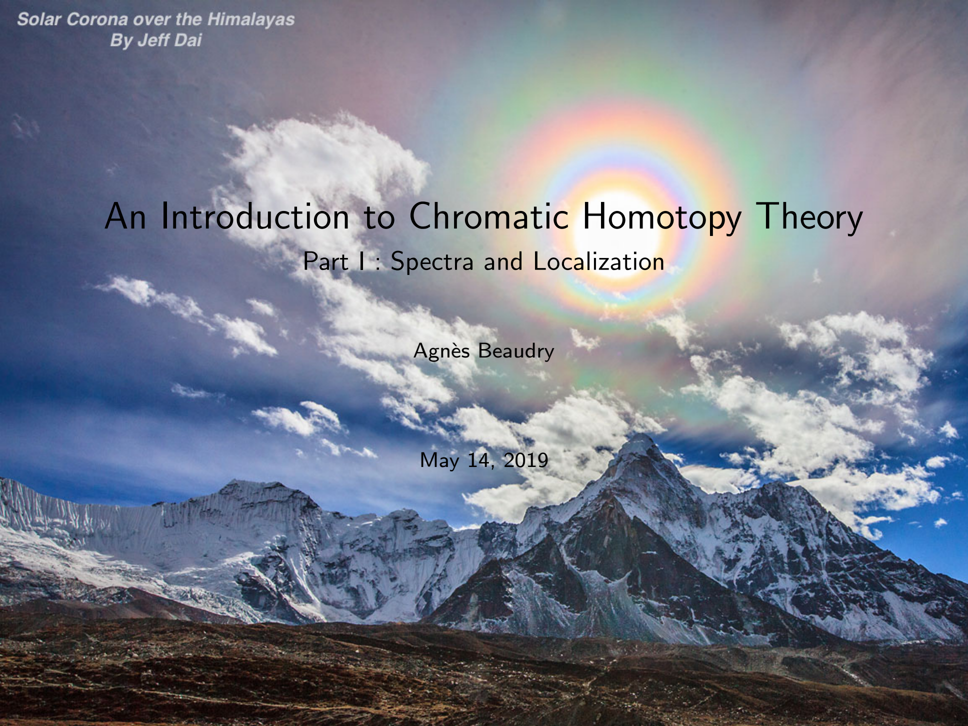

An Introduction to Chromatic Homotopy Theory

Part I : Spectra and Localization

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Outline of the Course

(I) Spectra and Localization

(II) Complex Orientations and the Morava K-Theories $\rightsquigarrow$ MU

(III) The Chromatic Filtration $\rightsquigarrow$ Conjectures

(IV) Morava E-Theory and the Stabilizer Group $\rightsquigarrow$ $K(n)$-local

Part I – Spectra and Localization

(1) Cohomology Theories

(2) Spectra

(3) Stable Homotopy Category Set-up.

(4) Bousfield Localization

(5) What is Chromatic Homotopy Theory? Motivation

Cohomology with Coefficients in $G \in \mathrm{Ab}$

Reduced

$$\widetilde{HG}^*(-) : \mathrm{CW}_+^{op} \longrightarrow \mathrm{Ab}$$

↑ based CW-complexes

1. (Homotopy) If $f \simeq g$, then $\widetilde{HG}^*(f) = \widetilde{HG}^*(g)$.
2. (Additivity)
$$\widetilde{HG}^*\left(\coprod_{i \in I} X_i\right) \cong \prod_{i \in I} \widetilde{HG}^*(X_i)$$
3. (Exactness) For $A \subseteq X$ a subcomplex, the following sequence is exact:
$$\widetilde{HG}^*(X/A) \to \widetilde{HG}^*(X) \to \widetilde{HG}^*(A)$$
4. (Suspension) For each n, there is a natural isomorphism
$$\widetilde{HG}^n(X) \xrightarrow{\cong} \widetilde{HG}^{n+1}(\Sigma X).$$
5. (Dimension) $\widetilde{HG}^*(S^0) = G$ in $* = 0$.

In fact, it is *representable*

$$\widetilde{HG}^n(X) \cong [X, K(G,n)], \qquad K(G,n) \xrightarrow{\simeq} \Omega K(G, n+1).$$

Eilenberg MacLane space $\qquad \pi_* K(G,n) = \begin{cases} G & * = n \\ 0 & * \neq n \end{cases}$

Eilenberg-Steenrod Axioms

A *reduced cohomology theory* is a functor

$$\tilde{E}^*: \mathsf{CW}_+^{op} \longrightarrow \mathsf{Ab}$$

which satisfies the following axioms

1. (Homotopy) If $f \simeq g$ then $\tilde{E}^*(f) = \tilde{E}^*(g)$.
2. (Additivity)

$$\tilde{E}^*\left(\coprod_{i \in I} X_i\right) \cong \prod_{i \in I} \tilde{E}^*(X_i)$$

3. (Exactness) For $A \subseteq X$ a subcomplex, the following sequence is exact:

$$\tilde{E}^*(X/A) \to \tilde{E}^*(X) \to \tilde{E}^*(A)$$

4. (Suspension) For each n, there is a natural isomorphism

$$\tilde{E}^n(X) \xrightarrow{\cong} \tilde{E}^{n+1}(\Sigma X).$$

5. ~~(Dimension) $\tilde{E}^*(S^0) = G$ in $* = 0$.~~ $\Rightarrow$ $\tilde{H}G^\cdot \cong \tilde{E}^\cdot$

$$\boxed{E^*(X) = \tilde{E}^*(X_+).}$$

The Brown Representability Theorem

Let E be a cohomology theory. There is a sequence of based spaces E_n, $n \geqslant 0$ with weak equivalences

$$\omega_n \colon E_n \xrightarrow{\simeq} \Omega E_{n+1}. \quad \text{suspension iso.}$$

such that

$$\boxed{\widetilde{E}^n(X) \cong [X, E_n].}$$

The adjunction

$$[X, \overset{\mathrm{Maps}_*(S^1, Y)}{\overset{\shortparallel}{\Omega Y}}] \cong [\underset{X \wedge S^1}{\Sigma X}, Y].$$

gives the suspension isomorphism

$$\widetilde{E}^n(X) \cong [X, E_n] \xrightarrow[\omega_n \circ -]{\cong} [X, \Omega E_{n+1}] \cong [\Sigma X, E_{n+1}] \cong \widetilde{E}^{n+1}(\Sigma X)$$

The Category of Spectra Sp

Objects. An *(Ω-)spectrum* E is a sequence of based spaces E_n, $n \geqslant 0$ with weak equivalences

$$\boxed{\omega_n \colon E_n \xrightarrow{\simeq} \Omega E_{n+1}}$$

Morphisms. A map of $f \colon E \to F$ is a sequence of maps $f_n \colon E_n \to F_n$ such that the diagram commutes:

$$\begin{tikzcd} E_n \arrow[r, "f_n"] \arrow[d, "\omega_n"'] & F_n \arrow[d, "\omega_n"] \\ \Omega E_{n+1} \arrow[r, "\Omega f_{n+1}"] & \Omega F_{n+1} \end{tikzcd}$$

We denote the category of spectra by Sp.

Spectrification

For a sequence $E = \{E_n : n \geqslant 0\}$ and inclusions $\omega_n \colon E_n \hookrightarrow \Omega E_{n+1}$,

$E_0 \to \Omega E_1 \to \Omega^2 E_2 \to \cdots$

$$\mathbb{L}E_n = \varinjlim_k \Omega^k E_{n+k}, \qquad \mathbb{L}\omega_n = \varinjlim_k \Omega^k \omega_{n+k}$$

is a spectrum. This is called *spectrification*.

Ordinary Cohomology with Coefficients in G

$HG^*(-)$ is represented by

$$HG_n = K(G, n) \qquad \omega_n \colon K(G, n) \xrightarrow{\simeq} \Omega K(G, n+1).$$

Complex K-Theory

By Bott Periodicity, $\Omega U \simeq \mathbb{Z} \times BU$ and $\Omega(\mathbb{Z} \times BU) \simeq U$. Complex K-theory $K^*(-)$, is represented by

$$K = \{\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots\}.$$

Suspension Spectra

For a based space X, the *suspension spectrum* is the spectrification of

$$(\Sigma^\infty X)_n = \Sigma^n X \qquad \omega_n \colon \Sigma^n X \to \Omega \Sigma^{n+1} X$$

where ω_n is the adjoint to the identity

$$\sigma_n \colon \Sigma \Sigma^n X \xrightarrow{=} \Sigma^{n+1} X.$$

We often write X for $\Sigma^\infty X$. The $\boxed{\textit{sphere spectrum} \text{ is } S^0 = \Sigma^\infty S^0.}$

Complex Cobordism

MU is the spectrification of the sequence

$$\{MU(1), \Sigma MU(1), MU(2), \Sigma MU(2), \ldots\}$$

where $MU(n)$ is the Thom space of the canonical complex vector bundle

$p \longmapsto [p]$

$$\gamma_n \to Gr_n(\mathbb{C}^\infty) \simeq BU(n).$$

Note:

$$\begin{tikzcd}
i^*\gamma_{n+1} \cong \gamma_n \oplus \mathbb{C} \arrow[r] \arrow[d] & \gamma_{n+1} \arrow[d] \\
BU(n) \arrow[r, "i"] & BU(n+1)
\end{tikzcd}$$

$\to \mathbb{R}^2$

The map $\omega_{2n+1} \colon \Sigma MU(n) \to MU(2n)$ is adjoint to:

$$\Sigma^2 MU(n) \cong Th(i^*\gamma_{n+1}) \xrightarrow{\sigma_{2n+1}} Th(\gamma_{n+1}) = MU(n+1)$$

The map $\omega_{2n} \colon MU(n) \to \Omega\Sigma MU(n)$ is adjoint to the identity.

MU(pt)

Homotopy Groups

If E is a spectrum, then the *rth homotopy group* of E is

$$\pi_r E = \varinjlim_n \pi_{r+n} E_n.$$

A map $f\colon E \to F$ is a weak equivalence if $\pi_* f$ is an isomorphism.

Examples

- The *stable homotopy groups of spheres are* are

$$\pi_r^s = \pi_r \Sigma^\infty S^0 \cong \varinjlim_n \pi_{r+n} S^n.$$

- $\pi_* HA \cong A$ concentrated in $* = 0$. $\simeq S^2$
- $\pi_* K \cong \mathbb{Z}[\beta^{\pm 1}]$ for $\beta \in \pi_2 K = K(\mathbb{C}P^1)$ the Bott class, i.e.,

$$\pi_{2r} K = \mathbb{Z}\{\beta^r\}, \qquad \pi_{2r+1} K = 0.$$

← has neg. homotopy

- $\pi_* MU \cong \mathbb{Z}[x_1, x_2, \ldots]$ for $x_n \in \pi_{2n} MU$ related to $[\mathbb{C}P^n]$.

$$\pi_0 MU = \mathbb{Z}\{1\}, \quad \pi_1 MU = 0, \quad \pi_2 MU = \mathbb{Z}\{x_1\}, \quad \pi_4 MU = \mathbb{Z}\{x_1^2, x_2\}.$$

Homotopy Category

Let $\mathcal{C}$ be a category and $\mathcal{W}$ a subcategory such that

- All isomorphisms of $\mathcal{C}$ are in $\mathcal{W}$,
- If 2 out of 3 of $\{f, g, g \circ f\}$ are in $\mathcal{W}$, then so is the third.

The *homotopy category of* $\mathcal{C}$ is a category $Ho(\mathcal{C}, \mathcal{W})$ and a functor

if it exists

$$\iota \colon \mathcal{C} \to Ho(\mathcal{C}, \mathcal{W})$$

such that, for $F \colon \mathcal{C} \to \mathcal{D}$ which maps $\mathcal{W}$ to isomorphisms, there is

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\Downarrow\simeq"] & \mathcal{D} \\ Ho(\mathcal{C}, \mathcal{W}) \arrow[ur, dotted, "\exists! F_{\mathcal{W}}"'] & \end{tikzcd}$$

such that $F \xrightarrow{\cong} F_{\mathcal{W}} \circ \iota$.

Stable Homotopy Category

$\mathcal{SH} = Ho(Sp, \mathcal{W}_{S^0})$ for $\mathcal{W}_{S^0}$ the weak equivalences is called the *stable homotopy category*.

π_* isos.

Margolis

Ravenel MU

Nilpotence ....

- $[X, Y] := \mathcal{SH}(X, Y)$ are abelian groups
- Finite products and coproducts are equivalent in $\mathcal{SH}$
- Closed symmetric monoidal with unit S^0:

$(\Sigma^\infty X \wedge E)_n$

$X \wedge E_n$

$$- \wedge - : \mathcal{SH} \times \mathcal{SH} \to \mathcal{SH} \qquad \boxed{F(-,-)} : \mathcal{SH}^{op} \times \mathcal{SH} \to \mathcal{SH}$$

$\otimes$

$$\boxed{F(X \wedge Y, Z) \cong F(X, F(Y, Z))}$$

Spectrify

Margolis

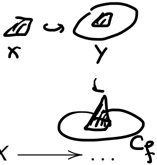
- Triangulated: $\Sigma : \mathcal{SH} \to \mathcal{SH}$ with

$$\Sigma X = X \wedge S^1 \qquad \Sigma^{-1} X = F(S^1, X) .= \Omega X$$

- Distinguished triangles

cofiber sequence

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X$$



where $C_f = Y \cup_f (X \wedge [0, 1])$ is the cofiber of f.

- Long exact sequences for $[-, Z]$ and $[Z, -]$. For example:

$$\dots \longrightarrow \pi_n X \longrightarrow \pi_n Y \longrightarrow \pi_n C_f \longrightarrow \pi_{n-1} X \longrightarrow \dots$$

Models for Spectra

There are other choices $(\mathcal{C}, \mathcal{W})$ with $\mathcal{SH} = Ho(\mathcal{C}, \mathcal{W})$. In particular, there are closed symmetric monoidal models for Sp.

EKMM $\mathbb{S}$-Module

Symmetric Spectra SS

Orthogonal Spectra

∞-infinity cat of Spectra

Homology and Cohomology

If E is a spectrum, then E-homology is the functor $\widetilde{E}_* : \mathcal{SH} \to \mathrm{Ab}$

$$X \mapsto \widetilde{E}_n(X) := \pi_n(E \wedge X)$$

and E-cohomology is the functor $E^* : \mathcal{SH}^{op} \to \mathrm{Ab}$

$$X \mapsto \widetilde{E}^n(X) := \pi_{-n}F(X, E) = [X, \Sigma^n E].$$

We let

$$E_n := \widetilde{E}_n(S^0) = \pi_n E = \widetilde{E}^{-n}(S^0) =: E^{-n}$$

Stable Homotopy Groups as a Homology Theory

For $E = S^0$, this gives:

$$\widetilde{S^0}_*(X) = \pi_*(X)$$

Weak equivalences are $\widetilde{S^0}_*$-isomorphisms and $\mathcal{SH}$ is obtained from Sp by inverting these.

X space

$$\pi_r^s X = \operatorname{colim}_{n+r} \pi_n X$$

Bousfield Localization

A map $f: X \to Y$ is an *E-equivalences* if

$$\widetilde{E}_*(f): \widetilde{E}_*(X) \xrightarrow{\cong} \widetilde{E}_*(Y)$$

Let $\mathcal{W}_E \subseteq Sp$ be the subcategory of E-equivalences and

$$\mathcal{SH}_E := Ho(Sp, \mathcal{W}_E).$$

" E-local "

Bousfield

- X is *E-acyclic* if $\widetilde{E}_*(X) = 0$.
- Y is *E-local* if $[X, Y] = 0$ when X is E-acyclic.

Let Sp_E be the full subcategory whose objects are E-local spectra. In fact:

$$\mathcal{SH}_E \cong Ho(Sp_E, \mathcal{W} \cap Sp_E).$$

Exercise

A map $f: X \to Y$ in Sp_E is a weak equivalence iff it is an E-equivalence.

Universal Property of Localization

An *E-localization* is an E-equivalence $\eta\colon X \to L_E X$ for $L_E X \in Sp_E$. These exist and are unique in $\mathcal{SH}$:

$\uparrow E$. iso

$$L_E\colon \mathcal{SH} \to \mathcal{SH}_E \subseteq \mathcal{SH}, \qquad \eta\colon 1_{\mathcal{SH}} \to L_E.$$

If $f\colon X \to Y$ with $Y \in Sp_E$, then

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] & Y \\ L_E X \arrow[ur, dotted, "f_E"'] & \end{tikzcd}$$

and f_E is unique in $\mathcal{SH}$. There is a distinguished triangle

$$C_E X \to X \to L_E X \to \Sigma C_E X$$

Initial

where $C_E X$ is the terminal E_*-acyclic spectrum with a map to X.

Exercise

There is a natural transformation $L_{F\vee E} \to L_E$ and,

Ravenel' Nilpotence...

Bousfield loc.

$$\boxed{L_E \simeq L_E L_{E\vee F} \simeq L_{E\vee F} L_E.}$$

Localization and Completion

For G an abelian group, let SG a a Moore spectrum

cofiber

$$\coprod_{j\in J} S^0\{r_j\} \longrightarrow \coprod_{i\in I} S^0\{g_i\} \longrightarrow SG \longrightarrow \Sigma \coprod_{j\in J} S^0\{r_j\}$$

for a presentation $\bigoplus_{j\in J} \mathbb{Z}\{r_j\} \to \bigoplus_{i\in I} \mathbb{Z}\{g_i\} \to G$.

- If $G = \mathbb{Z}_{(p)}$ or $\mathbb{Q}$, then $L_{SG}X \simeq SG \wedge X$ and
$$\pi_* L_{SG} X \cong \pi_* X \otimes G.$$
The *p-localization* of X is:
$$X_{(p)} := L_{S\mathbb{Z}_{(p)}} X$$
The *rationalization* of X is:
$$X_{\mathbb{Q}} := L_{S\mathbb{Q}} X$$
- If the groups $\pi_* X$ are finitely generated, then
$$\pi_* L_{S\mathbb{Z}/p} X \cong \pi_* X \otimes \mathbb{Z}_p = \lim_{\leftarrow i} \mathbb{Z}/p^i$$
Bousfield
where $\mathbb{Z}_p$ is the p-adic integers. The *p-completion* of X is
$$\boxed{X_p := L_{S\mathbb{Z}/p} X.}$$

The Sphere

For the *p-local* sphere $S^0_{(p)} = L_{S\mathbb{Z}_{(p)}} S^0$,

$$\pi_n S^0_{(p)} \cong \begin{cases} \mathbb{Z}_{(p)} & n = 0 \\ \mathrm{Tor}_p(\pi_n^s) & n > 0. \end{cases}$$

$\mathbb{Z}_p$ S_p

For the rarional sphere $S^0_{\mathbb{Q}} = L_{S\mathbb{Q}} S^0$,

$$\pi_n S^0_{\mathbb{Q}} \cong \begin{cases} \mathbb{Q} & n = 0 \\ 0 & n > 0. \end{cases}$$

Connective Spectra

A spectrum is *connective* if $\pi_r X = 0$ for $r < 0$. For such X

$$L_{HG} X \simeq L_{SG} X.$$

Eilenberg MacLane

$S_p^0 = L_{H\mathbb{Z}/p} S^0$

Adams Operations

For $k \geqslant 0$, there are *unstable* operations

$$\psi^k : K^*(X) \to K^*(X).$$

They give stable operations

Adams Blue Book

$$\psi^k : K_p \to K_p$$

for $k \in \mathbb{Z}_p^\times$ The action of ψ^k on $\pi_* K_p = \mathbb{Z}_p[\beta^{\pm 1}]$ is determined by

ring homo.

$\mathbb{Z}_p$-algebra

$$\psi^k(\beta) = k\beta.$$

K-Theory Localization

Let p be odd and $K\mathbb{Z}/p = K \wedge S\mathbb{Z}/p$ so that

Adams Baird Ravenel Bousfield

$$\pi_* K\mathbb{Z}/p \cong \mathbb{Z}/p[\beta^{\pm 1}].$$

There is a distinguished triangle

$$\boxed{L_{K\mathbb{Z}/p} S^0} \longrightarrow K_p \xrightarrow{\psi^\ell - 1} K_p \longrightarrow \Sigma L_{K\mathbb{Z}/p} S^0$$

for ℓ a topological generator of $\mathbb{Z}_p^\times$. $\cong C_{p-1} \times \mathbb{Z}_p$

$\pi_* L_{K\mathbb{Z}/p} S^0$

Image of J

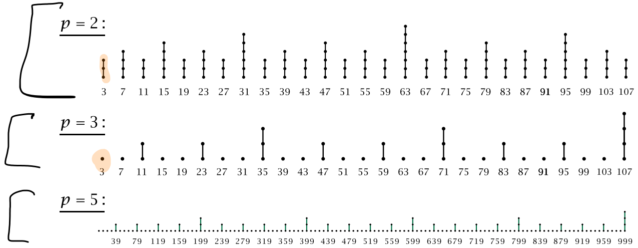
Real Bott Periodicity gives $\pi_{3+4k}O \cong \mathbb{Z}$. The J homomorphism

$$J\colon \pi_n O \to \pi_n^s$$

Hatcher Vector bundles

has image

n	3	7	11	15	19	23	27	31	35	39	43
	$\mathbb{Z}_{24}$	$\mathbb{Z}_{240}$	$\mathbb{Z}_{504}$	$\mathbb{Z}_{480}$	$\mathbb{Z}_{264}$	$\mathbb{Z}_{65520}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{16320}$	$\mathbb{Z}_{28728}$	$\mathbb{Z}_{13200}$	$\mathbb{Z}_{552}$



Exercise

Compute $\pi_* L_{K\mathbb{Z}/p} S^0$ and compare with the p-component of $\mathrm{im}(J)$.

detects imJ.

Next Time

Fix a prime p. There are spectra $K(0), K(1), K(2), \ldots, K(\infty)$ called the Morava K-theories:

$$L_n X := L_{K(0) \vee \ldots \vee K(n)} X$$

In fact,

$$\boxed{L_{K(0)} \cong L_{H\mathbb{Q}} \qquad L_{K(1)} \cong L_{K\mathbb{Z}/p} \qquad L_{K(\infty)} \cong L_{H\mathbb{Z}/p}}$$

The *chromatic convergence theorem* (Hopkins–Ravenel) states:

$$S^0_{(p)} \cong \overset{\text{holim}}{\cancel{\varprojlim_n}} L_n S^0$$

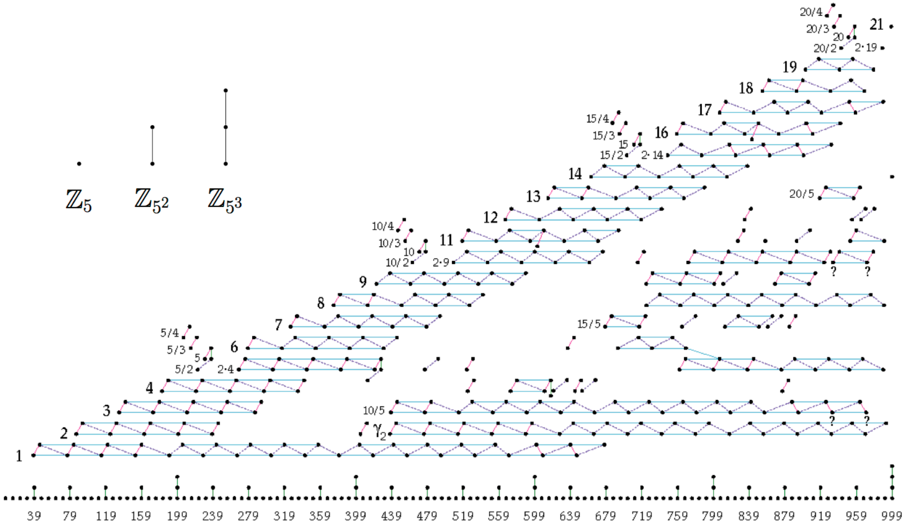
~~in $\mathcal{SH}$~~. $\boxed{\text{The } n\text{th chromatic layer is } L_n S^0.}$

Chromatic homotopy theory studies the $S^0_{(p)}$ via this filtration.

Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$\pi_* S_{(5)}$

$p = 5$



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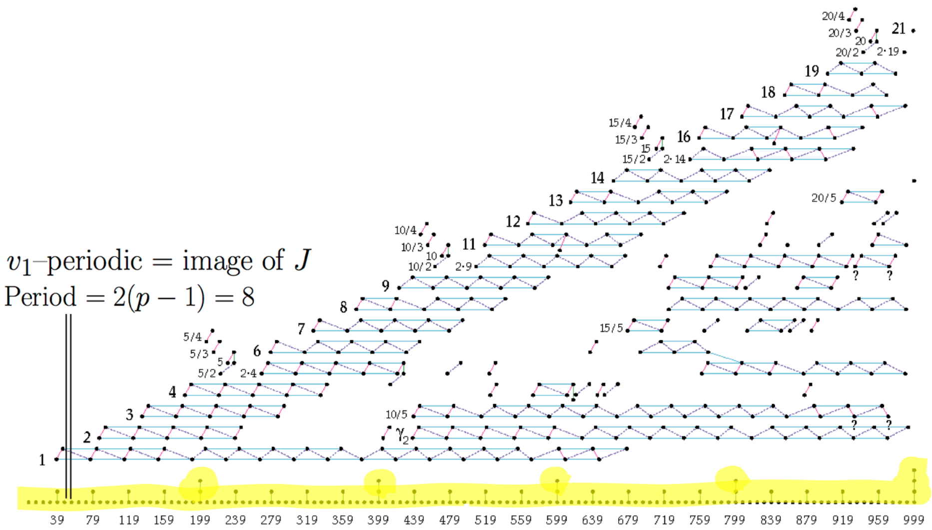
$\pi_* S_{(5)}$

$p = 5$



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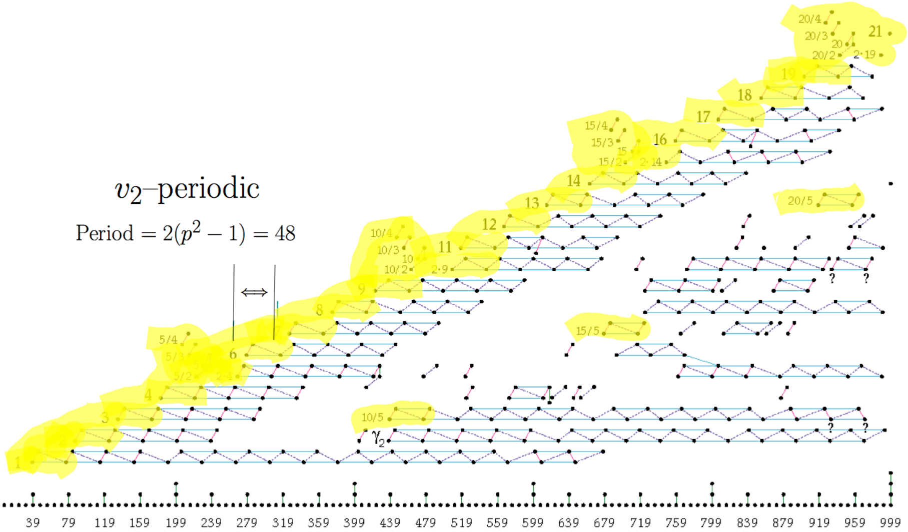
$\pi_* S_{(5)}$

$p = 5$



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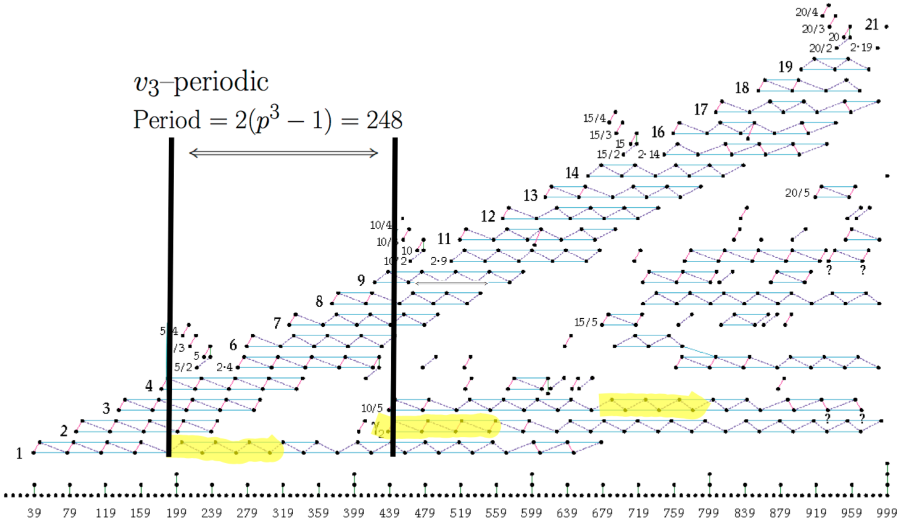
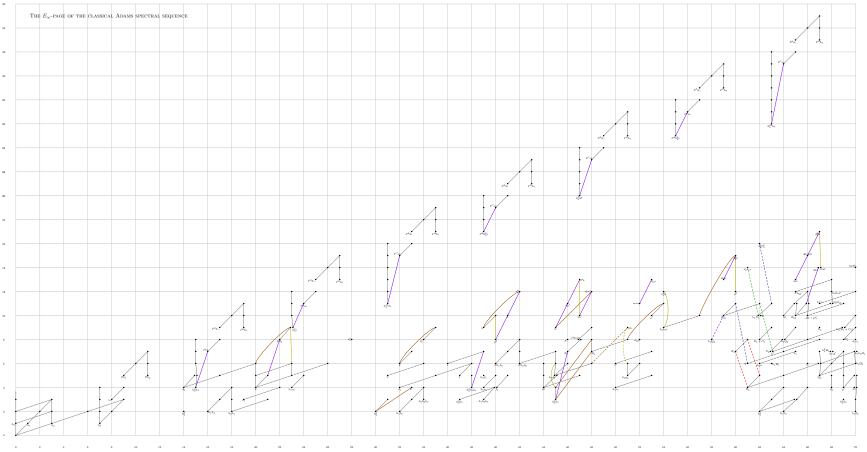
$\pi_* S_{(5)}$

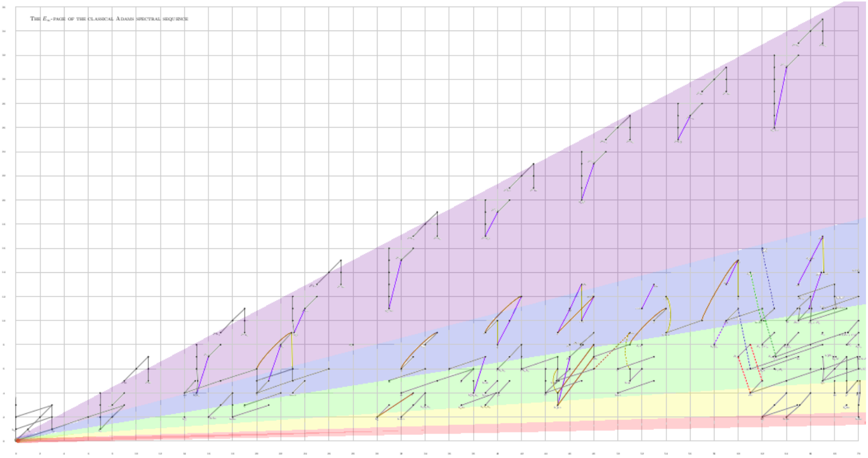
$p = 5$



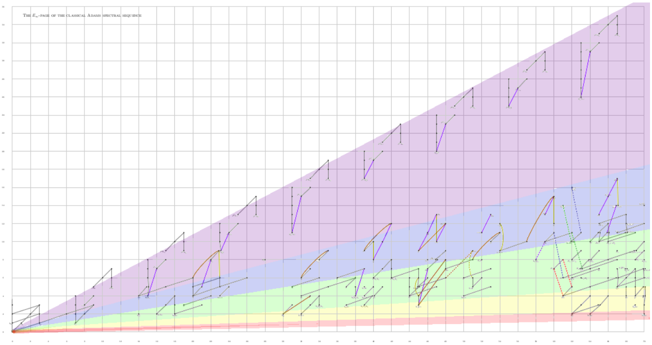
$\pi_* S_{(2)}$ (Illustration by Isaksen)



$\pi_* S_{(2)}$ (Illustration by Isaksen)



$\pi_* S_{(2)}$ (Illustration by Isaksen)



Telescope Conjecture (Ravenel)

The first n-rays are detected by $L_n S^0$.

Chromatic Splitting Conjecture (Hopkins)

The gluing data for the chromatic layers is simple.


Thank you!
Doug
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