Warm Up
Simplify each expression. Assume all variables are positive.

1. $2\sqrt{27x} + 3\sqrt{12x}$
   $$12\sqrt{3x}$$

2. $\sqrt[3]{72y^5}$
   $$6y^2\sqrt[3]{2y}$$

3. $\sqrt[3]{(x + 2)^3}$
   $$x + 2$$

4. $\sqrt[4]{2(48y)}$
   $$4\sqrt[4]{6y}$$

Write each expression in radical form.

5. $(x + 6)^{\frac{1}{2}}$
   $$\sqrt{x + 6}$$

6. $(3y + 4)^{\frac{3}{5}}$
   $$\sqrt[5]{(3y + 4)^3}$$
Objective

Solve radical equations and inequalities.
Vocabulary

radical equation
radical inequality
A radical equation contains a variable within a radical. Recall that you can solve quadratic equations by taking the square root of both sides. Similarly, radical equations can be solved by raising both sides to a power.
Solving Radical Equations

<table>
<thead>
<tr>
<th>Steps</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Isolate the radical.</td>
<td>$\sqrt[3]{x} - 2 = 0$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt[3]{x} = 2$</td>
</tr>
<tr>
<td>2. Raise both sides of the equation to</td>
<td>$\left(\sqrt[3]{x}\right)^3 = (2)^3$</td>
</tr>
<tr>
<td>the power equal to the index of the</td>
<td></td>
</tr>
<tr>
<td>radical.</td>
<td></td>
</tr>
<tr>
<td>3. Simplify and solve.</td>
<td>$x = 8$</td>
</tr>
</tbody>
</table>

Remember!

For a square root, the index of the radical is 2.

$$\sqrt{x + 1} = \sqrt[2]{x + 1}$$
Example 1A: Solving Equations Containing One Radical

Solve each equation.

\[ 5 + \sqrt{x + 1} = 16 \]

\[ \sqrt{x + 1} = 16 - 5 \]

\[ \sqrt{x + 1} = 11 \]

\[ (\sqrt{x + 1})^2 = (11)^2 \]

\[ x + 1 = 121 \]

\[ x = 120 \]

Check

\[ \frac{5 + \sqrt{x + 1}}{16} = 16 \]

\[ 5 + \sqrt{120} + 1 \]

\[ 5 + \sqrt{121} \]

\[ 16 \]

\[ 16 \]

\[ \checkmark \]
Example 1B: Solving Equations Containing One Radical

Solve each equation.

\[ 7^{\frac{3}{5}}x - 7 = 84 \]

Divide by 7.

\[ \frac{7^{\frac{3}{5}}x - 7}{7} = \frac{84}{7} \]

Simplify.

\[ 7^{\frac{1}{5}}x - 7 = 12 \]

Simplify.

\[ \left( 7^{\frac{1}{5}}x - 7 \right)^3 = (12)^3 \]

Cube both sides.

\[ 5x - 7 = 1728 \]

Simplify.

\[ 5x = 1735 \]

Solve for \( x \).

\[ x = 347 \]

Check

\[ 7^{\frac{3}{5}}(347) - 7 = 84 \]

\[ 7^{\frac{3}{5}}1728 = 84 \]

84 84 ✓
Solve the equation.

\[ 4 + \sqrt{x - 1} = 5 \]

\[ \sqrt{x - 1} = 5 - 4 \quad \text{Subtract 4.} \]

\[ \sqrt{x - 1} = 1 \quad \text{Simplify.} \]

\[ (\sqrt{x - 1})^2 = (1)^2 \quad \text{Square both sides.} \]

\[ x - 1 = 1 \quad \text{Simplify.} \]

\[ x = 2 \quad \text{Solve for } x. \]

Check

\[ 4 + \sqrt{x - 1} = 5 \]

\[ \begin{array}{c|c}
4 + \sqrt{2 - 1} & 5 \\
4 + \sqrt{1} & 5 \\
5 & 5 \\
\end{array} \]
Solving Radical Equations and Inequalities

Check It Out! Example 1b

Solve the equation.

\[ \sqrt[3]{3x - 4} = 2 \]

\[ (\sqrt[3]{3x - 4})^3 = (2)^3 \]

Cube both sides.

\[ 3x - 4 = 8 \]

Simplify.

\[ 3x = 12 \]

Solve for \( x \).

\[ x = 4 \]

Check

\[ \sqrt[3]{3(4) - 4} = 2 \]

\[ \sqrt[3]{8} = 2 \]

\[ 2 = 2 \]  \( \checkmark \)
Check It Out! Example 1c

Solve the equation.

\[ 6\sqrt{x + 10} = 42 \]

\[ \frac{6\sqrt{x + 10}}{6} = \frac{42}{6} \quad \text{Divide by 6.} \]

\[ (\sqrt{x + 10})^2 = (7)^2 \quad \text{Square both sides.} \]

\[ x + 10 = 49 \quad \text{Simplify.} \]

\[ x = 39 \quad \text{Solve for } x. \]

Check

\[ 6\sqrt{39 + 10} = 42 \]

\[ 6\sqrt{49} = 42 \quad \checkmark \]
Example 2: Solving Equations Containing Two Radicals

Solve \( \sqrt{7x + 2} = 3\sqrt{3x - 2} \)

Square both sides.

\[
\left( \sqrt{7x + 2} \right)^2 = \left( 3\sqrt{3x - 2} \right)^2
\]

Simplify.

\[
7x + 2 = 9(3x - 2)
\]

Distribute.

\[
7x + 2 = 27x - 18
\]

Solve for \( x \).

\[
20 = 20x
\]

\[
1 = x
\]
Example 2 Continued

Check

\[
\begin{array}{c|c}
\sqrt{7x} + 2 & = 3\sqrt{3x} - 2 \\
\sqrt{7(1)} + 2 & = 3\sqrt{3(1)} - 2 \\
3 & 3 \checkmark
\end{array}
\]
Solve each equation.

\[ \sqrt{8x + 6} = 3\sqrt{x} \]

\[ \left( \sqrt{8x + 6} \right)^2 = \left(3\sqrt{x}\right)^2 \]

\[ 8x + 6 = 9x \]

\[ 8x + 6 = 9x \]

\[ 6 = x \]

**Check**

\[ \sqrt{8(6) + 6} = 3\sqrt{6} \]

\[ \sqrt{8(6) + 6} | 3\sqrt{6} \]

\[ 3\sqrt{6} | 3\sqrt{6} \]
Check It Out! Example 2b

Solve each equation.

\[ \sqrt[3]{x + 6} = 2\sqrt[3]{x - 1} \]
\[ (\sqrt[3]{x + 6})^3 = (2\sqrt[3]{x - 1})^3 \]
\[ x + 6 = 8(x - 1) \]
\[ x + 6 = 8x - 8 \]
\[ 14 = 7x \]
\[ 2 = x \]

Cube both sides.
Simplify.
Distribute.
Solve for x.

Check

\[ \sqrt[3]{2 + 6} = \sqrt[3]{2^3} - 1 \]
\[ 8 \]
\[ 2 \]

2 2  ✓
Raising each side of an equation to an even power may introduce extraneous solutions.

**Helpful Hint**

You can use the intersect feature on a graphing calculator to find the point where the two curves intersect.
Solve \( \sqrt{-3x + 33} = 5 - x \).

**Method 1** Use a graphing calculator.

Let \( Y_1 = \sqrt{-3x + 33} \) and \( Y_2 = 5 - x \).

The graphs intersect in only one point, so there is exactly one solution.

The solution is \( x = -1 \).
Example 3 Continued

Method 2  Use algebra to solve the equation.

Step 1  Solve for \( x \).

\[
\sqrt{-3x + 33} = 5 - x
\]

\[
\left(\sqrt{-3x + 33}\right)^2 = (5 - x)^2
\]

\[
-3x + 33 = 25 - 10x + x^2
\]

\[
0 = x^2 - 7x - 8
\]

\[
0 = (x - 8)(x + 1)
\]

\[
x - 8 = 0 \quad \text{or} \quad x + 1 = 0
\]

\[
x = 8 \quad \text{or} \quad x = -1
\]
Example 3 Continued

**Method 2** Use algebra to solve the equation.

**Step 2** Use substitution to check for extraneous solutions.

\[
\sqrt{-3x + 33} = 5 - x
\]

<table>
<thead>
<tr>
<th>(\sqrt{-3(8) + 33})</th>
<th>5 - 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-3</td>
</tr>
</tbody>
</table>

Because \(x = 8\) is extraneous, the only solution is \(x = -1\).
Check It Out! Example 3a

Solve each equation.

\[ \sqrt{2x + 14} = x + 3 \]

**Method 1** Use a graphing calculator.

Let \( Y_1 = \sqrt{2x + 14} \) and \( Y_2 = x + 3 \).

The graphs intersect in only one point, so there is exactly one solution.

The solution is \( x = 1 \).
Check It Out! Example 3a Continued

Method 2  Use algebra to solve the equation.

Step 1  Solve for $x$.

$$\sqrt{2x + 14} = x + 3$$

$$\left(\sqrt{2x + 14}\right)^2 = (x + 3)^2$$

$$2x + 14 = x^2 + 6x + 9$$

$$0 = x^2 + 4x - 5$$

$$0 = (x + 5)(x - 1)$$

$$x + 5 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = -5 \quad \text{or} \quad x = 1$$
Check It Out! Example 3a Continued

Method 1  Use algebra to solve the equation.

Step 2  Use substitution to check for extraneous solutions.

\[ \sqrt{2x + 14} = x + 3 \]

\[ \begin{array}{c|c|c}
\sqrt{2(1) + 14} & 1 + 3 \\
2 & 4 & 4 \checkmark \\
\end{array} \]

Because \( x = -5 \) is extraneous, the only solution is \( x = 1 \).
Check It Out! Example 3b

Solve each equation.

\[ \sqrt{-9x + 28} = -x + 4 \]

**Method 1** Use a graphing calculator.

Let \( Y_1 = \sqrt{-9x + 28} \)

and \( Y_2 = -x + 4 \).

The graphs intersect in two points, so there are two solutions.

The solutions are \( x = -4 \) and \( x = 3 \).
Check It Out! Example 3b Continued

**Method 2** Use algebra to solve the equation.

**Step 1** Solve for $x$.

\[
\sqrt{-9x + 28} = -x + 4
\]

Square both sides.

\[
\left(\sqrt{-9x + 28}\right)^2 = (-x + 4)^2
\]

Simplify.

\[
-9x + 28 = x^2 - 8x + 16
\]

Write in standard form.

\[
0 = x^2 + x - 12
\]

Factor.

\[
0 = (x + 4)(x - 3)
\]

Solve for $x$.

\[
x + 4 = 0 \quad \text{or} \quad x - 3 = 0
\]

\[
x = -4 \quad \text{or} \quad x = 3
\]
Check It Out! Example 3b Continued

Method 1  Use algebra to solve the equation.

Step 2  Use substitution to check for extraneous solutions.

\[ \sqrt{-9x + 28} = -x + 4 \]

\[
\begin{array}{c|c}
\sqrt{-9(-4) + 28} & -(-4) + 4 \\
8 & 8 \checkmark \\
\end{array}
\]

\[
\begin{array}{c|c}
\sqrt{-9(3) + 28} & -(3) + 4 \\
1 & 1 \checkmark \\
\end{array}
\]
Remember!

To find a power, multiply the exponents.

\[
\left( (x + 12)^{\frac{1}{2}} \right)^2 \\
(x + 12)^{\frac{1}{2}} \cdot 2 \\
x + 12
\]
Example 4A: Solving Equations with Rational Exponents

Solve each equation.

\[(5x + 7)^{\frac{1}{3}} = 3\]

\[\sqrt[3]{5x + 7} = 3\]  \hspace{1cm} \text{Write in radical form.}

\[\left(\sqrt[3]{5x + 7}\right)^3 = (3)^3\]  \hspace{1cm} \text{Cube both sides.}

\[5x + 7 = 27\]  \hspace{1cm} \text{Simplify.}

\[5x = 20\]  \hspace{1cm} \text{Factor.}

\[x = 4\]  \hspace{1cm} \text{Solve for } x.\]
Example 4B: Solving Equations with Rational Exponents

\[ 2x = (4x + 8)^{\frac{1}{2}} \]

\textbf{Step 1} Solve for } x.

\[ (2x)^2 = [(4x + 8)^{\frac{1}{2}}]^2 \]

\[ 4x^2 = 4x + 8 \]

\[ 4x^2 - 4x - 8 = 0 \]

\[ 4(x^2 - x - 2) = 0 \]

\[ 4(x - 2)(x + 1) = 0 \]

\[ 4 \neq 0, \quad x - 2 = 0 \text{ or } x + 1 = 0 \]

\[ x = 2 \text{ or } x = -1 \]
Example 4B Continued

Step 2  Use substitution to check for extraneous solutions.

\[
\begin{array}{c|c}
2x &= (4x + 8)^{\frac{1}{2}} \\
2(2) &= (4(2) + 8)^{\frac{1}{2}} \\
4 &= 16^{\frac{1}{2}} \\
&= 4 \\
4 &= 4 \checkmark
\end{array}
\quad
\begin{array}{c|c}
2x &= (4x + 8)^{\frac{1}{2}} \\
2(-1) &= (4(-1) + 8)^{\frac{1}{2}} \\
-2 &= 4^{\frac{1}{2}} \\
-2 &= 2 \times
\end{array}
\]

The only solution is \( x = 2 \).
Solve each equation.

\[(x + 5)^{\frac{1}{3}} = 3\]

\[\sqrt[3]{x + 5} = 3\]  \hspace{1cm} \text{Write in radical form.}

\[(\sqrt[3]{x + 5})^3 = (3)^3\] \hspace{1cm} \text{Cube both sides.}

\[x + 5 = 27\] \hspace{1cm} \text{Simplify.}

\[x = 22\] \hspace{1cm} \text{Solve for x.}
Check It Out! Example 4b

\[(2x + 15)^{\frac{1}{2}} = x\]

**Step 1** Solve for \(x\).

\[
\left[(2x + 15)^{\frac{1}{2}}\right]^2 = (x)^2
\]

\[
2x + 15 = x^2
\]

\[
x^2 - 2x - 15 = 0
\]

\[
(x - 5)(x + 3) = 0
\]

\[
x - 5 = 0 \text{ or } x + 3 = 0
\]

\[
x = 5 \text{ or } x = -3
\]
Check It Out! Example 4b Continued

**Step 2** Use substitution to check for extraneous solutions.

\[
\begin{align*}
(2x + 15)^{\frac{1}{2}} &= x \\
(2(5) + 15)^{\frac{1}{2}} &= 5 \\
(10 + 15)^{\frac{1}{2}} &= 5 \\
5 &\quad 5 \checkmark
\end{align*}
\]

\[
\begin{align*}
(2x + 15)^{\frac{1}{2}} &= x \\
(2(-3) + 15)^{\frac{1}{2}} &= -3 \\
(-6 + 15)^{\frac{1}{2}} &= -3 \\
3 &\quad -3 \times
\end{align*}
\]

The only solution is \( x = 5 \).
Check It Out! Example 4c

\[ 3(x + 6)^{\frac{1}{2}} = 9 \]

\[ [3(x + 6)^{\frac{1}{2}}]^2 = (9)^2 \]

\[ 9(x + 6) = 81 \]

\[ 9x + 54 = 81 \]

\[ 9x = 27 \]

\[ x = 3 \]

*Raise both sides to the reciprocal power.*

*Simplify.*

*Distribute 9.*

*Simplify.*

*Solve for x.*
A **radical inequality** is an inequality that contains a variable within a radical. You can solve radical inequalities by graphing or using algebra.

**Remember!**

A radical expression with an even index and a negative radicand has no real roots.
Example 5: Solving Radical Inequalities

Solve \( \sqrt{2x - 6} + 3 \leq 9 \).

**Method 1** Use a graph and a table.

On a graphing calculator, let \( Y_1 = \sqrt{2x - 6} + 3 \) and \( Y_2 = 9 \). The graph of \( Y_1 \) is at or below the graph of \( Y_2 \) for values of \( x \) between 3 and 21. Notice that \( Y_1 \) is undefined when \( x < 3 \).

The solution is \( 3 \leq x \leq 21 \).
Example 5 Continued

Method 2 Use algebra to solve the inequality.

Step 1 Solve for $x$.

\[
\sqrt{2x - 6} + 3 \leq 9
\]

Subtract 3.

\[
\sqrt{2x - 6} \leq 6
\]

Square both sides.

\[
(\sqrt{2x - 6})^2 \leq (6)^2
\]

Simplify.

\[
2x \leq 42
\]

Solve for $x$.

\[
x \leq 21
\]
Example 5 Continued

Method 2 Use algebra to solve the inequality.

Step 2 Consider the radicand.

\[ 2x - 6 \geq 0 \quad \text{The radicand cannot be negative.} \]

\[ 2x \geq 6 \quad \text{Solve for x.} \]

\[ x \geq 3 \]

The solution of \( \sqrt{2x - 6} + 3 \leq 9 \) is \( x \geq 3 \) and \( x \leq 21 \), or \( 3 \leq x \leq 21 \).
Check It Out! Example 5a

**Solve** $\sqrt{x - 3} + 2 \leq 5$.

**Method 1** Use a graph and a table.

On a graphing calculator, let $Y_1 = \sqrt{x - 3} + 2$ and $Y_2 = 5$. The graph of $Y_1$ is at or below the graph of $Y_2$ for values of $x$ between 3 and 12. Notice that $Y_1$ is undefined when $< 3$.

The solution is $3 \leq x \leq 12$. 
Check It Out! Example 5a Continued

Method 2 Use algebra to solve the inequality.

Step 1 Solve for \(x\).

\[
\sqrt{x - 3} + 2 \leq 5
\]

Subtract 2.

\[
\sqrt{x - 3} \leq 3
\]

Square both sides.

\[
\left(\sqrt{x - 3}\right)^2 \leq (3)^2
\]

Simplify.

\[
x - 3 \leq 9
\]

Solve for \(x\).

\[
x \leq 12
\]
Method 2 Use algebra to solve the inequality.

Step 2 Consider the radicand.

\[ x - 3 \geq 0 \quad \text{The radicand cannot be negative.} \]

\[ x \geq 3 \quad \text{Solve for } x. \]

The solution of \( \sqrt{x-3} + 2 \leq 5 \) is \( x \geq 3 \) and \( x \leq 12 \), or \( 3 \leq x \leq 12 \).
Solve $\sqrt[3]{x} + 2 \geq 1$.

**Method 1** Use a graph and a table.

On a graphing calculator, let $Y_1 = \sqrt[3]{x} + 2$ and $Y_2 = 1$. The graph of $Y_1$ is at or above the graph of $Y_2$ for values of $x$ greater than $-1$. Notice that $Y_1$ is undefined when $< -2$.

The solution is $x \geq -1$. 
Check It Out! Example 5b Continued

Method 2 Use algebra to solve the inequality.

Step 1 Solve for \(x\).

\[
\sqrt[3]{x + 2} \geq 1
\]

\[
(\sqrt[3]{x + 2})^3 \geq (1)^3 \quad \text{Cube both sides.}
\]

\[
x + 2 \geq 1 \quad \text{Solve for } x.
\]

\[
x \geq -1
\]
Check It Out! Example 5b Continued

Method 2 Use algebra to solve the inequality.

Step 2 Consider the radicand.

\[ x + 2 \geq 1 \quad \text{The radicand cannot be negative.} \]

\[ x \geq -1 \quad \text{Solve for } x. \]

The solution of \( \sqrt[3]{x+2} \geq 1 \) is \( x \geq -1 \).
Example 6: Automobile Application

The time $t$ in seconds that it takes a car to travel a quarter mile when starting from a full stop can be estimated by using the formula $t = 5.825\sqrt[3]{\frac{w}{P}}$, where $w$ is the weight of the car in pounds and $P$ is the power delivered by the engine in horsepower. If the quarter-mile time from a 3590 lb car is 13.4 s, how much power does its engine deliver? Round to the nearest horsepower.
Example 6 Continued

Use the formula to determine the amount of horsepower the 3590 lb car has if it finishes the quarter-mile in 13.4s.

\[ t = 5.825 \frac{3\sqrt{W}}{P} \]

\[ 13.4 = 5.825 \frac{3\sqrt{3590}}{P} \]

\[ (13.4)^3 = \left( 5.825 \frac{3\sqrt{3590}}{P} \right)^3 \]

\[ 2406.104 \approx 197.646 \left( \frac{3590}{P} \right) \]

\[ 2406.104P \approx 709,548.747 \]

\[ P \approx 295 \]

The engine delivers a power of about 295 hp.
Check It Out! Example 6

The speed \( s \) in miles per hour that a car is traveling when it goes into a skid can be estimated by using the formula \( s = \sqrt{30fd} \), where \( f \) is the coefficient of friction and \( d \) is the length of the skid marks in feet.

A car skids to a stop on a street with a speed limit of 30 mi/h. The skid marks measure 35 ft, and the coefficient of friction was 0.7. Was the car speeding? Explain.

Use the formula to determine the greatest possible length of the driver’s skid marks if he was traveling 30 mi/h.
Check It Out! Example 6 Continued

\[ s = \sqrt{30fd} \]

\[ 30 = \sqrt{30(0.7)d} \quad \text{Substitute 30 for } s \text{ and } 0.7 \text{ for } f. \]

\[ 30 = \sqrt{21d} \quad \text{Simplify.} \]

\[ (30)^2 = (\sqrt{21d})^2 \quad \text{Square both sides.} \]

\[ 900 = 21d \quad \text{Simplify.} \]

\[ 43 \approx d \quad \text{Solve for } d. \]

If the car were traveling 30 mi/h, its skid marks would have measured about 43 ft. Because the actual skid marks measure less than 43 ft, the car was not speeding.
Lesson Quiz: Part I

Solve each equation or inequality.

1. $7 + \sqrt{2x + 4} = 13$ \hspace{1cm} x = 16

2. $3\sqrt[3]{x + 4} = \sqrt[3]{2x - 17}$ \hspace{1cm} x = -5

3. $\sqrt{2x + 12} = x + 2$ \hspace{1cm} x = 2

4. $4(\sqrt{x - 5})^2 = 12$ \hspace{1cm} x = 14

5. $\sqrt[3]{x + 5} \geq 4$ \hspace{1cm} x \geq 59
Lesson Quiz: Part II

6. The radius \( r \) in feet of a spherical water tank can be determined by using the formula

\[
r = \frac{3V}{4\pi},
\]

where \( V \) is the volume of the tank in cubic feet. To the nearest cubic foot, what is the volume of a spherical tank with a radius of 32 ft?

137,258 ft\(^3\)