The motivic slice spectral sequence

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Motivation

For a scheme X let

$$\mathcal{K}(X) = \Omega \mathrm{BQ}(\mathrm{Vect}_X)$$

be Quillen's K-theory space. It is the loop space of the classifying space of the categorical group completion of the exact category of vector bundles over X. Its homotopy groups are the algebraic K-groups of X.

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For a scheme X let

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For example, if $X = \mathrm{Spec}(R)$ is an affine scheme, $K_0(X)$ is the Grothendieck group of finitely generated projective R-modules.
If $X = \mathrm{Spec}(F)$ is the spectrum of a field,

$$K_0(F) \overset{\dim}{\cong} \mathbb{Z} \quad K_1(F) \cong F^\times \quad K_2(F) \cong F^\times \otimes F^\times / \langle u \otimes (1-u) \rangle.$$

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The only fields for which all K-groups are known explicitly are finite, thanks to Quillen. The situation for topological K-theory is much better. One reason is the Atiyah-Hirzebruch spectral sequence which employs singular cohomology to compute topological K-theory.

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Conjecture (Beilinson 1982)

"I hope very much, that in fact it [motivic cohomology] exists, and may be defined by elementary means. ... One should have Atiyah-Hirzebruch spectral sequence, converging to Quillen's K-theory at least for smooth schemes."

Properties of K-theory

Theorem (Quillen 1972)

If X is regular, the projection $X \times \mathbb{A}^1 \to X$ induces an equivalence $\mathcal{K}(X) \to \mathcal{K}(X \times \mathbb{A}^1)$.

Properties of K-theory

Theorem (Quillen 1972)

If X is regular, the projection $X \times \mathbb{A}^1 \to X$ induces an equivalence $\mathbb{K}(X) \to \mathbb{K}(X \times \mathbb{A}^1)$.

For the next property, a *Nisnevich square* is a pullback square

$$\begin{array}{ccc} p^{-1}(U) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

of schemes, where $U \hookrightarrow X$ is an open embedding and $p\colon Y \to X$ is an étale morphism such that the induced morphism $Y \setminus p^{-1}(U) \to X \setminus U$ of reduced closed subschemes is an isomorphism.

Properties of K-theory

Theorem (Thomason-Trobaugh 1990)

For every Nisnevich square of regular schemes, the square

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{K}(U) \\ \downarrow & & \downarrow \\ \mathcal{K}(Y) & \longrightarrow & \mathcal{K}(p^{-1}(U)) \end{array}$$

is a homotopy pullback square.

This theorem holds for quasi-compact and quasi-separated schemes, provided $\mathcal{K}$ is interpreted as a not necessarily connective spectrum.

The Nisnevich topology

Let $f\colon \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ be the étale morphism induced by the field extension $\mathbb{R} \hookrightarrow \mathbb{C}$, and let $x \in \mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec}\big(\mathbb{R}[t]\big)$ be the closed point given by the prime ideal $(t^2 + 1)$. It has residue field $\mathbb{C}$.

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$$f^{-1}\big(\mathbb{A}^1_{\mathbb{R}} \smallsetminus \{x\}\big) = \mathbb{A}^1_{\mathbb{C}} \smallsetminus f^{-1}(x) = \mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(t+i),(t-i)\}$$

which shows that

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(t+i),(t-i)\} & \longrightarrow & \mathbb{A}^1_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathbb{A}^1_{\mathbb{R}} \smallsetminus \{(t^2+1)\} & \longrightarrow & \mathbb{A}^1_{\mathbb{R}} \end{array}$$

is **not** a Nisnevich square.

The Nisnevich topology

However, the pullback square

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbb{C}} \setminus \{(t+i),(t-i)\} & \longrightarrow & \mathbb{A}^1_{\mathbb{C}} \setminus \{(t-i)\} \\ \downarrow & & \downarrow \\ \mathbb{A}^1_{\mathbb{R}} \setminus \{(t^2+1)\} & \longrightarrow & \mathbb{A}^1_{\mathbb{R}} \end{array}$$

is a Nisnevich square. Both reduced closed complements are Spec($\mathbb{C}$), and f induces the identity.

Why the Nisnevich topology?

The Nisnevich topology (invented by Nisnevich in 1989 as "completely decomposed topology") sits between the Zariski topology and the étale topology. It shares the good properties of both and avoids the bad properties of both.

	Zariski	Nisnevich	étale
smooth implies locally $\mathbb{A}^d$	false	true	true
f_* is exact for f finite	false	true	true
fields are points	true	true	false
cohom. dim. is Krull dim.	true	true	false
K-theory has descent	true	true	false

The motivic stable homotopy category (Morel-Voevodsky)

Let S be a scheme. The motivic stable homotopy category SH(S) of S contains $\mathbb{P}^1$*-spectra* or *motivic spectra* as objects:

- $\mathbf{E} = (\mathbf{E}_0, \mathbf{E}_1, \ldots, \mathbf{E}_n \ldots)$ and structure maps $\mathbf{E}_n \wedge \mathbb{P}^1 \to \mathbf{E}_{n+1}$, where
- $\mathbf{E}_n \colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{sSet}_\bullet$ is a pointed simplicial presheaf on the category Sm_S of smooth S-schemes.
- Any smooth S-scheme with a rational point (like $(\mathbb{P}^1, \infty)$) defines a (representable discrete) pointed simplicial presheaf on Sm_S.
- The smash product of pointed simplicial presheaves is $B \wedge C = B \times C / B \vee C$.

Motivic suspension spectra

Every smooth S-scheme X (which may not have a rational point) defines a motivic suspension spectrum

$$\Sigma^\infty X_+ = (X_+, X_+ \wedge \mathbb{P}^1, X_+ \wedge \mathbb{P}^1 \wedge \mathbb{P}^1, \dots)$$

with identities as structure maps, where $X_+ = X \coprod \mathrm{Spec}(S)$. In particular, $\mathbf{1}_S := \Sigma^\infty S_+$ is the *motivic sphere spectrum over* S.

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$$\Sigma^\infty L = (L, L \wedge \mathbb{P}^1, L \wedge \mathbb{P}^1 \wedge \mathbb{P}^1, \dots)$$

with identities as structure maps, where $L\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{sSet}_\bullet$ is the constant pointed simplicial presheaf with value L. Note that $\Sigma^\infty(S, \mathrm{id}_S) = \Sigma^\infty * = *$ is the zero object in $\mathrm{SH}(S)$.

Equivalences

Equivalences in the motivic stable homotopy category are determined by the following conditions:

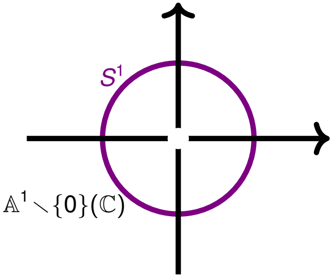
$\mathbb{A}^1$-invariance The projection $\Sigma^\infty(X \times \mathbb{A}^1 \to X)_+$ is an equivalence for every $X \in \mathrm{Sm}_S$.

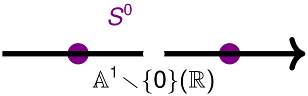
Nisnevich descent Every Nisnevich square of smooth S-schemes induces a homotopy pushout square of $\mathbb{P}^1$-spectra.

$\mathbb{P}^1$-stability The functor $\mathbb{P}^1 \wedge -$ is an equivalence.

Spheres

Let Σ denote suspension with the simplicial circle. $S^1 = \Delta^1/\partial\Delta^1$. Then $\mathbb{A}^1$-invariance and Nisnevich descent imply that $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_{\mathrm{m}}$, where $\mathbb{G}_{\mathrm{m}} = (\mathbb{A}^1 \setminus \{0\}, 1)$. Let $\Sigma^{1+(1)}$ denote suspension with $S^{1+(1)} := \mathbb{P}^1$. Then for every $s, w \in \mathbb{Z}$, there exists $S^{s+(w)} \in \mathrm{SH}(S)$ and the corresponding suspension functor $\Sigma^{s+(w)}$.





For example, $\mathbb{A}^n \setminus \{0\} \simeq S^{n-1+(n)}$, and $\mathbb{A}^n/\mathbb{A}^n \setminus \{0\} \simeq \mathbb{P}^n/\mathbb{P}^{n-1} \simeq S^{n+(n)}$.

Structural properties of $\mathrm{SH}(S)$

The motivic stable homotopy category $\mathrm{SH}(S)$ admits the following structures:

- A closed symmetric monoidal structure $(\mathbf{D}, \mathbf{E}) \mapsto \mathbf{D} \wedge \mathbf{E}$, with unit $\mathbf{1}_S$.
- A compatible triangulated structure, with shift functor Σ and homotopy cofiber sequences defining distinguished triangles.
- A six functor formalism expanding on the base change functor $f^* \colon \mathrm{SH}(R) \to \mathrm{SH}(S)$ for $f \colon S \to R$. The functor f^* is strong symmetric monoidal, always has a right adjoint f_*, and a left adjoint $f_\sharp$ if f is smooth.

Homotopy groups and sheaves

For a $\mathbb{P}^1$-spectrum $\mathbf{E} \in \mathrm{SH}(S)$ and integers s, w, let

$$\pi_{s+(w)}\mathbf{E} := [\Sigma^{s+(w)}\mathbf{1}, \mathbf{E}]$$

and let

$$\pi_{s+(\star)}\mathbf{E} = \bigoplus_{w \in \mathbb{Z}} \pi_{s+(w)}\mathbf{E}$$

denote the direct sum.

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denote the direct sum.
The associated Nisnevich sheaf of $U \mapsto [\Sigma^{s+(w)}U_+, \mathbf{E}]$ for $U \in \mathrm{Sm}_S$ is denoted $\underline{\pi}_{s+(w)}\mathbf{E}$, which gives rise to $\underline{\pi}_{s+(\star)}\mathbf{E}$.
If $S = \mathrm{Spec}(F)$ is the spectrum of a field, $\underline{\pi}_{s+(\star)}\mathbf{E}(F) = \pi_{s+(\star)}\mathbf{E}$.

Algebraic K-theory **KGL**

Voevodsky et. al. constructed a $\mathbb{P}^1$-spectrum

$$\mathbf{KGL} = (\mathcal{K}, \mathcal{K}, \mathcal{K}, \dots)$$

via the structure map $\mathcal{K} \wedge \mathbb{P}^1 \to \mathcal{K}$ which corresponds to multiplication with $[\mathcal{O}] - [\mathcal{O}(-1)]$. If S is regular, it represents Quillen's higher algebraic K-groups:

$$[\Sigma^{p+(q)} X_+, \mathbf{KGL}] \cong K_{p-q}^{\text{Quillen}}(X)$$

Algebraic bordism **MGL**

Let **MGL** be Voevodsky's Thom $\mathbb{P}^1$-spectrum, with $\mathbf{MGL}_n$ the Thom space of the tautological vector bundle over the infinite Grassmannian $\mathrm{Gr}_n = \mathrm{BGL}_n$. The structure maps are the obvious ones. Its "universal" orientation induces a graded ring homomorphism from the Lazard ring:

$$\mathbb{L} = \mathbb{Z}[x_1, x_2, \ldots] \to \pi_{*+(\star)}\mathbf{MGL} \qquad \deg(x_k) = k + (k)$$

Voevodsky's motivic Eilenberg-MacLane spectrum

Let $S = \mathrm{Spec}(F)$ for a field F. For any abelian group A there is a motivic Eilenberg-MacLane spectrum $\mathbf{H}A$ over F, representing motivic cohomology with coefficients in A. In particular, for every smooth F-variety X one has

$$[X_+, \Sigma^{s+(w)}\mathbf{H}A] = H^{s+w,w}(X; A)$$
$$H^{2n,n}(X; \mathbb{Z}) \cong \mathrm{CH}^n(X)$$
$$H^{n,n}(F; \mathbb{Z}) \cong K_n^{\mathrm{Milnor}}(F)$$

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A nice motivic Eilenberg-MacLane spectrum which is invariant under base change exists over any scheme, by pulling back Spitzweck's motivic Eilenberg-MacLane spectrum over $\mathrm{Spec}(\mathbb{Z})$.

Major question?

What is $\pi_{*+(\star)}\mathbf{1}_S$?

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Partial answers by: Morel, Hopkins, Isaksen, Dugger, Levine, Ananyevskiy, Panin, Guillou, Ormsby, Østvær, Heller, Wilson, Bachmann, ...

Ring structure on $\pi_{*+(\star)}\mathbf{1}_S$

Since $\mathbf{1}$ is the unit in a symmetric monoidal category, $\pi_{*+(\star)}\mathbf{1}_S$ admits a ring structure via smash product. It coincides with the ring structure defined by composition.

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$$\alpha \cdot \beta = (-1)^{st} \epsilon^{wx} \beta \cdot \alpha$$

where $\alpha \in \pi_{s+(w)}\mathbf{1}$ and $\beta \in \pi_{t+(x)}\mathbf{1}$.

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$$\alpha \cdot \beta = (-1)^{st}\epsilon^{wx}\beta \cdot \alpha$$

where $\alpha \in \pi_{s+(w)}\mathbf{1}$ and $\beta \in \pi_{t+(x)}\mathbf{1}$. In particular, $\pi_{s+(\star)}\mathbf{E}$ is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ϵ-commutative ring $\pi_{0+(\star)}\mathbf{1}_S$ for every motivic spectrum $\mathbf{E} \in \mathrm{SH}(S)$.

Obvious maps of spheres

Units Let $u \in \mathcal{O}_S^\times$ be a unit. Viewed as a morphism $S \to \mathbb{A}^1 \setminus \{0\}$, it defines a map $[u]\colon \mathbf{1} \to \Sigma^{(1)}\mathbf{1}$, hence $[u] \in \pi_{(-1)}\mathbf{1}$.

Hopf map The canonical morphism $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ defines a map $\eta\colon \Sigma^{1+(2)}\mathbf{1} \to \Sigma^{1+(1)}\mathbf{1}$, hence $\eta \in \pi_{(1)}\mathbf{1}$.

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Not so obvious relations

The following relations hold:

Steinberg For every $u \in \mathcal{O}_S^\times$ with $1 - u \in \mathcal{O}_S^\times$, one has $[u][1-u] = 0 \in \pi_{(-2)}\mathbf{1}$.

Commutativity For every $u \in \mathcal{O}_S^\times$, one has $[u]\eta = \eta[u] \in \pi_0\mathbf{1}$.

Twisted logarithm For every $u, v \in \mathcal{O}_S^\times$, one has $[uv] = [u] + [v] + \eta[u][v] \in \pi_{(-1)}\mathbf{1}$.

Hyperbolic plane $\eta(\eta[-1] + 1) = -\eta \in \pi_{(1)}\mathbf{1}$.

Milnor-Witt K-theory

Let F be a field. Let $K_\star^{\mathrm{MW}}(F)$ denote the $\mathbb{Z}$-graded associative ring generated by elements $[u], u \in F \smallsetminus \{0\}$, of degree 1 and an element η of degree -1, subject to the following relations:

1. $[u] \cdot [1-u] = 0$ for all $u \in F \smallsetminus \{0,1\}$
2. $[u] \cdot \eta = \eta \cdot [u]$ for all $u \in F \smallsetminus \{0\}$:
3. $[uv] = [u] + [v] + \eta \cdot [u] \cdot [v]$ for all $u, v \in F \smallsetminus \{0\}$:
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Moreover, $K_0^{\mathrm{MW}}(F)$ is the Grothendieck-Witt ring of symmetric bilinear forms over F, and $K_n^{\mathrm{MW}}(F)$ is isomorphic to the Witt ring of symmetric bilinear forms over F for $n < 0$.

Partial answers: The zeroth line

$K^{\mathrm{MW}}_{\star}(F)$ is designed to produce a homomorphism

$$K^{\mathrm{MW}}_{\star}(F) \to \pi_{0-(\star)}\mathbf{1}_F.$$

Theorem (Morel)

This homomorphism is an isomorphism

$$K^{\mathrm{MW}}_{n} \xrightarrow{\cong} \pi_{0-(n)}\mathbf{1}$$

of graded rings for any field.

In particular, $\pi_{0+(0)}\mathbf{1}$ is the Grothendieck-Witt ring of symmetric bilinear forms, and $\pi_{0+(n)}\mathbf{1}$ is the Witt ring of symmetric bilinear forms for $n > 0$.

Morel's homotopy t-structure

- For $q \in \mathbb{Z}$, $\mathrm{SH}_{\geq q}(S)$ is the full subcategory of $\mathrm{SH}(S)$ closed under homotopy colimits and extensions, which contains $\Sigma^{q+(n)}\Sigma^{\infty}X_{+}$, $X \in \mathrm{Sm}_S$, $n \in \mathbb{Z}$.
- A $\mathbb{P}^1$-spectrum in $\mathrm{SH}_{\geq q}(X)$ is called q-connective.
- If $S = \mathrm{Spec}(F)$ where F is a field, $\mathbf{E}$ is q-connective if and only if $\underline{\pi}_{s+(\star)}\mathbf{E} = 0$ for all $s < q$.

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1 and **MGL** are connective (0-connective). Over a field, $\mathbf{H}\mathbb{Z}$ is connective. For all $q \in \mathbb{Z}$, $\pi_{q+(q)}\mathbf{KGL}_F = \mathbb{Z}$, whence **KGL** is not q-connective.

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The homotopy t-structure is exhaustive and Hausdorff.

Voevodsky's slice filtration

- For $q \in \mathbb{Z}$, $\Sigma^{(q)}\,\mathrm{SH}^{\mathrm{eff}}(S)$ is the full subcategory of $\mathrm{SH}(S)$ closed under homotopy colimits and extensions, which contains $\Sigma^{n+(q)}\Sigma^{\infty}X_{+}$, $X \in \mathrm{Sm}_S$, $n \in \mathbb{Z}$.
- $i_q\colon \Sigma^{(q)}\,\mathrm{SH}^{\mathrm{eff}}(S) \hookrightarrow \mathrm{SH}(S)$ has a right adjoint, denoted
- $r_q\colon\ \mathrm{SH}(S) \to \Sigma^{(q)}\,\mathrm{SH}^{\mathrm{eff}}(S)$ (Neeman)
- $\mathrm{f}_q := i_q \circ r_q\colon\ \mathrm{SH}(S) \to \mathrm{SH}(S)$

Motivic spectra in $\Sigma^{(q)}\,\mathrm{SH}^{\mathrm{eff}}(S)$ are called q-effective.

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- $\mathsf{f}_q := i_q \circ r_q\colon\ \mathrm{SH}(S) \to \mathrm{SH}(S)$

Motivic spectra in $\Sigma^{(q)}\,\mathrm{SH}^{\mathrm{eff}}(S)$ are called q-effective. Both $\mathbf{1}$ and **MGL** are effective, but **KGL** is not q-effective for any $q \in \mathbb{Z}$. Any $\mathbf{E} \in \mathrm{SH}(F)$ induces a natural homotopy cofiber sequence

$$\mathsf{f}_{q+1}\mathbf{E} \to \mathsf{f}_q\mathbf{E} \to \mathsf{s}_q\mathbf{E} \to \Sigma\mathsf{f}_{q+1}\mathbf{E}$$

defining the q-th *slice* $\mathsf{s}_q\mathbf{E}$ of $\mathbf{E}$.

Voevodsky's slice filtration

The slice filtration is triangulated and exhaustive. However, it is not Hausdorff: Let F be of characteristic not two, and let

$$\tau := -1 \in \pi_{0-(1)}\mathbf{H}\mathbb{Z}/2 = \ker\big(F^\times \xrightarrow{x \mapsto x^2} F^\times\big).$$

Then $* \neq \mathbf{H}\mathbb{Z}/2[\tau^{-1}] \in \Sigma^{q+(q)}\,\mathrm{SH}^{\mathrm{eff}}(F)$ for all $q \in \mathbb{Z}$.

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Then $* \neq \mathbf{H}\mathbb{Z}/2[\tau^{-1}] \in \Sigma^{q+(q)}\,\mathrm{SH}^{\mathrm{eff}}(F)$ for all $q \in \mathbb{Z}$. This $\mathbb{P}^1$-spectrum is not q-connective for any q. On the other hand, the η-inverted motivic sphere spectrum $\mathbf{1}[\eta^{-1}]$ is connective, but not q-effective for any $q \in \mathbb{Z}$.

Slices and smash product

Theorem (Pelaez 2008, Gutiérrez-R.-Spitzweck-Østvær 2010)

The slice filtration is multiplicative: There are natural pairings

$$\mathsf{f}_p\mathbf{D} \wedge \mathsf{f}_q\mathbf{E} \to \mathsf{f}_{p+q}(\mathbf{D} \wedge \mathbf{E}) \quad \text{and} \quad \mathsf{s}_p\mathbf{D} \wedge \mathsf{s}_q\mathbf{E} \to \mathsf{s}_{p+q}(\mathbf{D} \wedge \mathbf{E}).$$

In particular, if $\mathbf{E}$ *is a motivic ring spectrum, so are* $\mathsf{f}_0\mathbf{E}$ *and* $\mathsf{s}_0\mathbf{E}$*, and* $\mathsf{f}_*\mathbf{E}$ *and* $\mathsf{s}_*\mathbf{E}$ *are graded motivic ring spectra.*

Every motivic spectrum is a module over $\mathbf{1}$, whence every slice is a module over $\mathsf{s}_0\mathbf{1}$.

Spitzweck's very effective slice filtration

- For $q \in \mathbb{Z}$, $\Sigma^{q+(q)}\,\mathrm{SH}^{\mathrm{veff}}(S)$ is the full subcategory of $\mathrm{SH}(S)$ closed under homotopy colimits and extensions, which contains $\Sigma^{q+(q)}\Sigma^\infty X_+$, $X \in \mathrm{Sm}_S$.
- $vi_q \colon \Sigma^{q+(q)}\,\mathrm{SH}^{\mathrm{veff}}(F) \hookrightarrow \mathrm{SH}(F)$ has a right adjoint, denoted
- $vr_q \colon\ \mathrm{SH}(F) \to \Sigma^{q+(q)}\,\mathrm{SH}^{\mathrm{veff}}(F)$ (Neeman)
- $\mathrm{vf}_q := vi_q \circ vr_q \colon\ \mathrm{SH}(F) \to \mathrm{SH}(F)$

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- $\mathrm{vf}_q := vi_q \circ vr_q \colon\ \mathrm{SH}(F) \to \mathrm{SH}(F)$

Motivic spectra in $\Sigma^{q+(q)}\,\mathrm{SH}^{\mathrm{veff}}(S)$ are called very q-effective. Both **1** and **MGL** are very effective. The effective motivic spectrum $\Sigma^{-1}\mathbf{1}$ is not very effective, but very -1-effective.

Spitzweck's very effective slice filtration

Again any $\mathbf{E} \in \mathrm{SH}(F)$ yields a natural homotopy cofiber sequence

$$\mathrm{vf}_{q+1}\mathbf{E} \to \mathrm{vf}_q\mathbf{E} \to \mathrm{vs}_q\mathbf{E} \to \Sigma\mathrm{vf}_{q+1}\mathbf{E}$$

defining the q-th *very effective slice* of $\mathbf{E}$. The very effective slice filtration is exhaustive and Hausdorff. However, it is not triangulated. In particular, the very effective slices are often harder to determine than the slices.

The slice spectral sequence

For every $\mathbf{E} \in \mathrm{SH}(F)$ and every integer n there is a slice spectral sequence

$$\pi_{p+(n)}\mathsf{s}_q(\mathbf{E}) \Longrightarrow \pi_{p+(n)}\mathbf{E}$$

which might converge strongly to the exhaustive filtration

$$\mathrm{Image}\big(\pi_{p+(n)}\mathsf{f}_q(\mathbf{E}) \to \pi_{p+(n)}\mathbf{E}\big).$$

The first differential is (induced by) the following composition:

$$\mathsf{s}_q\mathbf{E} \to \Sigma\mathsf{f}_{q+1}\mathbf{E} \to \Sigma\mathsf{s}_{q+1}\mathbf{E}$$

The motivic Atiyah-Hirzebruch spectral sequence

The next lecture will provide that the slice spectral sequence for **KGL** is the motivic Atiyah-Hirzebruch spectral sequence Beilinson hoped for. It will also give information on the slice spectral sequence for the motivic sphere spectrum.

Three filtrations

For a set M of objects in $\mathrm{SH}(S)$, let $\langle M \rangle$ denote the full subcategory of $\mathrm{SH}(S)$ closed under homotopy colimits and extensions containing M. Fix $q \in \mathbb{Z}$.

q-connective	q-effective	very q-effective
$\langle \Sigma^{q+(n)} \Sigma^\infty X_+ \rangle^{n \in \mathbb{Z}}_{X \in \mathrm{Sm}_S}$	$\langle \Sigma^{n+(q)} \Sigma^\infty X_+ \rangle^{n \in \mathbb{Z}}_{X \in \mathrm{Sm}_S}$	$\langle \Sigma^{q+(q)} \Sigma^\infty X_+ \rangle_{X \in \mathrm{Sm}_S}$
Hausdorff	**not** Hausdorff	Hausdorff
not triangulated	triangulated	**not** triangulated
$\Sigma^q \mathbf{1}[\eta^{-1}]$	$\mathbf{H}\mathbb{Z}/2[\tau^{-1}]$	
$\mathbf{E}_{\geq q}$	$\mathrm{f}_q \mathbf{E}$	$\mathrm{vf}_q \mathbf{E}$
$EM(\underline{\pi}_{q+(\star)} \mathbf{E})$	$\mathrm{s}_q \mathbf{E}$	$\mathrm{vs}_q \mathbf{E}$

A very q-effective motivic spectrum is q-effective and q-connective.

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More generally, one may insert a motivic spectrum **D** to obtain

$$[\mathbf{D}, \mathsf{s}_q\mathbf{E}] \Longrightarrow [\mathbf{D}, \mathbf{E}].$$

The first differential is (induced by) the following composition:

$$\mathsf{s}_q\mathbf{E} \to \Sigma\mathsf{f}_{q+1}\mathbf{E} \to \Sigma\mathsf{s}_{q+1}\mathbf{E}$$

The slice spectral sequence

The first differential

$$\pi_{p+(\star)}\mathsf{s}_q\mathbf{E} \to \pi_{p-1+(\star)}\mathsf{s}_{q+1}\mathbf{E}$$

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is a $\pi_{0-(\star)}\mathbf{1}_F \cong K_\star^{\mathrm{MW}}(F)$-module homomorphism.
(The identification of $\mathsf{s}_0\mathbf{1}$ will imply that all differentials in the slice spectral sequence are even $K_\star^{\mathrm{Milnor}}(F)$-homomorphisms.)

The motivic Atiyah-Hirzebruch spectral sequence

Recall **KGL** $= (\mathcal{K}, \mathcal{K}, \dots)$, so $\mathbb{P}^1 \wedge$ **KGL** $\simeq$ **KGL** (Bott periodicity).

Theorem (Levine 2005, Voevodsky 2000)

Let F be a field. Then $\mathsf{s}_0\mathbf{KGL} \simeq \mathbf{H}\mathbb{Z}$ *in* $\mathrm{SH}(F)$. *In particular,* $\mathsf{s}_q\mathbf{KGL} \simeq \Sigma^{q+(q)}\mathbf{H}\mathbb{Z}$ *for all* $q \in \mathbb{Z}$ *by Bott periodicity. The associated slice spectral sequence converges strongly.*

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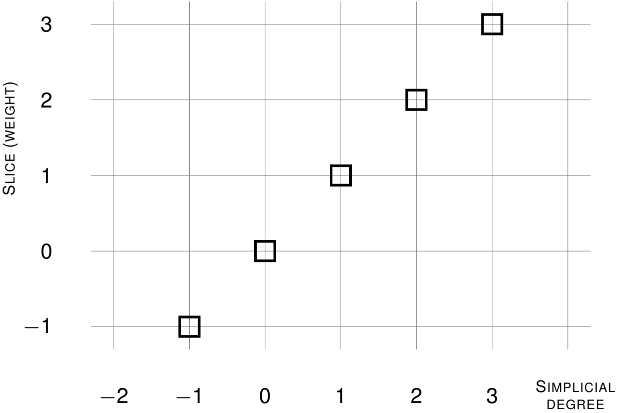
Levine's proof uses his homotopy coniveau filtration. Note that

$$\mathsf{f}_{q+1}(\mathbb{P}^1 \wedge \mathbf{E}) \simeq \mathbb{P}^1 \wedge \mathsf{f}_q(\mathbf{E}) \quad \text{and} \quad \mathsf{s}_{q+1}(\mathbb{P}^1 \wedge \mathbf{E}) \simeq \mathbb{P}^1 \wedge \mathsf{s}_q(\mathbf{E}).$$

Slices of **KGL** compare well with the homotopy groups of the corresponding topological spectrum KU.

A picture of $s_*\mathbf{KGL}$

Squares denote suspensions of $\mathbf{H}\mathbb{Z}$.



Slices of motivic cohomology

Motivic cohomology vanishes in negative weights, thus $\mathsf{f}_1\mathbf{H}\mathbb{Z} \simeq *$. Hence if $\mathbf{H}\mathbb{Z}$ is effective, the maps

$$\mathbf{H}\mathbb{Z} \leftarrow \mathsf{f}_0\mathbf{H}\mathbb{Z} \rightarrow \mathsf{s}_0\mathbf{H}\mathbb{Z}$$

are isomorphisms. Over a field of characteristic zero, effectivity can be shown by expressing $\mathbf{H}\mathbb{Z}$ via infinite symmetric products of spheres.

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For every field F, the motivic spectrum $\mathbf{H}\mathbb{Z}$ coincides with its zero slice.

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Theorem (Voevodsky 2003, Levine 2005)

For every field F, the motivic spectrum $\mathbf{H}\mathbb{Z}$ coincides with its zero slice.

Work of Spitzweck and Bachmann-Hoyois generalizes this to any Dedekind domain.
Slices of $\mathbf{H}\mathbb{Z}$ compare well with the homotopy groups of the corresponding topological Eilenberg-MacLane spectrum $\mathrm{H}\mathbb{Z}$.

Algebraic bordism **MGL**

Let **MGL** be Voevodsky's motivic Thom spectrum. Its "universal" orientation induces a graded ring homomorphism:

$$\mathbb{L} = \mathbb{Z}[x_1, x_2, \dots] \to \pi_{*+(\star)}\mathbf{MGL} \qquad \deg(x_k) = k + (k)$$

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induced by the canonical orientation of $\mathbf{H}\mathbb{Z}$ *is an equivalence. The same is true over any field, at least after inverting the exponential characteristic e.*

Sketch of proof of Theorem (Hopkins-Morel, Hoyois)

- $\Phi \wedge \mathbf{H}\mathbb{Z}$ is an equivalence, because
  - $\Phi \wedge \mathbf{H}\mathbb{Q}$ is an equivalence by motivic Landweber exactness
  - $\Phi \wedge \mathbf{H}\mathbb{Z}/\ell$ is an equivalence by motivic Steenrod algebra computation, cellularity, "motives are $\mathbf{H}\mathbb{Z}$-modules", provided $\ell \neq e$ is prime.
- $\mathbf{MGL}/(x_1, x_2, \dots)$ is $\mathbf{H}\mathbb{Z}$-local, because
  - $\mathbf{MGL}_{\leq 0} \to \mathbf{H}\mathbb{Z}_{\leq 0}$ is an equivalence, the map on nontrivial homotopy sheaves being $\underline{\pi}_{0-(\star)}\mathbf{MGL} \cong \underline{K}_{\star}^{\text{Milnor}} \cong \underline{\pi}_{0-(\star)}\mathbf{H}\mathbb{Z}$
  - homotopy t-structure truncations of $\mathbf{MGL}/(x_1, x_2, \dots)$ are $\mathbf{MGL}_{\leq 0}$-local by GRSØ.

Slices of **MGL**

Consequences for a field of exponential characteristic e:

- $\mathbf{H}\mathbb{Z}[e^{-1}]$ is (very) effective
- $\mathsf{s}_0\mathbf{1}[e^{-1}] = \mathsf{s}_0\mathbf{MGL}[e^{-1}] = \mathbf{H}\mathbb{Z}[e^{-1}]$
- $\mathsf{s}_q\mathbf{MGL}[e^{-1}] \cong \Sigma^{q+(q)}\mathbf{H}\mathbb{Z}[e^{-1}] \otimes \mathbb{L}_q$ for all q
- $\mathsf{vf}_q\mathbf{MGL} = \mathsf{f}_q\mathbf{MGL}$ and $\mathsf{vs}_q\mathbf{MGL} = \mathsf{s}_q\mathbf{MGL}$ for all q

Slices of **MGL** compare well with the homotopy groups of the corresponding topological spectrum MU.

Slices of **MGL**

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The identification $\mathsf{s}_0(\mathbf{1}) \simeq \mathsf{s}_0(\mathbf{H}\mathbb{Z}) \simeq \mathbf{H}\mathbb{Z}$ holds over a Dedekind domain (Bachmann-Hoyois).

Slices of $\mathbf{MGL}$

Consequences for a field of exponential characteristic e:

- $\mathbf{H}\mathbb{Z}[e^{-1}]$ is (very) effective
- $s_0\mathbf{1}[e^{-1}] = s_0\mathbf{MGL}[e^{-1}] = \mathbf{H}\mathbb{Z}[e^{-1}]$
- $s_q\mathbf{MGL}[e^{-1}] \cong \Sigma^{q+(q)}\mathbf{H}\mathbb{Z}[e^{-1}] \otimes \mathbb{L}_q$ for all q
- $vf_q\mathbf{MGL} = f_q\mathbf{MGL}$ and $vs_q\mathbf{MGL} = s_q\mathbf{MGL}$ for all q

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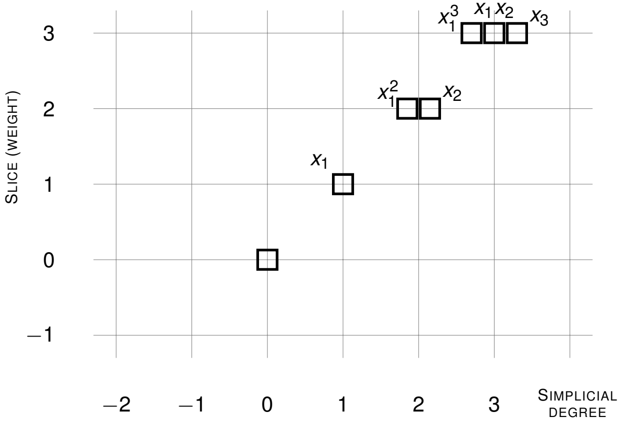
The identification $s_0(\mathbf{1}) \simeq s_0(\mathbf{H}\mathbb{Z}) \simeq \mathbf{H}\mathbb{Z}$ holds over a Dedekind domain (Bachmann-Hoyois).

Convention: From now on, $[e^{-1}]$ may be removed from the notation. Hence over a field of positive characteristic e, we may implicitly invert e.

A picture of s_*MGL

- Squares denote suspensions of $\mathbf{H}\mathbb{Z}$.
- s_q**MGL** is the sum of all "dots" on the q-th horizontal line.



Slices of **KGL** via **MGL**

Let **kgl** $:= \mathrm{vf}_0$**KGL**. The canonical orientation on **KGL** defines a map **MGL** $\to$ **KGL** which factors over **kgl**. Over a field one obtains equivalences

$$\mathbf{kgl} \simeq \mathbf{MGL}/(x_2, x_3, \ldots) \quad \text{and} \quad \mathbf{KGL} \simeq \mathbf{MGL}/(x_2, x_3, \ldots)[x_1^{-1}]$$

at least after inverting its exponential characteristic e.

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at least after inverting its exponential characteristic e.

Consequences:

- $\mathrm{vf}_q\mathbf{KGL} = \mathrm{f}_q\mathbf{KGL} = \Sigma^{q+(q)}\mathbf{kgl}$
- $\mathrm{vs}_q\mathbf{KGL} = \mathrm{s}_q\mathbf{KGL} = \Sigma^{q+(q)}\mathbf{H}\mathbb{Z}$ for all q
- $\operatorname{holim}_q \mathrm{f}_q\mathbf{KGL} \simeq *$ (slice convergence)

Slices of the motivic sphere spectrum $\mathbf{1}$

The **MGL**-Adams resolution of **1** computes its slices.

$$\mathbf{1} \longrightarrow \mathbf{MGL} \rightrightarrows \mathbf{MGL} \wedge \mathbf{MGL} \rightrightarrows \mathbf{MGL} \wedge \mathbf{MGL} \wedge \mathbf{MGL} \cdots$$

Lemma

The canonical map $s_q\mathbf{1} \to \operatorname{holim}_{\Delta} s_q\mathbf{MGL}^{\wedge\bullet+1}$ *is an equivalence.*

Slices of the motivic sphere spectrum $\mathbf{1}$

Lemma

The canonical map $\mathsf{s}_q\mathbf{1} \to \operatorname{holim}_{\Delta} \mathsf{s}_q\mathbf{MGL}^{\wedge\bullet+1}$ *is an equivalence.*

Proof.

Consider $\phi(\mathbf{E}, q) \colon \mathsf{s}_q\mathbf{E} \to \operatorname{holim}_{\Delta} \mathsf{s}_q(\mathbf{E} \wedge \mathbf{MGL}^{\wedge\bullet+1})$, which is an equivalence if $\mathbf{E}$ is an $\mathbf{MGL}$-module. In the homotopy cofiber sequence

$$\mathbf{1} \to \mathbf{MGL} \to \overline{\mathbf{MGL}} \to \Sigma\mathbf{1}$$

$\overline{\mathbf{MGL}}$ is 1-effective, whence $\overline{\mathbf{MGL}}^{\wedge m}$ is m-effective. The map $\phi(\overline{\mathbf{MGL}}^{\wedge m}, q)$ is an equivalence for $q < m$. Since s_q is a triangulated functor, downward induction on m applies. □

Slices of the motivic sphere spectrum $\mathbf{1}$

Theorem (Voevodsky-Levine, RSØ)

$$\mathsf{s}_q(\mathbf{1}) \simeq \Sigma^{q+(q)}\mathbf{H}\mathbb{Z} \otimes \bigoplus_{p\in\mathbb{Z}} \mathrm{Ext}^{p-q,2q}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*)$$

This identification is compatible with the canonical multiplicative structures on each side.

Sketch of proof:

$$\begin{aligned}
\mathsf{s}_q\mathbf{1} &\simeq \operatorname*{holim}_{\Delta} \mathsf{s}_q\mathbf{MGL}^{\wedge\bullet+1} \\
&\simeq \operatorname*{holim}_{\Delta} \Sigma^{q+(q)}\mathbf{H}\mathbb{Z} \otimes \pi_{2q}\mathrm{MU}^{\wedge\bullet+1} \\
&\simeq \Sigma^{q+(q)}\mathbf{H}\mathbb{Z} \otimes \mathrm{Tot}\left(\pi_{2q}\mathrm{MU}^{\wedge\bullet+1}\right) \\
&\simeq \Sigma^{q+(q)}\mathbf{H}\mathbb{Z} \otimes \bigoplus_{p\in\mathbb{Z}} \mathrm{Ext}^{p-q,2q}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*)
\end{aligned}$$

Slices of the motivic sphere spectrum $\mathbf{1}$

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Slices of $\mathbf{1}$ do **not** compare well with the homotopy groups of the corresponding topological spectrum $\mathbb{S}$.

Slices of the motivic sphere spectrum **1**

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This identification is compatible with the canonical multiplicative structures on each side.

Slices of **1** do **not** compare well with the homotopy groups of the corresponding topological spectrum $\mathbb{S}$.
Since homotopy theorists (Zahler, Ravenel, ...) computed $\mathrm{Ext}^{s,t}_{\mathsf{MU}_*\mathsf{MU}}(\mathsf{MU}_*, \mathsf{MU}_*)$ in a certain range, small slices $\mathsf{s}_q(\mathbf{1})$ are known explicitly, and some summands are known in all slices.

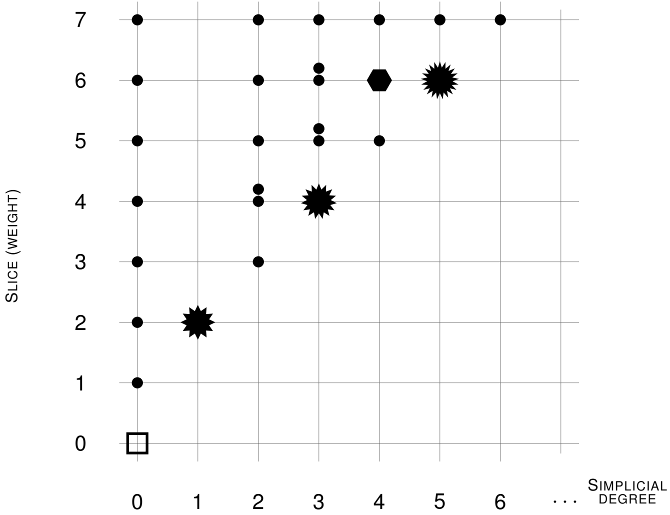
Zahler: The ANSS for the spheres

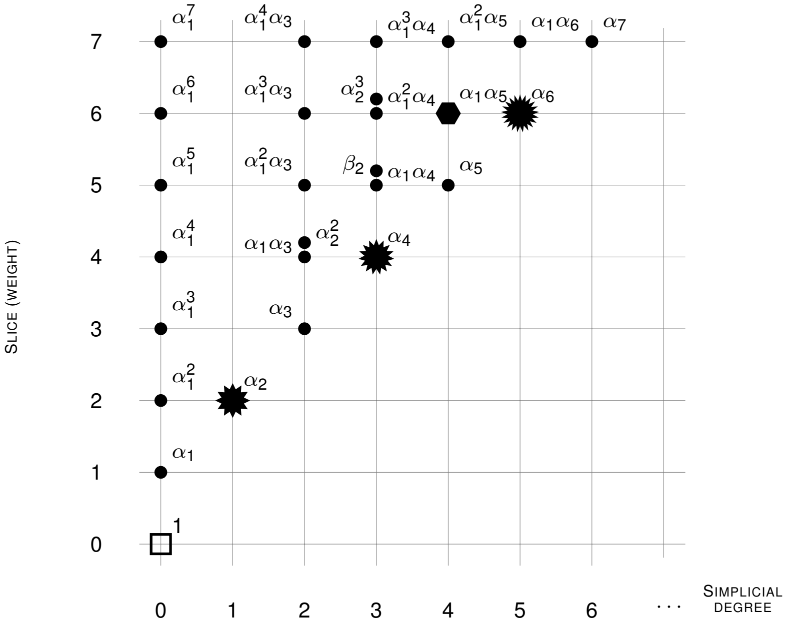
Table 2. Spectral sequences for $_2\pi_*^S$

14		$_2E_2^{*,*}$ (Novikov Sequence)														Z_2		0		Z_2		Z_2	
13														Z_2		0		Z_2		Z_2		Z_2
12													Z_2		0		Z_2		Z_2		Z_2	
11												Z_2		0		Z_2		Z_2		Z_2		Z_2
10											Z_2		0		Z_2		Z_2		Z_2		Z_2	
9										Z_2		0		Z_2		Z_2		Z_2		Z_2		Z_2
8									Z_2		0		Z_2		Z_2		Z_2		Z_2		Z_2	
7								Z_2		0		Z_2		Z_2		Z_2		Z_2		Z_2		
6							Z_2		0		Z_2		Z_2		Z_2		Z_2		Z_2			
5						Z_2		0		Z_2		Z_2		Z_2		Z_2		Z_2				
4					Z_2		0		Z_2		Z_2		Z_2		Z_2		Z_2					
3				Z_2		0		Z_2		$2Z_2$		Z_2		Z_2		$2Z_2$		$3Z_2$				
↑ 2			Z_2		0	d_3	$2Z_2$		$2Z_2$		Z_2	d_3	Z_2		$3Z_2$		$2Z_2$		$Z_2 + Z_4$			
s 1		Z_2		Z_4		Z_2		Z_{16}		Z_2		Z_8		Z_2		Z_{32}		Z_2		Z_8		Z_2
0	Z																					
$t-s$ →	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$_2\pi^S_{t-s}$	Z	Z_2	Z_2	Z_8	0	0	Z_2	Z_{16}	$2Z_2$	$3Z_2$	Z_2	Z_8	0	0	$2Z_2$	$Z_{32}+Z_2$	$2Z_2$	$4Z_2$	Z_8+Z_2	Z_8+Z_2	Z_8	$2Z_2$

A picture of $s_q\mathbf{1}$ for $q \leq 7$

Square: $\mathbf{H}\mathbb{Z}$, small circle: $\mathbf{H}\mathbb{Z}/2$

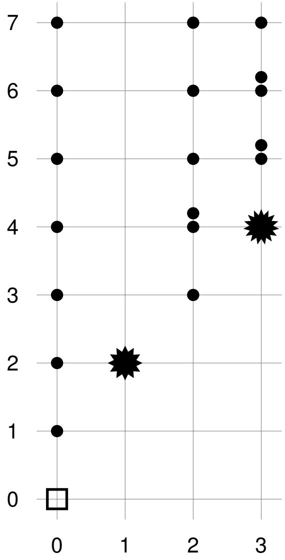



SLICE (WEIGHT)
SIMPLICIAL DEGREE
0 1 2 3 4 5 6 ...
0 1 2 3 4 5 6 7
1
α_1
α_1^2
α_2
α_1^3
α_3
α_1^4
$\alpha_1\alpha_3$
α_2^2
α_4
α_1^5
$\alpha_1^2\alpha_3$
β_2
$\alpha_1\alpha_4$
α_5
α_1^6
$\alpha_1^3\alpha_3$
α_2^3
$\alpha_1^2\alpha_4$
$\alpha_1\alpha_5$
α_6
α_1^7
$\alpha_1^4\alpha_3$
$\alpha_1^3\alpha_4$
$\alpha_1^2\alpha_5$
$\alpha_1\alpha_6$
α_7

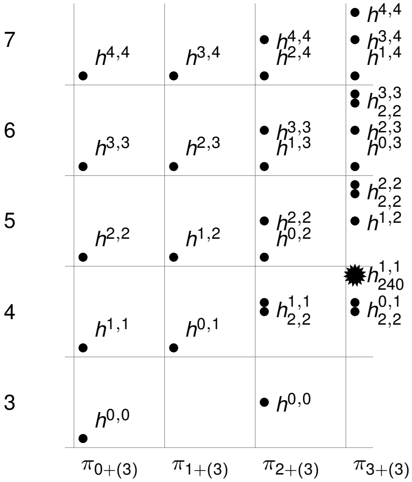
The slice spectral sequences of **1**

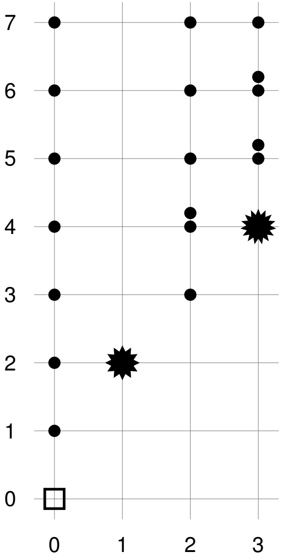
The following abbreviations are used:

- Integral motivic cohomology: $H^{s,w} = \pi_{w-s+(-w)}\mathbf{H}\mathbb{Z}$
- If $\ell > 2$ is a natural number, $h_{\ell}^{s,w} = (\mathbf{H}\mathbb{Z}/\ell)^{s,w}(F)$ denotes motivic cohomology with coefficients in $\mathbb{Z}/\ell$.
- $h^{s,w} = (\mathbf{H}\mathbb{Z}/2)^{s,w}(F)$
- $h_{2,2}^{s,w} = \left(\mathbf{H}(\mathbb{Z}/2)^{2}\right)^{s,t}(F)$
- Recall $\tau = -1 \in h^{0,1}$ and $\rho = \overline{-1} \in h^{1,1}$

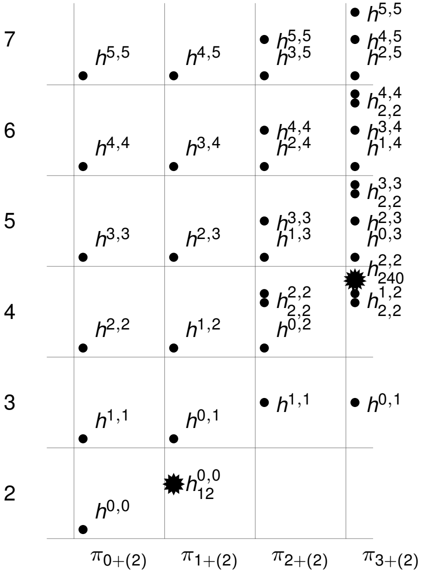


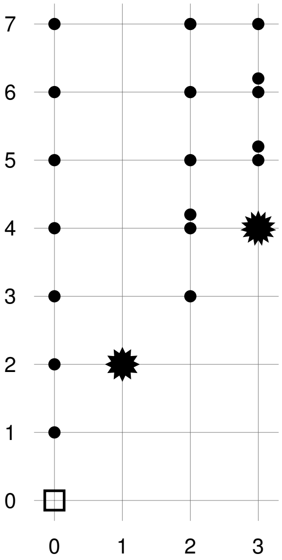
$\pi_{k+(3)} s_q \mathbf{1}$



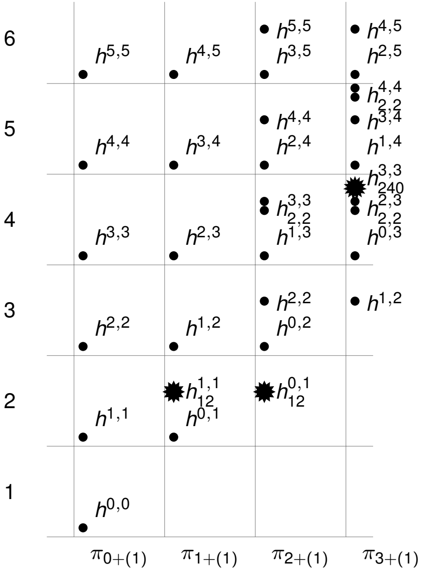


$\pi_{k+(2)} \mathbf{s}_q \mathbf{1}$

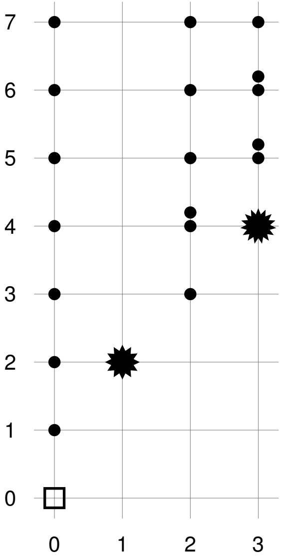


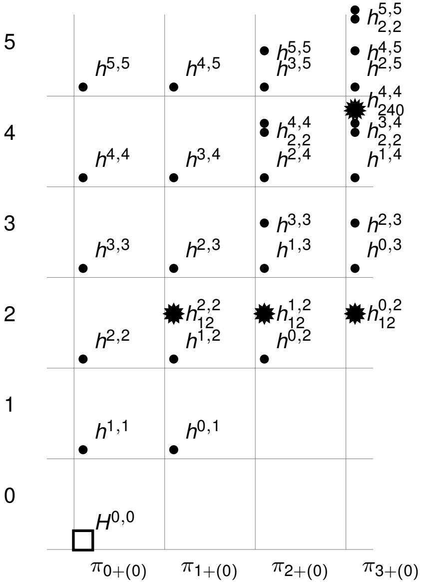


$\pi_{k+(1)} \mathrm{S}_q \mathbf{1}$

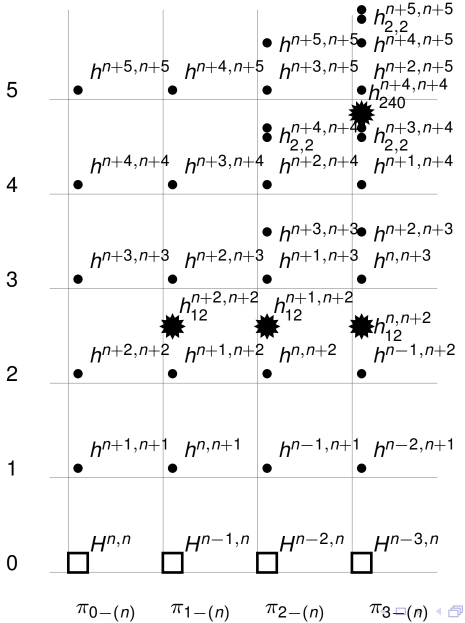


$\pi_{k+(0)} \mathbf{S}_q \mathbf{1}$





The $-n$-th slice spectral sequence for **1**



Hermitian K-theory **KQ**

From now on, F is a field of characteristic not two. Let **KQ** be Hornbostel's hermitian K-theory spectrum. It is a motivic ring spectrum with the following properties:

- There is a homotopy cofiber sequence:

$$\Sigma^{(1)}\mathbf{KQ} \xrightarrow{\eta\wedge\mathbf{KQ}} \mathbf{KQ} \xrightarrow{u} \mathbf{KGL} \xrightarrow{v} \Sigma^{1+(1)}\mathbf{KQ}$$

- The forgetful map u is a ring map, and v factors as $\mathbf{KGL} \xrightarrow{\beta} \Sigma^{1+(1)}\mathbf{KGL} \xrightarrow{\Sigma^{1+(1)}h} \Sigma^{1+(1)}\mathbf{KQ}$, h the hyperbolic map.

Slices of **KQ**

These properties and the determination of $s_*\mathbf{KGL}$ imply:

Theorem (R.-Østvær 2013)

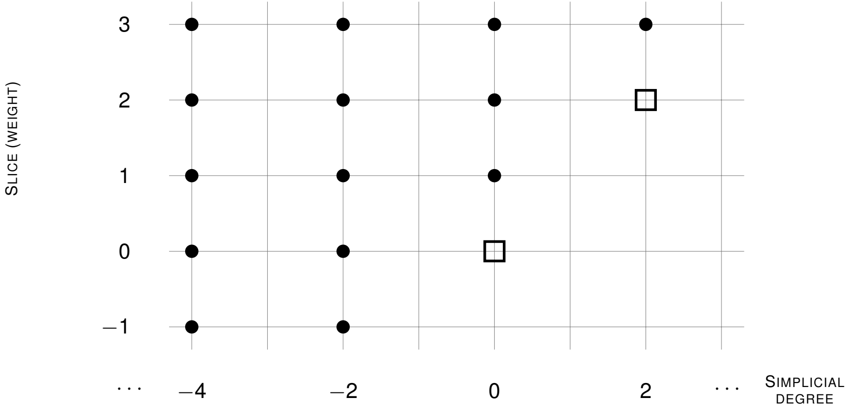
Let F be a field of $\mathrm{char}(F) \neq 2$ *and $q \in \mathbb{Z}$. The q-th slice of the hermitian K-theory spectrum* **KQ** *over F is given as*

$$\Sigma^{(q)} \begin{cases} (\Sigma^q \mathbf{H}\mathbb{Z}) \vee \bigvee_{i<\frac{q}{2}} \Sigma^{2i}\mathbf{H}\mathbb{Z}/2 & q \equiv 0 \bmod 2 \\ \bigvee_{i<\frac{q+1}{2}} \Sigma^{2i}\mathbf{H}\mathbb{Z}/2 & q \equiv 1 \bmod 2 \end{cases}$$

Slices of **KQ** do **not** compare well with the homotopy groups of the corresponding topological spectrum KO.

A picture of $s_*\mathbf{KQ}$

Small circles denote suspensions of $\mathbf{H}\mathbb{Z}/2$.



The motivic Steenrod algebra

Recall that the first slice differential is (induced by) the following composition:

$$\mathsf{s}_q\mathbf{E} \to \Sigma \mathsf{f}_{q+1}\mathbf{E} \to \Sigma \mathsf{s}_{q+1}\mathbf{E}$$

In the previous examples all slices are (sums of) motivic Eilenberg-MacLane spectra.

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For every prime ℓ, Voevodsky determined all endomorphisms of $\mathbf{H}\mathbb{Z}/\ell$ in SH(F) over a field F with char(F) $= 0$. This was extended to char(F) $\neq \ell$ by Hoyois-Kelly-Østvær, and to Dedekind rings having ℓ as a unit by Spitzweck.

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The following arguments require endomorphisms of weight ≤ 2.

The first differential for **KGL**/2

$$
\begin{array}{ccc}
\mathsf{s}_q\mathbf{KGL}/2 & \xrightarrow{d^1_q(\mathbf{KGL}/2)} & \Sigma\mathsf{s}_{q+1}\mathbf{KGL}/2 \\
\cong\downarrow & & \downarrow\cong \\
\Sigma^{q+(q)}\mathbf{H}\mathbb{Z}/2 & \xrightarrow{\mathsf{Sq}^2\,\mathsf{Sq}^1+\mathsf{Sq}^1\,\mathsf{Sq}^2} & \Sigma^{q+2+(q+1)}\mathbf{H}\mathbb{Z}/2
\end{array}
$$

- Bott periodicity for **KGL**
- Voevodsky's motivic Steenrod algebra
- Motivic Adem relations
- Slice convergence for **KGL**
- Suslin's computation of $\pi_{s+(w)}\mathbf{KGL}/2$ over $\mathbb{R}$
- Base change from $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$

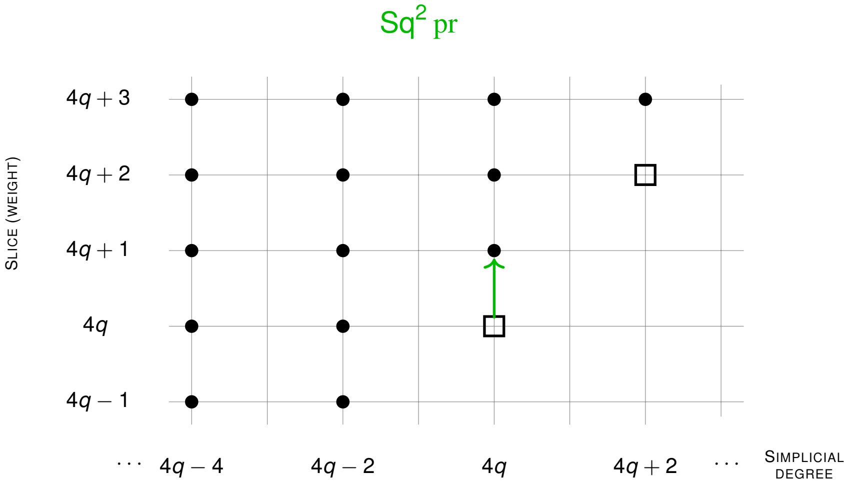
The first differential for **KQ**

$\mathbf{1} \to \mathbf{KQ} \xrightarrow{\text{forget}} \mathbf{KGL}$ is a ring map and s_0 preserves ring maps.

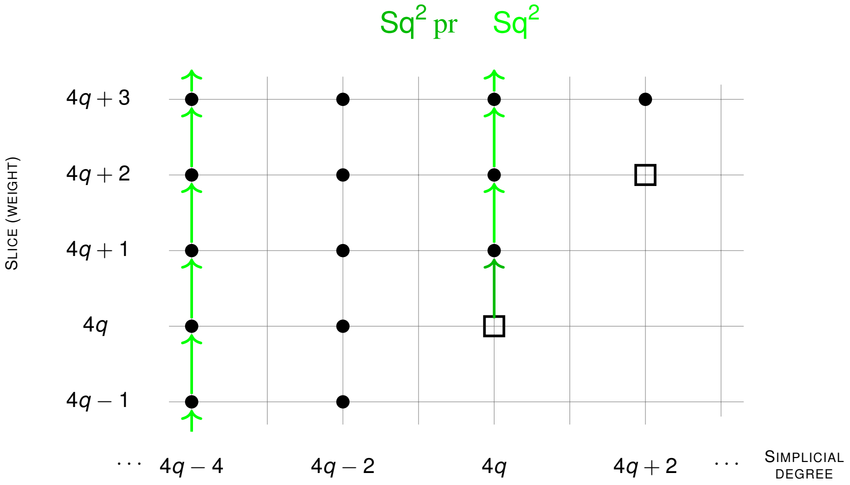
$$\begin{array}{ccccc} s_0\mathbf{KQ} & \xrightarrow{\text{pr}} & \mathbf{H}\mathbb{Z} = s_0\mathbf{KGL} & \xrightarrow{\text{pr}} & s_0\mathbf{KGL}/2 \\ \downarrow d_0^1(\mathbf{KQ}) & & \downarrow \mathrm{Sq}^2 \circ \text{pr} & & \downarrow d^1(\mathbf{KGL}/2) \\ \Sigma s_1\mathbf{1} & \xrightarrow{\text{pr}} & \Sigma^{(1)}\mathbf{H}\mathbb{Z}/2 & \xrightarrow{\mathrm{Sq}^1} & \Sigma s_1\mathbf{KGL}/2 \end{array}$$

Hence $d_0^1(\mathbf{KQ})$ restricted to $\mathbf{H}\mathbb{Z}$ is $\mathrm{Sq}^2 \circ \text{pr}$.

The first differential for **KQ**



The first differential for **KQ**



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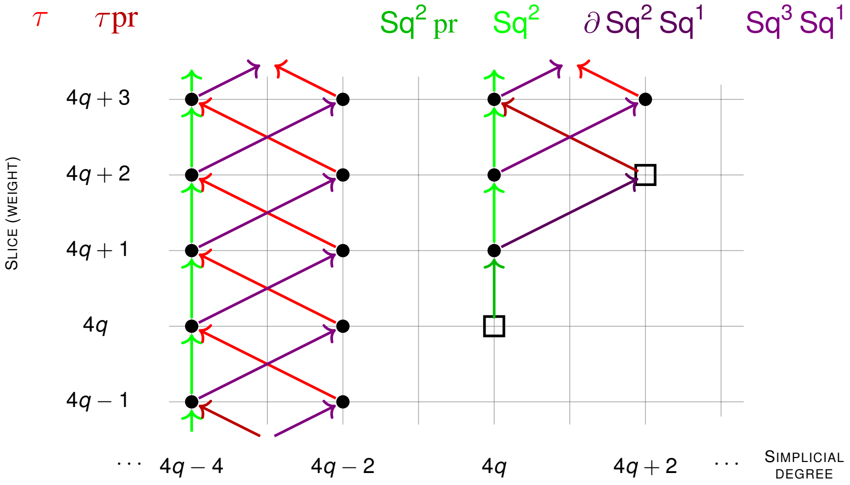
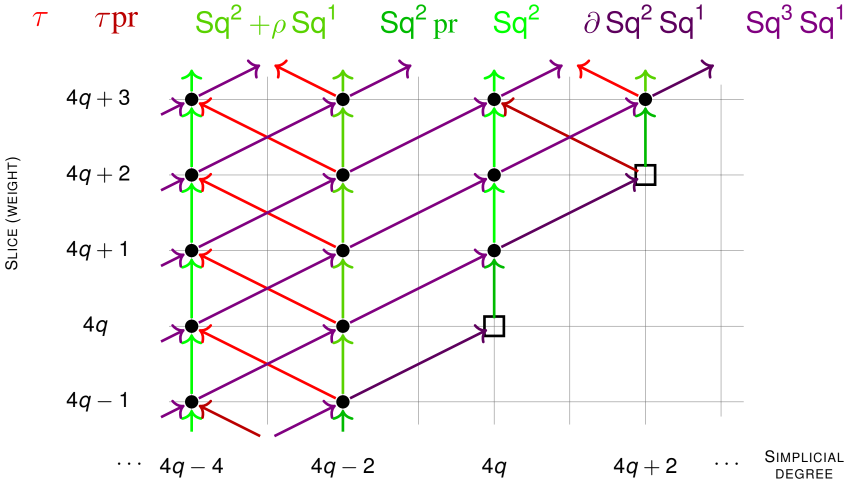


The first differential for **KQ**



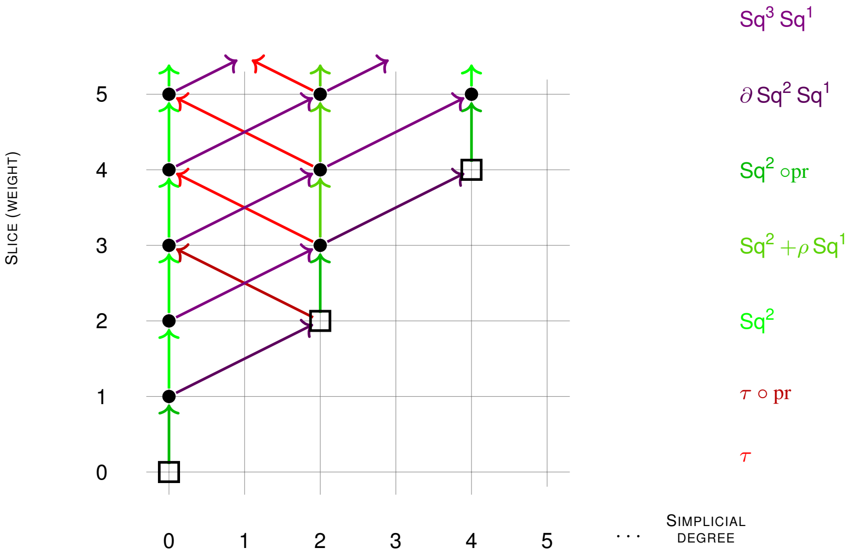
Restriction to very effective covers

Since **1** is very effective, the unit map $\mathbf{1} \to \mathbf{KQ}$ factors over $\mathbf{kq} := \mathrm{vf}_0\mathbf{KQ}$. Taking very effective covers produces a homotopy cofiber sequence

$$\Sigma^{(1)}\mathbf{kq} \xrightarrow{\eta \wedge \mathbf{kq}} \mathbf{kq} \xrightarrow{u} \mathbf{kgl} \xrightarrow{v} \Sigma^{1+(1)}\mathbf{kq}$$

where $\mathbf{kgl} = \mathrm{vf}_0\mathbf{KGL} \simeq \mathrm{f}_0\mathbf{KGL}$.

The first differential for **kq**



The unit map $\mathbf{1} \to \mathbf{kq}$ on slices

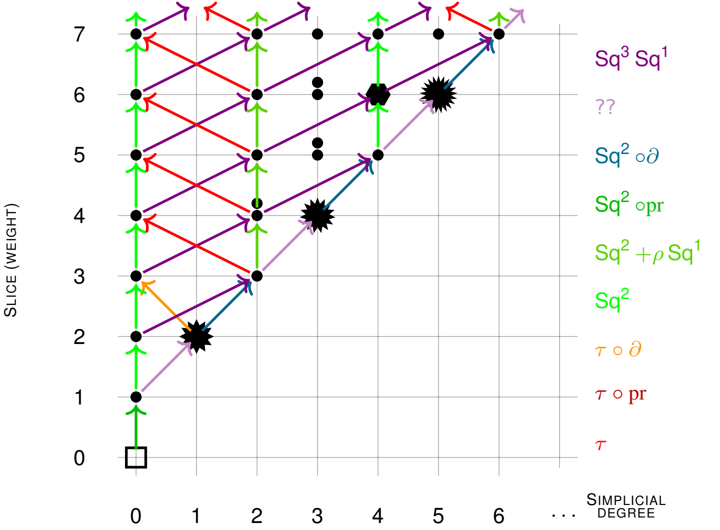
Lemma

Let $q \geq 1$. The map $s_{2q-1}\mathbf{1} \to s_{2q-1}\mathbf{kq}$ is the inclusion on the summand $\Sigma^{2q-2+(2q-1)}\mathbf{H}\mathbb{Z}/2$ corresponding to $\alpha_{2q-1} \in \mathrm{Ext}^{1,4q-2}_{MU_*MU}(MU_*, MU_*)$.
The map $s_{2q}\mathbf{1} \to s_{2q}\mathbf{KQ}$ sends the summand $\Sigma^{2q-1+(2q)}\mathbf{H}\mathbb{Z}/a_{2q}$ corresponding to $\alpha_{2q} \in \mathrm{Ext}^{1,4q}_{MU_*MU}(MU_*, MU_*)$ to the summand $\Sigma^{2q+(2q)}\mathbf{H}\mathbb{Z}$ in such a way that the composition

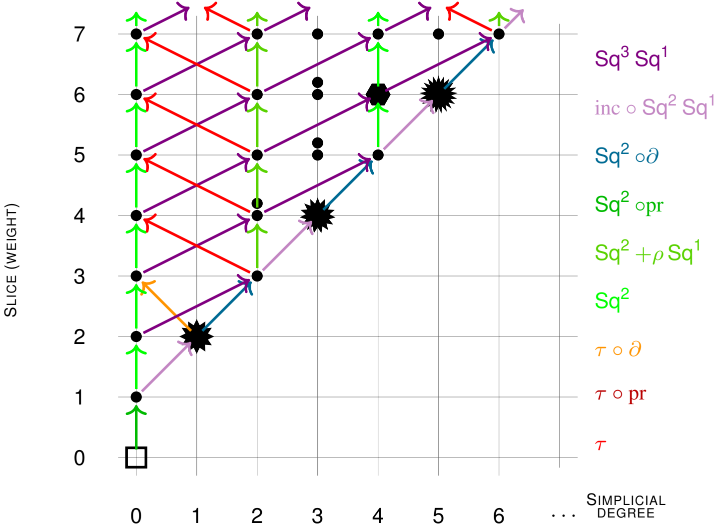
$$\Sigma^{2q-1+(2q)}\mathbf{H}\mathbb{Z}/a_{2q} \to \Sigma^{2q+(2q)}\mathbf{H}\mathbb{Z} \xrightarrow{\mathrm{pr}} \Sigma^{2q+(2q)}\mathbf{H}\mathbb{Z}/2$$

coincides with the unique nontrivial element.

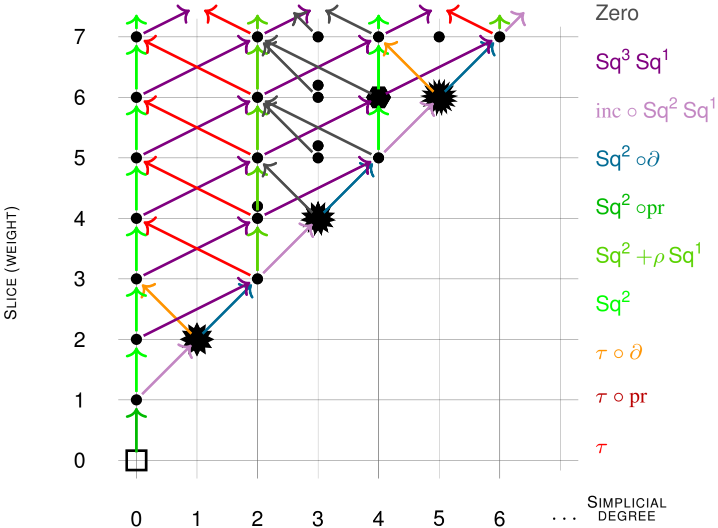
The first differential for **1** by comparison with **kq** ...



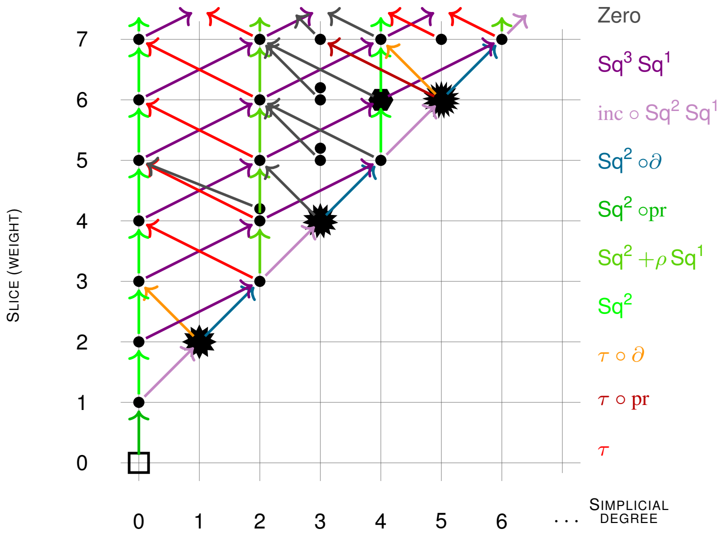
... and multiplicative properties ...



... and Adem relations ...



... and topology (which detects τ).



The E^2-page for **1**

5	$h^{n+5,n+5}$		
4	$h^{n+4,n+4}$		$h^{n+4,n+4}$
3	$h^{n+3,n+3}$	$h^{n+2,n+3}/\tau\partial h_{12}^{n+1,n+2}$	$h^{n+1,n+3}/\mathrm{Sq}^2 h^{n-1,n+2}$
2	$h^{n+2,n+2}$	$h_{12}^{n+2,n+2}$ $h^{n+1,n+2}/\mathrm{Sq}^2 h^{n-1,n+1}$	$\ker(h_{12}^{n+1,n+2} \xrightarrow{\partial} h^{n+2,n+2})$ $h^{n,n+2}/\mathrm{Sq}^2 h^{n-2,n+1}$
1	$h^{n+1,n+1}$	$h^{n,n+1}/\mathrm{Sq}^2\mathrm{pr}H^{n-2,n}$	$\ker(h^{n-1,n+1} \xrightarrow{\mathrm{Sq}^2} h^{n+1,n+2})$
0	$H^{n,n}$	$H^{n-1,n}$	$\ker(H^{n-2,n} \xrightarrow{\mathrm{Sq}^2\mathrm{pr}} h^{n,n+1})$
	$\pi_{0-(n)}\mathbf{1}$	$\pi_{1-(n)}\mathbf{1}$	$\pi_{2-(n)}\mathbf{1}$

Milnor conjecture on Galois cohomology (Voevodsky)

Multiplication with

$$\tau \cdot - \colon h^{s,w} \to h^{s,w+1}$$

is an isomorphism for all $0 \leq s \leq w$, whence every $a \in h^{s,w}$ is of the form $a = \tau^{w-s} b$ with $b \in h^{s,s} \cong K_s^{\mathrm{Milnor}}/2$.

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$$\mathrm{Sq}^1(\tau) = \rho, \mathrm{Sq}^2(\tau) = 0, \mathrm{Sq}^2(\tau^2) = \tau\rho^2$$

one obtains

$$(\mathrm{Sq}^1 \colon h^{s,w} \to h^{s+1,w}) \cong \begin{cases} 0 & w - s \text{ even} \\ \rho \cdot - & w - s \text{ odd} \end{cases}$$

$$(\mathrm{Sq}^2 \colon h^{s,w} \to h^{s+2,w+1}) \cong \begin{cases} 0 & w - s \equiv 0, 1 (4) \\ \rho^2 \cdot - & w - s \equiv 2, 3 (4) \end{cases}$$

Higher differentials for $\mathbf{1}$

Lemma

Let F be a field of characteristic different from two. Consider the slice spectral sequence of $\mathbf{1}_F$.

1. *All differentials ending in the column for $\pi_{0+(\star)}\mathbf{1}_F$ are zero.*
2. *All differentials of degree ≥ 2 ending in the column for $\pi_{1+(\star)}\mathbf{1}_F$ are zero.*

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Proof ingredients: The first statement follows by comparison with **kq**. The second statement: A theorem of Orlov-Vishik-Voevodsky shows that one may reduce from F to fields of small cohomological dimension, or $F = \mathbb{R}$. For the latter one uses real realization. Triviality of one d^2 follows from the multiplicative structure. For differentials originating in s_0 one compares with appropriate motivic Moore spectra, like $\mathbf{1}/12 + 6\eta[-1]$.

The E^∞-page for **1**

	$\pi_{0-(n)}\mathbf{1}$	$\pi_{1-(n)}\mathbf{1}$	$\pi_{2-(n)}\mathbf{1}$
5	$h^{n+5,n+5}$		
4	$h^{n+4,n+4}$		$h^{n+4,n+4}$
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2	$h^{n+2,n+2}$	$h_{12}^{n+2,n+2}$; $h^{n+1,n+2}/\mathrm{Sq}^2 h^{n-1,n+1}$	$\ker(h_{12}^{n+1,n+2} \xrightarrow{\partial} h^{n+2,n+2})$; $h^{n,n+2}/\mathrm{Sq}^2 h^{n-2,n+1}$
1	$h^{n+1,n+1}$	$h^{n,n+1}/\mathrm{Sq}^2\mathrm{pr}H^{n-2,n}$	$\ker(h^{n-1,n+1} \xrightarrow{\mathrm{Sq}^2} h^{n+1,n+2})$
0	$H^{n,n}$	$H^{n-1,n}$	$\ker(H^{n-2,n} \xrightarrow{\mathrm{Sq}^2\mathrm{pr}} h^{n,n+1})$

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Let g_q be defined by the natural homotopy cofiber sequence:

$$\mathrm{f}_q(\mathbf{E}) \to \mathbf{E} \to \mathrm{g}_q(\mathbf{E}) \to \Sigma \mathrm{f}_q(\mathbf{E})$$

The natural map $\mathrm{f}_{q+1} \to \mathrm{f}_q$ induces a natural map $\mathrm{g}_{q+1} \to \mathrm{g}_q$ whose homotopy fiber is s_q.

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$$\mathrm{sc}(\mathbf{E}) := \operatorname{holim}_q \mathsf{g}_q(\mathbf{E})$$

denote the *slice completion* of **E**. Then

$$\operatorname{holim}_q \mathsf{f}_q(\mathbf{E}) \to \mathbf{E} \to \mathrm{sc}(\mathbf{E}) \to \Sigma \operatorname{holim}_q \mathsf{f}_q(\mathbf{E})$$

is a homotopy cofiber sequence. The slice spectral sequence for **E** converges conditionally to $\mathrm{sc}(\mathbf{E})$.

Slice completion

$\mathbf{E}$ is *slice complete* if $\mathbf{E} \to \mathrm{sc}(\mathbf{E})$ is an equivalence. Since

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- **KGL** and **MGL** are slice complete.
- $\mathrm{sc}\big(\mathbf{H}\mathbb{Z}/2[\tau^{-1}]\big) \simeq *$, whence $\mathbf{H}\mathbb{Z}/2[\tau^{-1}]$ is not slice complete.
- The motivic spectrum $\mathbf{KQ}[\eta^{-1}]$ represents Balmer's higher Witt groups. Every slice of $\mathbf{KQ}[\eta^{-1}]$ is a sum of suspensions of $\mathbf{H}\mathbb{Z}/2$. Hence all slices of $\mathbf{KQ}[\eta^{-1}]/3$, and also $\mathrm{sc}(\mathbf{KQ}[\eta^{-1}]/3)$, are contractible. But $\mathbf{KQ}[\eta^{-1}]/3$ is not contractible for $F \subset \mathbb{R}$.

Levine's convergence theorem

Recall $\mathbf{E} \in \mathrm{SH}(S)$ is compact if $[\mathbf{E}, -]$ commutes with direct sums.

Theorem (Levine 2011)

Let F be a field of finite cohomological dimension. Then every compact motivic spectrum in $\mathrm{SH}(F)$ is slice complete, after inverting the exponential characteristic of F.

This is a very strong theorem, but we would like to have a convergence result which applies to all fields.

Slice completion and η-completion

Since η is trivial on $\mathsf{s}_0\mathbf{1} = \mathbf{H}\mathbb{Z}$, it is trivial on every slice. It follows that $\mathrm{sc}(\mathbf{E})$ is η-complete for every motivic spectrum $\mathbf{E}$ satisfying $\mathsf{f}_e(\mathbf{E}) = \mathbf{E}$ for some integer e. This implies the easy part of the following:

Theorem (R.-Spitzweck-Østvær 2016)

Let $\mathbf{E}$ *be a cellular motivic spectrum of finite type over a field. Then there is a natural equivalence:*

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Hence the slice spectral sequence for $\mathbf{1}$ computes $\mathbf{1}^{\wedge}_{\eta}$.

Cellular motivic spectra

Attaching a cell to a motivic spectrum **E** refers to taking the homotopy cofiber of some map

$$\Sigma^{s+(w)}\mathbf{1} \to \mathbf{E}$$

in SH(F). The homotopy cofiber **D** consists of **E** and a cell of *dimension* $s + 1 + (w)$ and *weight* w. A motivic spectrum **E** is *cellular* if it is the homotopy colimit of a sequence

$$* = \mathbf{E}_{-1} \to \mathbf{E}_0 \to \mathbf{E}_1 \to \cdots \to \mathbf{E}_n \to \cdots$$

in which $\mathbf{E}_n$ is obtained by attaching cells to $\mathbf{E}_{n-1}$ for every n.

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in which $\mathbf{E}_n$ is obtained by attaching cells to $\mathbf{E}_{n-1}$ for every n. A cellular motivic spectrum **E** is of *finite type* if

- there exists an integer k such that **E** contains no cells in dimension $s + (w)$ with $s < k$, and
- for every integer n, **E** contains at most finitely many cells of dimension $n + (w)$.

Cellular of finite type

- **1** is cellular of finite type.
- $\mathbb{P}^\infty$ is not compact, but cellular of finite type.
- **MGL** is cellular of finite type.
- Theorem (Hopkins-Morel, Hoyois) implies that $\mathbf{H}\mathbb{Z}$ is cellular of finite type over a field of characteristic zero. If ℓ is a prime different from char(F), then $\mathbf{H}\mathbb{Z}/\ell$ is cellular of finite type.
- $\mathbf{1}[\eta^{-1}]$ is cellular, but not of finite type.
- **KGL** and **KQ** are cellular, but not of finite type.
- If X is an elliptic curve, X is not cellular.

Finiteness conditions on slices

The basic problem in proving the convergence theorem is that f_q, and hence g_p, does not necessarily commute with homotopy limits (although it commutes with homotopy colimits by a result of Spitzweck).

Definition

A motivic spectrum **E** is called *slice-finitary* if for every $n \in \mathbb{Z}$ there exist natural numbers $a_n \leq b_n$ and a finite collection $\{C_{n,0}, C_{n,1}, \ldots, C_{n,b_n})$ of finitely generated abelian groups with the following properties:

1. The n-th slice $\mathrm{s}_n\mathbf{E}$ is weakly equivalent to the sum $\Sigma^{a_n+(n)}\mathbf{H}C_{n,0} \vee \Sigma^{a_n+1+(n)}\mathbf{H}C_{n,1} \vee \cdots \vee \Sigma^{b_n+(n)}\mathbf{H}C_{n,b_n}$.
2. The sequence (a_n) is non-decreasing and diverges to $+\infty$.
3. There exists an integer e such that $\mathrm{f}_e\mathbf{E} = \mathbf{E}$.

Slice-finitary motivic spectra

Example

- **MGL** is slice-finitary, with $a_n = n$. (Also $b_n - n$.)
- **1** is not slice-finitary. ($a_n = 0$, $b_n = n - 1$ for $n > 0$.)
- The Moore spectrum $\mathbf{1}/p$ is slice-finitary for p odd.
- $\mathbf{1}[\frac{1}{2}]$ is not slice-finitary.
- $\mathrm{f}_1\mathbf{1}[\frac{1}{2}]$ is slice-finitary.

The reason in the third and fifth case is that at the prime p,

$$\mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \cong 0 \text{ for } 2s(p-1) > t.$$

Slice completion for slice-finitary motivic spectra

Lemma

Let $\mathbf{E}$ *be a slice-finitary motivic spectrum over a field which is cellular of finite type. Then* $\mathrm{sc}(\mathbf{E})$ *and* $\mathrm{holim}_q \mathsf{f}_q(\mathbf{E})$ *are cellular of finite type, respectively. The slices of* $\mathrm{holim}_q \mathsf{f}_q(\mathbf{E})$ *are trivial.*

Problem: $\mathbf{1}$ is not slice-finitary. However, the following holds:

Lemma

The motivic spectrum $\mathbf{1}/\eta$ *is slice-finitary.*

The topological Adams-Novikov spectral sequence

In his 1972 Annals paper, Zahler stated: "We hope to prove that $h^{n-1}\alpha_{k+1}$ generates $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}^{n,2(n+k)}(\mathrm{BP}_*, \mathrm{BP}_*) \cong \mathbb{Z}/2$ for all k (n sufficiently large) ..." Zahler had already accomplished this for $k \leq 6$.

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Theorem (Andrews-Miller 2014)

Let $k \geq 2$. Then $\mathrm{Ext}^{n,2(n+k)}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) = \{0, \alpha_1^{n-1}\alpha_{k+1}\}$ for n sufficiently large.

This theorem implies that $\mathbf{1}/\eta$ is slice-finitary. The remaining ingredient is a connectivity statement:

Lemma

The canonical map $\mathbf{1}/\eta \to \mathrm{sc}(\mathbf{1}/\eta)$ is 1-connective.

Slice completion and η-completion

Lemma

Let **E** *be a cellular motivic spectrum of finite type over a field. The canonical map*

$$\mathbf{E}/\eta \to \mathrm{sc}(\mathbf{E}/\eta)$$

is an equivalence.

Proof.

$$\begin{array}{ccccc}
\mathrm{holim}_q\, \mathsf{f}_q(\mathbf{E}/\eta) & \longrightarrow & \mathbf{E}/\eta & \longrightarrow & \mathrm{sc}(\mathbf{E}/\eta) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{sc}(\mathrm{holim}_q\, \mathsf{f}_q(\mathbf{E}/\eta)) & \longrightarrow & \mathrm{sc}(\mathbf{E}/\eta) & \longrightarrow & \mathrm{sc}(\mathrm{sc}(\mathbf{E}/\eta))
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\downarrow & & \downarrow{\scriptstyle \mathrm{conn}(\mathbf{E})+1} & & \downarrow{\scriptstyle \simeq} \\
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The slice convergence theorem

By induction, $\mathbf{E}/\eta^k \to \mathrm{sc}(\mathbf{E}/\eta^k)$ is an equivalence for **E** cellular of finite type and $k \in \mathbb{N}$.

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By induction, $\mathbf{E}/\eta^k \to \mathrm{sc}(\mathbf{E}/\eta^k)$ is an equivalence for $\mathbf{E}$ cellular of finite type and $k \in \mathbb{N}$. Since $\mathbf{E}^{\wedge}_{\eta}$ is the homotopy limit of

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the induced map $\mathbf{E}^{\wedge}_{\eta} \to \big(\mathrm{sc}(\mathbf{E})\big)^{\wedge}_{\eta}$ is an equivalence.

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the induced map $\mathbf{E}^{\wedge}_{\eta} \to \big(\mathrm{sc}(\mathbf{E})\big)^{\wedge}_{\eta}$ is an equivalence. As mentioned already,

$$\mathrm{sc}(\mathbf{E}) \xrightarrow{\simeq} \big(\mathrm{sc}(\mathbf{E})\big)^{\wedge}_{\eta}$$

which completes the proof. In particular, the slice spectral sequence for **1** determines $\mathbf{1}^{\wedge}_{\eta}$.

An arithmetic square

Consider the homotopy pullback square

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{1}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \mathbf{1}^{\wedge}_{\eta} & \longrightarrow & \mathbf{1}^{\wedge}_{\eta}[\eta^{-1}] \end{array}$$

and the induced long exact sequence of homotopy groups. The slice spectral sequence allows to conclude that $\pi_{1+(w)}\mathbf{1}^{\wedge}_{\eta} \cong 0$ for $w \geq 3$, and that $\pi_{2+(w)}\mathbf{1}^{\wedge}_{\eta} \cong 0$ for $w \geq 5$. Hence

$$\pi_{1+(w)}\mathbf{1}^{\wedge}_{\eta}[\eta^{-1}] = \pi_{2+(w)}\mathbf{1}^{\wedge}_{\eta}[\eta^{-1}] = 0$$

for all w.

Another vanishing result

Theorem (R. 2016)

For all w, $\pi_{1+(w)}\mathbf{1}[\eta^{-1}] = \pi_{2+(w)}\mathbf{1}[\eta^{-1}] = 0$.

Guillou-Isaksen: $\pi_{3+(w)}\mathbf{1}[\eta^{-1}]$ is never zero.
Nevertheless, the map $\pi_{3+(w)}\mathbf{1}^{\wedge}_{\eta}[\eta^{-1}] \to \pi_{2+(w)}\mathbf{1}$ is zero. Hence for all w:

$$\pi_{1+(w)}\mathbf{1} = \pi_{1+(w)}\mathbf{1}^{\wedge}_{\eta} \qquad \pi_{2+(w)}\mathbf{1} = \pi_{2+(w)}\mathbf{1}^{\wedge}_{\eta}$$

The 0-line for $\mathbf{1}$

The behaviour of the unit $\mathbf{1} \to \mathbf{kq}$ on slices shows that

$$\pi_{0+(\star)}\mathbf{1}^{\wedge}_{\eta} \to \pi_{0+(\star)}\mathbf{kq}^{\wedge}_{\eta}$$

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is an isomorphism. Injectivity of the map

$$\pi_{0-(\star)}\mathbf{1} \to \pi_{0-(\star)}\mathbf{kq} \cong K^{\mathrm{MW}}_{\star}$$

is equivalent to the η-filtration on $\pi_{0-(\star)}\mathbf{1}$ being Hausdorff.

The E^∞-page for **1**

	$\pi_{0-(n)}\mathbf{1}$	$\pi_{1-(n)}\mathbf{1}$	$\pi_{2-(n)}\mathbf{1}$
5	$h^{n+5,n+5}$		
4	$h^{n+4,n+4}$		$h^{n+4,n+4}$
3	$h^{n+3,n+3}$	$h^{n+2,n+3}/\tau\partial h_{12}^{n+1,n+2}$	$h^{n+1,n+3}/\mathrm{Sq}^2 h^{n-1,n+2}$
2	$h^{n+2,n+2}$	$h_{12}^{n+2,n+2}$ $h^{n+1,n+2}/\mathrm{Sq}^2 h^{n-1,n+1}$	$\ker(h_{12}^{n+1,n+2} \xrightarrow{\partial} h^{n+2,n+2})$ $h^{n,n+2}/\mathrm{Sq}^2 h^{n-2,n+1}$
1	$h^{n+1,n+1}$	$h^{n,n+1}/\mathrm{Sq}^2\mathrm{pr}H^{n-2,n}$	$\ker(h^{n-1,n+1} \xrightarrow{\mathrm{Sq}^2} h^{n+1,n+2})$
0	$H^{n,n}$	$H^{n-1,n}$	$\ker(H^{n-2,n} \xrightarrow{\mathrm{Sq}^2\mathrm{pr}} h^{n,n+1})$

The 1-line for $\mathbf{1}$

The behaviour of the unit $\mathbf{1} \to \mathbf{kq}$ on slices shows that

$$\pi_{1-(\star)}\mathbf{1} \cong \pi_{1-(\star)}\mathbf{1}^{\wedge}_{\eta} \to \pi_{1-(\star)}\mathbf{kq}^{\wedge}_{\eta} \cong \pi_{1-(\star)}\mathbf{kq}$$

is surjective.

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is surjective. Its kernel is an extension of the $K^{\mathrm{MW}}_{\star}$-modules

$$h^{\star+2,\star+3}/\tau\partial h_{12}^{\star+1,\star+2} \cong h^{\star+2,\star+2}/\partial h_{12}^{\star+1,\star+2}$$

and

$$h_{12}^{\star+2,\star+2} \cong K^{\mathrm{MW}}_{2+\star}/(\eta, 12).$$

One can check that the kernel is even a $K^{\mathrm{Milnor}}_{\star}$-module. The resulting Ext group is $\mathbb{Z}/2$ over any field of characteristic not 2. Topological realization (or alternatively the multiplicative structure) implies that the extension is the unique nontrivial one.

The 1-line for $\mathbf{1}$

Theorem (R.-Spitzweck-Østvær 2016)

Let F be a field of exponential characteristic $e \neq 2$ and $w \in \mathbb{Z}$. The unit map $\mathbf{1} \to \mathbf{kq}$ induces a surjection $\pi_{1+(w)}\mathbf{1} \to \pi_{1+(w)}\mathbf{kq}$. After inverting e, its kernel is given as follows:

$$0 \to K_{2-w}^{\mathrm{Milnor}}/24 \to \pi_{1+(w)}\mathbf{1} \to \pi_{1+(w)}\mathbf{kq} \to 0$$

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$$0 \to K^{\mathrm{Milnor}}_{2-w}/24 \to \pi_{1+(w)}\mathbf{1} \to \pi_{1+(w)}\mathbf{kq} \to 0$$

Let $\nu \in \pi_{1+(2)}\mathbf{1}$ denote the map $S^{3+(4)} \to S^{2+(2)}$ obtained via the Hopf construction on $\mathrm{SL}_2 \simeq \mathbb{A}^2 \smallsetminus \{0\} \simeq S^{1+(2)}$. Let $\eta_{\mathrm{top}} \in \pi_{1+(0)}\mathbf{1}$ denote the topological first Hopf map.

- ν generates $\pi_{1+(2)}\mathbf{1} = \mathbb{Z}/24$ ($w = 2$) and $K^{\mathrm{Milnor}}_{2-*}/24$.
- $12\nu = \eta^2\eta_{\mathrm{top}}$.
- η_{top} and ν do not generate $\pi_{1+(\star)}\mathbf{1}$ in general: $\pi_{1-(2)}\mathbf{1}_{\mathbb{Q}[\sqrt{-1}]}$ contains $\mathbb{Z}$, coming from $H^{1,2}$.

Finitely presented $K^{\mathrm{MW}}_{\star}$-modules

One can check that the $K^{\mathrm{MW}}_{\star}$-module homomorphism

$$K^{\mathrm{MW}}_{2+\star} \oplus K^{\mathrm{MW}}_{\star} \to \pi_{1-(\star)}\mathsf{f}_1\mathbf{1}$$

sending (a, b) to $a \cdot \nu + b \cdot \eta_{\mathrm{top}}$ induces an isomorphism

$$K^{\mathrm{MW}}_{2+\star}\{\nu\} \oplus K^{\mathrm{MW}}_{\star}\{\eta_{\mathrm{top}}\}/(\eta\nu, 2\eta_{\mathrm{top}}, \eta^2\eta_{\mathrm{top}} - 12\nu) \cong \pi_{1-(\star)}\mathsf{f}_1\mathbf{1}$$

after inverting the exponential characteristic. The proof involves that $\pi_{1-(\star)}\mathsf{f}_1\mathbf{kq} \cong K^{\mathrm{MW}}_{\star}\{\eta_{\mathrm{top}}\}/(\eta^2\eta_{\mathrm{top}})$.

Finitely presented $K_\star^{\mathrm{MW}}$-modules

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after inverting the exponential characteristic. The proof involves that $\pi_{1-(\star)}\mathsf{f}_1\mathbf{kq} \cong K_\star^{\mathrm{MW}}\{\eta_{\mathrm{top}}\}/(\eta^2\eta_{\mathrm{top}})$.
The $K_\star^{\mathrm{MW}}$-module $\pi_{2-(\star)}\mathsf{f}_1\mathbf{1}$ is not finitely generated for many number fields, such as $\mathbb{Q}(\zeta_p)$ where $p \equiv -1 \bmod 8$ is prime.