, for $n = 3$, we'd like to choose $h : \Lambda^2(u(n)^*) \longrightarrow \Lambda^1(u(n)^*)$ such that h satisfies the Jacobi identity, dh is idempotent, and hd is idempotent. (The fact that we used the Lie algebra $u(n)$ here relies on a theorem we prove later in this paper).

One notes that, if h satisfies being a Gerstenhaber bracket, we have the sequence

$$g \times g \xrightarrow{[-,-]} g \xrightarrow{co-[-,-]} g \times g \xrightarrow{[-,-]} g$$

, where $[-,-]$ is used to denote the Gerstenhaber Lie bracket, co-$[-,-]$ denotes a co-Lie bracket (using the fact that the lie coalgebra is the dual structure to a Lie algebra), and the sequences

$$g \times g \xrightarrow{[-,-]} g \xrightarrow{co-[-,-]} g \times g, g \xrightarrow{co-[-,-]} g \times g \xrightarrow{[-,-]} g \tag{4.0.5}$$

are idempotent. This co-Lie bracket, which we'll call Δ, must satisfy the co-Jacobi identity. Thus, one possible approach to finding this chain homotopy h is to look back at the chain homotopy h for $n = 2$, solve for our co-Lie bracket Δ, and then dualize Δ to get h (and, hopefully, when one does this, we get that pr as a projection to a subcomplex of the Chevalley-Eilenberg DGA is exactly projection to the critical complex). Using the fact that $u(n) \cong mat_n(k)$, $n \times n$ matrices over the field k, when $\sqrt{-1} \in k$, one can pursue computing the co-Lie bracket Δ for $n = 2$ using the fact that the Lie bracket on $mat_n(k)$ is just the commutator bracket. However, even after this computation, one must decide about the proper way to extend the Gerstenhaber bracket to 3-cochains and higher so that hd and $dh + hd$ are idempotent (and also so that $h_{n-1} \circ h_n = 0$). This remains to be solved.

(5) Another attempt at proving Conjecture 4.13 is the following: Let $pr = \iota \circ pr$. Using the equation $dh + hd = id - pr$, we can immediately get that for $h_0 : \Lambda^0(g) \longrightarrow 0, h = 0$ on 0-cochains. The idea is to inductively solve for what h must be on all n-cochains. $h_1 : \Lambda^1(g) \longrightarrow \Lambda^0(g)$ is any k-linear map (recall that h is free on the 1-cochains, so this makes sense).

Now, let x be a 1-cochain. Since $(dh+hd)(x) = dh(x)+hd(x)$, and h is 0 on 1-cochains, we get that $h_2(d(x)) = x - pr(x)$ for all $x \in \Lambda^1(g)$. This tells us that h_2 is determined on 2-coboundaries, and free on other 2-cochains. In particular, the critical complex must contain all 1-cocycles. (If x is a 1-cocycle, we also get that $0 = d(h_1(x)) = x - pr(x)$).

Now, let $x \wedge y$ be a 2-cochain. Then $(dh+hd)(x \wedge y) = dh(x \wedge y) + h(d(x) \wedge y + x \wedge dy) = x \wedge y - (pr)(x \wedge y)$, where, by assuming that h satisfies being a Gerstenhaber bracket, we can take $dh(x \wedge y)$ to be $d[x,y]$, where $[x,y]$ denotes the Gerstenhaber bracket, and we make use of the Leibniz rule that d obeys. This gives that for $h_3 : \Lambda^3(g) \longrightarrow \Lambda^2(g)$, $h_3((dx) \wedge y + x \wedge (dy)) = x \wedge y - pr(x \wedge y) - d(h_2(x \wedge y))$ (and we can allow $h_2(x \wedge y) = [x,y]$, the Gerstenhaber bracket), meaning that h_3 is determined on 3-coboundaries, and free on the other 3-cochains. Also, if $x \wedge y$ is a 2-cocyle, we have that $d(h_2(x \wedge y)) = x \wedge y - pr(x \wedge y)$, or that $d(h_2(d(x)) = d(x) - pr(d(x)) = d(x - pr(x))$.

Now, let $(dh+hd)(x \wedge y \wedge z) = dh(x \wedge y \wedge z) + hd(x \wedge y \wedge z)$ for $x \wedge y \wedge z$ a 3-cochain (or a 3-differential form, if you'd like). By the same considerations as before, $h_4(d(x \wedge y \wedge z)) = x \wedge y \wedge z - pr(x \wedge y \wedge z) - d(h_3(x \wedge y \wedge z))$, so h_4 is determined on 4-coboundaries, and free on other 4-cochains. In particular, if $x \wedge y \wedge z$ is a 3-cocycle, $d(h_3(x \wedge y \wedge z)) = x \wedge y \wedge z - pr(x \wedge y \wedge z)$.

So, we see that at each stage of solving for h inductively, we can describe the behavior of h on coboundaries and cocycles. One can ask how to extend h, then, to all n-cochains. One idea is to exploit an idea coming from Hodge theory: one can decompose by the Hodge decomposition $\Lambda^{n-1} = ker(d^n) \bigoplus im(d^n)$ and put this together with the equation $H^n(g) = \frac{ker(d^n)}{im(d^n)}$. However, this requires more thought about how to discuss objects like holomorphic forms in our setting, and would require us to lift everything we're doing to $\mathbb{C}$.

(6) Our latest attempt at solving for the chain homotopy h (or proving its existence for all n) in order to prove Conjecture 4.13 is that, based on our chain homotopy h for $n = 2$ and our general picture given by Figure 4.14 of our general setup, we can narrow down our search for h to the following: Can we choose $h_n : \Lambda_n \longrightarrow \Lambda_{n-1}$ for all n such that

(1) $h_{n-1} \circ h_n = 0$
(2) $\mathrm{pr}_n \circ h_n = 0$
(3) $\mathrm{h}_{n-1} \circ \iota_n = 0$
(4) $\mathrm{d}_n h_n + h_{n+1} d_{n+1} = id_{\Lambda^{n+1}} - (\iota_{n+1} \circ pr_{n+1})$

The only absolutely necessary equation that h must satisfy is (4). The idea, again, is to show that h exists inductively. Note that these equations also tell us that h is 0 on the critical complex. For $n = 0$, the equations give us that $h_0 = 0$. For $n = 1$, the equations give us that $h_1 d_1 h_1 = h_1$. For $n = 2$, the equations tell us that $h_2 d_2 h_2 = h_2$. (In particular, if $h_1(x)$ is a nonzero cocycle, then $d_1(h_1(x)) = 0$, and $h_2(x)$ can't be a nonzero cocycle).

The following conditions are able to recover everything about our general setup. If we can find an h such that $\Lambda^\bullet$ is a differential graded algebra, $h_{n-1} \circ h_n = 0$, and $hdh = h$, h will satisfy being the chain homotopy we want. Note that the condition $h \circ h = 0$ tells us that h would make our cochain complex by the differential d maps into a chain complex, as well. We also have this extra compatibility condition, $hdh = h$ for all n, between the chain complex and cochain complex.

One way to achieve our chain homotopy h by this latest attempt is to use the theory of Batalin-Vilkovisky algebras.

Definition 4.18. Let $C^\bullet$ be a cochain complex of R-modules. By a special homotopy on $C^\bullet$ we mean a differential $h : C^\bullet \longrightarrow C^{\bullet-1}$ such that $hdh = h$.
In other words: for each integer n, we have an R-linear map $h_n : C^n \longrightarrow C^{n-1}$ such that

- $h_n \circ h_{n+1} = 0$ for all n, and
- the square $\begin{array}{ccc} C^n & \xrightarrow{d^n} & C^{n+1} \\ {\scriptstyle h_n}\uparrow & & \downarrow{\scriptstyle h_n} \\ C^{n+1} & \xrightarrow{h_n} & C^n \end{array}$ commutes.

Definition 4.19 Recall that a Batalin-Vilkoviksy R-algebra, or BV R-algebra for short, is a graded commutative R-algebra A^* equipped with an R-linear operator $\Delta : A^* \longrightarrow A^{*-1}$ satisfying the axioms:

- $\Delta \circ \Delta = 0$, and
- $\Delta(xyz) - \Delta(xy)z + \Delta(x)yz - (-1)^{|x|}x\Delta(yz) - (-1)^{(|x|-1)|y|}y\Delta(xz) + (-1)^{|x|}x\Delta(y)z + (-1)^{|x|+|y|}xy\Delta(z) - \Delta(1)xyz = 0$ for all homogeneous $x, y, z \in A^*$.

One checks that this definition is equivalent to asking that A^* is a Gerstenhaber algebra equipped with an operator $\Delta : A^* \longrightarrow A^{*-1}$ such that
$[x, y] = (-1)^{|x|}\Delta(xy) - (-1)^{|x|}\Delta(x)y - x\Delta(y) + x\Delta(1)y$ and such that $\Delta(xy) = 0$.

Thus, one approach to showing that our chain homotopy h exists is to show possibly that $h = \Delta$, up to a multiplication by a scalar. If this is true, the last item one would have to check in order to prove Conjecture 4.13 is that, when one considers the equation $d_n h_n + h_{n+1} d_{n+1} = id_{\Lambda^{n+1}} - (\iota_{n+1} \circ pr_{n+1})$, the projection here to a subcomplex of the Λ_n DGA is really projection to the critical complex, as desired.

Conjecture 4.20 (Hopkins' conjecture) The cohomology of the diagonal line of slope -1 passing through bidegree $(0, 0)$ on the E_2 page in the $K(n)$-local Adams-Novikov spectral sequence for Smith-Toda $V(n-1)$ (or the cohomology of the Novikov diagonal for the E_2 term) is isomorphic to the cohomology of $U(n)$ $H^*(U(n)) \equiv E(x_1, x_3, ..., x_{2n-1})$, the exterior algebra on odd generators, where $x_i \in H^i$, $H^* = \bigoplus H^i$ over all i, and $U(n)$ is the group of $n \times n$ unitary matrices.

A classical result in Lie theory tells us that if X is a simply connected Lie group with Lie algebra $g(X)$, then the cohomology of the topological space X over the field K $H^*(X, K)$ is isomorphic to the cohomology of its Lie algebra over the field K $H^*(g(X), K)$. (In particular, $H^*_{dR}(G; \mathbb{R})$, the cohomology of the cochain complex of differential forms on G, is isomorphic to $H^*(G; \mathbb{R}) \cong H^*(g; \mathbb{R})$, the cohomology of the Chevalley-Eilenberg cochain complex of g,

with G a compact simply connected Lie group and g the Lie algebra of G, and $H^*_{dR}(G;\mathbb{R})$ is quasi-isomorphic by a chain-homotopy equivalence to $H^*(g;\mathbb{R})$).

Although $U(n)$ is not simply connected, this result still applies by a special property. The special property is that $U(n)$ is diffeomorphic to $SU(n)$ times $U(1)$, where $SU(n)$ is the Lie group of special unitary matrices,i.e., the unitary matrices with determinant equal to 1. The diffeomorphism $U(n) \cong SU(n) \times U(1)$ is actually quite natural; the determinant of any unitary matrix is a complex number of norm 1 (i.e., a complex number on the unit circle in the complex plane), and unitary 1×1 matrices are exactly just complex numbers of norm 1, so we get the short exact sequence

$$1 \longrightarrow SU(n) \longrightarrow U(n) \longrightarrow U(1) \longrightarrow 1 \tag{4.0.6}$$

, where the map $U(n) \longrightarrow U(1)$ is just the determinant map. This short exact sequence splits, yielding the diffeomorphism $U(n) \equiv SU(n) \times U(1)$. The same thing happens on the Lie algebra level, so we get that $u(n) = su(n) \times u(1)$. Since $SU(n)$ is a compact simply connected Lie group and since the cohomology of $U(1)$ and $u(1)$ are each just an exterior algebra on one generator in cohomological degree 1, we get isomorphisms

$$H^*(U(n);R) = H^*(SU(n) \times U(1);R) = H^*(SU(n);R) \otimes_R H^*(U(1);R) = \\ H^*(su(n);R) \otimes_R E(x_1) = H^*(su(n) \times u(1);R) = H^*(u(n);R)$$

Note that, $H^*(U(1);R)$ is just the exterior R-algebra $E(x_1)$ because $U(1)$ is homeomorphic to the circle S^1 (in fact, also diffeomorphic, but homeomorphic is enough to get the isomorphism in cohomology; in fact, even just a homotopy equivalence is enough to get the isomorphism in cohomology). The Lie group $U(1)$ is just the group of 1×1 unitary matrices. A 1×1 matrix just has a single entry, so suppose $[z]$ is a 1×1 unitary matrix. Then $[z]$ inverse must be equal to the conjugate-transpose of $[z]$, i.e., z^{-1} must be equal to the conjugate of z. It's a nice (and not hard) exercise in the algebra of the complex numbers to show that this is equivalent to z being on the unit circle in the complex plane! That's how you identify $U(1)$ as topologically just a circle. The cohomology of a circle is just an exterior algebra on a single generator in degree 1.

Thus, we have a good approach to proving Hopkins' conjecture: if we can show that the cohomology of the critical complex of $X_{p,n}$ agrees with the cohomology of either the Λ'_n or Λ''_n DGA, and then show that the $\overline{M}(n)$ or $\overline{N}(n)$ Lie algebra is isomorphic to the Lie algebra $u(n)$, we can prove Hopkins' conjecture. The $\overline{M}(n)$ Lie algebra is more likely to be isomorphic to the $u(n)$ Lie algebra; if you look at the tables for $n = 3$ in Example 4.10, you see that in the Lie algebra $\overline{N}(3)$, the element $x_{3,2}$ has the property that $[x_{3,2}, y] = 0$ for all y in the Lie algebra. That's what it means for $x_{3,2}$ to be in the center of the Lie algebra: the center of a Lie algebra is the vector space of all elements x with the property that $[x, y] = 0$ for all y in the Lie algebra. But, $x_{3,0}+x_{3,1}$ has the same property. So the center of $\overline{N}(3)$ must be at least two-dimensional, since we see two linearly independent elements in it, namely, $x_{3,2}$ and $x_{3,0} + x_{3,1}$. But that doesn't happen in $u(3)$ – Remember that $u(3)$ is the set of all 3×3 skew-Hermitian matrices, with Lie bracket given by $[M, N] = MN - NM$. So if a matrix M is in the center of $u(3)$, that's the same as saying that $MN = NM$ for all

N in $u(3)$. (That's why they call it the "center" of a Lie algebra: the center of a group is the set of all the elements that commute with everything else, and when you have a Lie algebra whose Lie bracket is a commutator, then the center of the Lie algebra is again all the elements that commute with everything else.) If you play around with a bit of linear algebra, you see quickly that the only skew-Hermitian matrices that commute with all other skew-Hermitian matrices are those which are imaginary multiples of the identity matrix. So the center of $u(3)$ is one-dimensional, with basis given by i times the identity matrix. So $u(3)$ can't be isomorphic to the $\overline{N}(3)$ Lie algebra! On the other hand, by the computations, the Lie algebra $\overline{M}(3)$ has one-dimensional center, generated by $x_{3,0} + x_{3,1} + x_{3,2}$, i.e., ζ_3. So we find it plausible that $\overline{M}(3)$ is isomorphic to $u(3)$, and in general that $\overline{M}(n)$ is isomorphic to $u(n)$.

Showing that the cohomology of the critical complex for $X_{p,n}$ agrees with the cohomology of the Lie algebra $\overline{M}(n)$ is contingent exactly on proving Conjecture 4.13. However, for the fact that the $\overline{M}(n)$ Lie algebra is isomorphic to the $u(n)$ Lie algebra, we're able to prove this.

Theorem 4.21 The Lie algebra $M(n)$ is isomorphic to the Lie algebra $u(n)$.

Proof. One can write down a basis for $u(n)$ consisting of matrices of three types - diagonal matrices with a single nonzero entry (which is necessarily imaginary), matrices which are zero on the diagonal and have two nonzero off-diagonal real entries (and then you're forced to have each entry be -1 times the other), and matrices which are zero on the diagonal and have two nonzero off-diagonal imaginary entries (and then you're forced to have each entry be equal to the other).

The Lie algebra $u(n)$ is naturally defined only over the real numbers, since if you multiply a skew-Hermitian matrix by a real number, it's still skew-Hermitian, but if you multiply a skew-Hermitian matrix by an imaginary number, it is usually not skew-Hermitian anymore. If you base-change $u(n)$ to the complex numbers (in other words, take the tensor product $u(n) \otimes_{\mathbb{R}} \mathbb{C}$), then you get $gl_n(C)$, the Lie algebra of all $n \times n$ matrices of complex numbers, with Lie bracket given by commutator bracket, i.e., $[M, N] = MN - NM$. As a vector space over $\mathbb{R}$, the basis for $u(n)$ we described earlier spans $u(n)$, but if you look at those same basis vectors and ask for their $\mathbb{C}$-linear span, as a linear subspace of $gl_n(\mathbb{C})$, you notice that you get everything: every $n \times n$ complex matrix is a $\mathbb{C}$-linear combination of those n^2 basis vectors. So $u(n) \otimes_{\mathbb{R}} \mathbb{C}$ agrees with $gl_n(\mathbb{C})$.

Thus, in order to show that the $\overline{M}(n)$ Lie algebra is isomorphic to the $u(n)$ Lie algebra, we have a slightly easier task: we'd really like to show that the $\overline{M}(n)$ Lie algebra is isomorphic to the $gl_n(\mathbb{C})$ Lie algebra by finding a bijection between $x_{i,j}$ basis vectors for $\overline{M}(n)$ and a suitable basis for $gl_n(\mathbb{C})$. This is easier, since the collection of all $n \times n$ complex matrices is more familiar and has more obvious symmetries than the collection of all $n \times n$ skew-Hermitian matrices. The question then is just to find a basis for $gl_n(\mathbb{C})$ over the complex numbers (so the basis will necessarily have n^2 elements in it) and a bijection f of that basis with the usual basis $x_{i,j}$ for $M(n)$, with the property that the Lie bracket of two basis vectors x, y in the chosen basis for $gl_n(\mathbb{C})$ agree with the Lie bracket of $f(x)$ and $f(y)$ in $M(n)$.

One such suitable basis for $gl_n(\mathbb{C})$ is $\{a_i^j | i, j = 1, 2, ..., n\}$ and Lie bracket given by

$$[a_i^j, a_k^l] = a_i^j * a_k^l - a_k^l * a_i^j = \delta_j^k a_l^i - \delta_l^i a_j^k \tag{4.0.7}$$

, where we take as a $\mathbb{C}$-linear basis for $gl_n\mathbb{C}$ the set of matrices which are zero except for a single entry which is 1, and a_i^j is the matrix which has the entry 1 in row i and column j and which has 0 as all other entries, and where δ is the Kronecker delta modulo n, i.e., $\delta_a^a = 1$ and $\delta_b^a = 0$ if $a \neq b$ modulo n. However, we see that this basis is almost exactly the same as the formula for the Lie bracket for $M(n)$ given by $[x_{i,j}x_{k,l}] = \delta_{i+j}^l x_{i+k,j} - \delta_{k+l}^j x_{i+k,l}$ (4.0.2). In fact, the bijection between basis elements a_i^j of $gl_n(\mathbb{C})$ and basis vectors $x_{i,j}$ of the $\overline{M}(n)$ Lie algebra is given by

$$f(a_i^j) = x_{i-j,j} \tag{4.0.8}$$

This holds since $[f(a_i^j), f(a_k^l)] = [x_{i-j,j}, x_{k-l,l}] = \delta_i^l x_{k-j,j} - \delta_k^j x_{i-l,l}$ (assuming that, without loss of generality, both deltas are equal to 1, and thus i=l and k=j) $= f(\delta_i^l a_k^j - \delta_k^j a_i^l)$, while on the other hand, $f([a_i^j, a_k^l]) = f(\delta_i^l a_k^j - \delta_k^j a_i^l)$ □

To review Hopkins' conjecture, we expect that the cohomology of the critical complex for $X_{p,n}$ agrees with $H^*(U(n); \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}[x_1, x_3, ..., x_{2n-1}]$, where $|x_i| = i$. First, we show that there exists a filtration on $u(n)$ such that the associated graded Lie algebra has Chevalley-Eilenberg DGA isomorphic to Λ_n, and such that the critical complex in $CE^\bullet(u(n))$ is isomprphic to the critical complex in Λ_n. Second, we show that $H^*(u(n); \mathbb{F}_p) \cong H^*(U(n); \mathbb{F}_p) \cong H^*(CE^\bullet(u(n)); \mathbb{F}_p)$. Third, we need to show that $H^*(u(n); \mathbb{F}_p)$ is concentrated in the critical complex. The first two steps are done, and the third step is contingent on proving Conjecture 4.13.

5 Future Research Project

We mentioned early on in the paper some computations of stable homotopy groups of spheres for low values of n. In particular, when $n = 1$, $\pi_* L_{E(1)} S$ is essentially the same as "im j", where j is the whitehead j-homomorphism. In fact, by Adams, the number of $\pi_{2n-1}(imj)$ is equal to the denominators of $\zeta(-i)$ (up to sign and a power of 2), where ζ denotes the Riemann-zeta function (in other words, for a positive integer n, $\zeta(1-2n)$ is equal to the number of $\pi_{4n-1}(L_{K(n)}S)$). In fact,

$$\zeta(1-2n) = \frac{\# K_{4n-2}(\mathbb{Z})}{\# K_{4n-1}(\mathbb{Z})} \tag{5.0.1}$$

up to sign and a power of 2 ,where K denotes a K-group, giving a result in algebraic K-theory. ((5.0.1) is known as the Lictenbaum-Quillen conjecture.)

A conjecture by Salch states that the Euler totient function ϕ is injective on the special class of denominators of the Riemann-zeta function given by $\zeta(1-2n)$, where n is a positive integer. (Note that, by Carmichael's totient function conjecture, ϕ is not injective in general). A future research project would be to try and prove this conjecture. Based on what Adams

discovered in the computation of stable homotopy groups of spheres for $n = 1$, we could take ϕ of both sides of the equation $\zeta(1-2n) = \pi_{4n-1}(L_{K(n)}S)$ (up to sign and a power of 2) to tell us about the $K(n)$-localization of stable homotopy groups of spheres. We could also take ϕ of both sides of equation (5.0.1) to tell us about the orders of certain K-groups.

The following are some considerations in trying to prove Salch's conjecture. For n a positive integer, $\zeta(1-n) = \frac{-B_n}{n}$, where B_n is the nth Bernoulli number (which, for n odd, is equal to 0). That is,

$$denom(\zeta(1-2n)) = denom(\frac{-B_{2n}}{2n}) \tag{5.0.2}$$

The denominators of the Bernoulli numbers are given by

$$denom(B_{2n}) = \prod_{\text{primes p such that p-1 divides 2n}} p \tag{5.0.3}$$

,due to Von Staudt-Clausen. In all,

$$denom(\zeta(1-2n)) = denom(\frac{numerator of B_{2n}}{2n \prod_{\text{primes p such that p-1 divides 2n}} p}) \tag{5.0.4}$$

In this formula, notice that, to fully evaluate the denominator, we need to know if some factors in the denominator cancel with terms in the numerator of B_{2n}. For this, we note that by Vonstaudt, if p is a factor of $2n$ and $p-1$ does not divide $2n$, then v_p(numerator of $B_{2n}) \geq v_p(2n)$, where $v_p(2n)$ is the largest integer m such that p^m divides $2n$. Thus, using this, the denominator of $\zeta(1-2n)$ for a positive integer n is equal to

$$denom(\zeta(1-2n)) = \prod_{\text{primes p such that p-1 divides 2n}} p^{v_p(2n)+1} \tag{5.0.5}$$

, and by the fact that $\phi(p^n) = (p-1)(p^{n-1})$ for a prime p,

$$\phi(denom(\zeta(1-2n))) = \prod_{\text{primes p such that p-1 divides 2n}} (p-1)p^{v_p(2n)} \tag{5.0.6}$$

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