CLASSICAL ALGEBRAIC NOVIKOV CHARTS AND \(C\)-MOTIVIC
ADAMS CHARTS FOR THE COFIBER OF \(\tau\)

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Abstract. This document presents large-format charts that display the algebraic Novikov spectral sequence that converges to the \(E_2\)-page of the classical Adams-Novikov spectral sequence at \(p = 2\). Up to reindexing, this spectral sequence is identical to the \(C\)-motivic Adams spectral sequence for the cofiber of \(\tau\). The charts are essentially complete through the 110-stem, including all differentials and all extensions by \(h_0\), \(h_1\), and \(h_2\). All information is derived from machine computation.

This document contains large-format charts that display the algebraic Novikov spectral sequence \([5]\) \([6]\). This spectral converges to \(\text{Ext}_{BP_*BP}(BP_*, BP_*)\), i.e., the \(E_2\)-page of the classical Adams-Novikov spectral sequence at \(p = 2\).

Up to reindexing, this spectral sequence is identical to the \(C\)-motivic Adams spectral sequence that converges to the stable motivic homotopy group of the cofiber of \(\tau\) \([2]\).

The charts are essentially complete through the 110-stem. This includes a complete description of all differentials, and all extensions by \(h_0\), \(h_1\), and \(h_2\).

The data presented in these charts was produced by the Wayne State University Grid high performance computing service. See \([1]\), \([4]\), and \([8]\) for more information on the origin of the information.

The charts are intended to be viewed electronically.

This document supersedes some of the charts in \([3, \text{Versions 1–4}]\).

1. Grading and naming conventions

The grading used in the charts is compatible with the standard grading conventions for Adams charts, in which the stem is indicated on the horizontal axis and the Adams filtration is indicated on the vertical axis. The motivic weight is suppressed.

This choice of grading is not compatible with the standard grading conventions for Adams-Novikov charts. Specifically, the Adams-Novikov filtration of an element equals twice its weight minus its stem.

The names of elements are incompatible with traditional notation for the algebraic Novikov spectral sequence, as in \([7, \text{Section 4.4}]\). Rather, names of elements come from the perspective of the \(C\)-motivic Adams spectral sequence for the cofiber.
of \( \tau \). An element \( x \) is the image of the element of the same name in the Adams spectral sequence for the bottom cell of the cofiber of \( \tau \), while an element \( \overline{x} \) maps to \( x \) in the Adams spectral sequence for the top cell. See [4] for more details about naming conventions.

2. \( h_1 \)-BOCKSTEIN SPECTRAL SEQUENCE

The charts on pages 5–7 indicate the elements of the algebraic Novikov \( E_2 \)-page that are non-zero after inverting \( h_1 \). More specifically, they depict the algebraic Novikov \( E_2 \)-page modulo elements that are annihilated by \( h_1^k \) for some \( k \geq 0 \). Equivalently, the chart indicates the \( \mathbb{C} \)-motivic Adams \( E_2 \)-page for the cofiber of \( \tau \), modulo elements that are annihilated by \( h_1^k \) for some \( k \geq 0 \).

The charts also show differentials in the \( h_1 \)-Bockstein spectral sequence that converges to the algebraic Novikov \( E_3 \)-page. In other words, the differentials correspond to algebraic Novikov \( d_2 \) differentials, ignoring terms that contain higher powers of \( h_1 \). For example, the \( h_1 \)-Bockstein differential \( d_0(\overline{e_0c_0}) = Pe_0 \) is a truncated version of the algebraic Novikov differential \( d_2(\overline{e_0c_0}) = Pe_0 + h_1^2c_0d_0 \). Meanwhile, there is also an algebraic Novikov differential \( d_2(\overline{Pe_0}) = Ph_1^2d_0 \), which does not appear as an \( h_1 \)-Bockstein differential because \( Pd_0 \) is hit by an earlier Bockstein differential. See [4] for more details about this \( h_1 \)-Bockstein spectral sequence.

(1) Black dots indicate copies of \( \mathbb{F}_2 \) that are in the image of inclusion of the bottom cell.
(2) Red dots and indicate copies of \( \mathbb{F}_2 \) that are detected by projection to the top cell.
(3) Black lines indicate extensions by \( h_1 \) that are in the image of inclusion of the bottom cell.
(4) Red lines indicate extensions by \( h_1 \) that are detected by projection to the top cell.
(5) Blue lines indicate extensions by \( h_1 \) that are hidden in the sense that their sources are detected by the top cell while their targets are detected by the bottom cell.
(6) Arrows of slope 1 indicate infinite towers of \( h_1 \) extensions.
(7) Light blue lines indicate \( h_1 \)-Bockstein differentials.
(8) Dashed light blue lines indicate the presence of differentials whose sources or targets are uncertain.

3. \( E_\infty \)-PAGE OF THE \( h_1 \)-BOCKSTEIN SPECTRAL SEQUENCE

The chart on page 8 presents the \( E_\infty \)-page of the \( h_1 \)-Bockstein spectral sequence discussed in Section 2. Elements in this chart represent elements of the algebraic Novikov \( E_3 \)-page, modulo terms involving higher powers of \( h_1 \).

See Section 2 for instructions on interpreting the chart. Additionally,

(1) Dashed black lines indicate the presence of extensions whose sources are uncertain. These uncertainties are created by uncertainties in associated differentials.

4. THE ALGEBRAIC NOVIKOV \( E_2 \)-PAGE

The charts on pages 9–12 present the \( E_2 \)-page of the algebraic Novikov spectral sequence, or equivalently the \( E_2 \)-page of the \( \mathbb{C} \)-motivic Adams spectral sequence for
the cofiber of $\tau$. These charts are complete through the 110-stem, with indicated exceptions.

1. Black dots indicate copies of $F_2$ that are in the image of inclusion of the bottom cell.
2. Red dots and indicate copies of $F_2$ that are detected by projection to the top cell.
3. Black lines indicate extensions by $h_0$, $h_1$, and $h_2$ that are in the image of inclusion of the bottom cell.
4. Red lines indicate extensions by $h_0$, $h_1$, and $h_2$ that are detected by projection to the top cell.
5. Blue lines indicate extensions by $h_0$, $h_1$, and $h_2$ that are hidden in the sense that their sources are detected by the top cell while their targets are detected by the bottom cell.
6. Arrows of slope 1 indicate infinite towers of $h_1$ extensions.
7. Light blue lines indicate differentials.
8. Dashed light blue lines indicate the presence of differentials whose sources or targets are uncertain. These uncertain differentials are also indicated on later charts.

For legibility, $h_1$-periodic $d_2$ differentials are not shown on this chart. The information from the $h_1$-Bockstein chart of Section 2 must be combined with this chart to obtain the full description of the algebraic Novikov $d_2$ differentials.

5. The algebraic Novikov $E_3$-page

The charts on pages 13–15 present the $E_3$-page of the algebraic Novikov spectral sequence, or equivalently the $E_3$-page of the $C$-motivic Adams spectral sequence for the cofiber of $\tau$. The charts are complete through the 110-stem, with indicated exceptions.

See Section 4 for instructions on interpreting the charts. Additionally,

1. Dashed black lines indicate the presence of extensions whose sources are uncertain. These uncertainties are created by uncertainties in associated differentials.

6. The algebraic Novikov $E_4$-page

The charts on pages 16–17 present the $E_4$-page of the algebraic Novikov spectral sequence, or equivalently the $E_4$-page of the $C$-motivic Adams spectral sequence for the cofiber of $\tau$. The charts are complete through the 110-stem, with indicated exceptions.

See Section 4 for instructions on interpreting the charts.

7. The algebraic Novikov $E_5$-page

The charts on pages 18–19 present the $E_5$-page of the algebraic Novikov spectral sequence, or equivalently the $E_5$-page of the $C$-motivic Adams spectral sequence for the cofiber of $\tau$. The charts are complete through the 110-stem, with indicated exceptions.

See Section 4 for instructions on interpreting the chart.

The charts show the $d_5$, $d_6$, $d_7$, and $d_8$ differentials. Through the 110-stem, there are no higher differentials.
8. The algebraic Novikov $E_\infty$-page

The charts on pages 20–21 present the $E_\infty$-page of the algebraic Novikov spectral sequence, or equivalently the $E_\infty$-page of the C-motivic Adams spectral sequence for the cofiber of $\tau$. The charts are complete through the 110-stem, with indicated exceptions.

1. Black and green dots detect elements of $\pi_*(C\tau)$ that are in the image of inclusion of the bottom cell. Black dots indicate values of inclusion of the bottom cell that are detected by Adams $E_\infty$-pages, while green dots indicate that there are filtration shifts.

2. Red and orange dots detect elements of $\pi_*(C\tau)$ that are detected by projection to the top cell. Red dots indicate values of projection to the top cell that are detected by Adams $E_\infty$-pages, while orange dots indicate that there are filtration shifts.

3. Purple dots detect elements of $\pi_*(C\tau)$ whose relationships to the bottom and top cells are uncertain.

4. Black, green, red, orange, and purple lines indicate extensions by $h_0$, $h_1$, and $h_2$ that are detected by the $E_\infty$-page.

5. Blue lines indicate hidden extensions by $2$, $\eta$, and $\nu$.

6. Arrows of slope 1 indicate infinite towers of $h_1$ extensions.

Beyond the 90-stem, all dots are black because we have not fully analyzed the values of inclusion of the bottom cell into $C\tau$ and of projection from $C\tau$ to the top cell.

References


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$E_2$-page of the algebraic Novikov spectral sequence