ANTIDIFFERENTIATION

Keeper
Honors Calculus

THEOREM: ANTIDERIVATIVE

THEOREM 1

The antiderivative of $f(x)$ is the set of functions $F(x) + C$ such that

$$\frac{d}{dx}[F(x) + C] = f(x)$$

The constant C is called the **constant of integration.**

NOTATION

$\int f(x)dx$ is used to represent the antiderivative of $f(x)$.

BASIC INTEGRATION FORMULAS

- $\int k\, dx = kx + C$
- $\int x^r\, dx = \frac{x^{r+1}}{r+1} + C,\ provided\ r \neq 1$
- $\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$
- $\int be^{ax} dx = \frac{b}{a} e^{ax} + C$

EXAMPLES - EVALUATE

$\int 8dx$ $8x + C$	$\int 3x^2 dx$ $x^3 + C$

EXAMPLES - EVALUATE

$\int e^x dx$ $e^x + C$	$\int \left(\frac{1}{x}\right) dx$ $\ln\|x\| + C$

EXAMPLES - EVALUATE

$\int x^7 dx$

$\frac{x^8}{8} + C$

$\int x^{99} dx$

$\frac{x^{100}}{100} + C$

EXAMPLES - EVALUATE

$\int \sqrt{x} dx$

$\frac{2}{3} x^{3/2} + C$

$\int \frac{1}{x^3} dx$

$\frac{1}{-2x^2} + C$

EXAMPLES - EVALUATE

$\int x^{10} dx$

$\frac{x^{11}}{11} + C$

$\int x^{200} dx$

$\frac{x^{201}}{201} + C$

EXAMPLES - EVALUATE

$\int \sqrt[6]{x} dx$

$\frac{6}{7} x^{7/6} + C$

$\int \frac{1}{x^4} dx$

$\frac{1}{-3x^3} + C$

EXAMPLES - EVALUATE

$\int e^{4x} dx$

$\frac{e^{4x}}{4} + C$

$\int e^{-3x} dx$

$\frac{e^{-3x}}{-3} + C$

or

$\frac{1}{-3e^{3x}} + C$

EXAMPLES - EVALUATE

$\int e^{\left(\frac{1}{2}\right)x} dx$

$2e^{\frac{1}{2}x} + C$

$\int x^5 dx$

$\frac{x^6}{6} + C$

MORE ANTIDERIVATIVE RULES

$\int kf(x)dx = k\int f(x)dx$

$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

EXAMPLES - EVALUATE

$\int (3x^5 + 7x^2 + 8)dx$

$\frac{x^6}{2} + \frac{7}{3}x^3 + 8x + C$

$\int \frac{4 + 3x + 2x^4}{x} dx$

$\int 4 \cdot \frac{1}{x} + 3 + 2x^3$

$4\ln|x| + 3x + \frac{x^4}{2} + C$

EXAMPLES - EVALUATE

$\int \frac{x^2 + 7x + 2}{x^2} dx$

$\int 1 + 7 \cdot \frac{1}{x} + 2x^{-2}$

$x + 7\ln|x| - \frac{2}{x} + C$

$\int \frac{\pi}{x} dx$

$\pi \ln|x| + C$

EXAMPLE

Find the function f such that $f'(x) = x^2$ and $f(-1) = 2$

$\int x^2 = \frac{x^3}{3} + C$

$2 = \frac{-1^3}{3} + C$

$2 = -\frac{1}{3} + C$

$2 + \frac{1}{3} = C$

$\frac{7}{3} = C$

$f(x) = \frac{x^3}{3} + \frac{7}{3}$

ANTIDERIVATIVES AS AREAS

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EXAMPLE 1

A vehicle travels at 50 mi/hr for 2 hr. How far has the vehicle traveled? Draw a picture to illustrate.

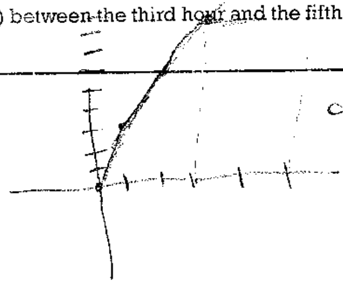
$2(50) = 100$ miles

EXAMPLE 2

The velocity of a moving object is given by the function $v(x) = 3x$, where x is in hours and v is in miles per hour. Use geometry to find the area under the graph, which is the distance the object has traveled:

a.) during the first 3 hr $(0 \le x \le 3)$;

b.) between the third hour and the fifth hour $(3 \le x \le 5)$.



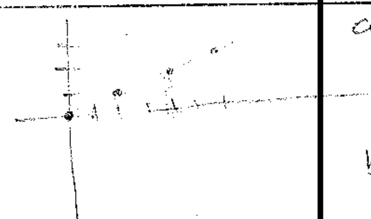
a) $\frac{1}{2}(3)(9) = \frac{27}{2}$ mi

b) $2(9) + \frac{1}{2}(2)(6)$

$18 + 6 = 24$ mi

YOU TRY!

- An object moves with a velocity of $v(t) = \frac{1}{2}t$ where t is in minutes and v is in feet per minute.
- a.) How far does the object travel during the first 30 min?
- b.) How far does the object travel between the first hour and the second hour?



a) $\frac{1}{2}(30)(15)$

225 feet

b) $60(30) + \frac{1}{2}60 \cdot 60$

$2(30)60$

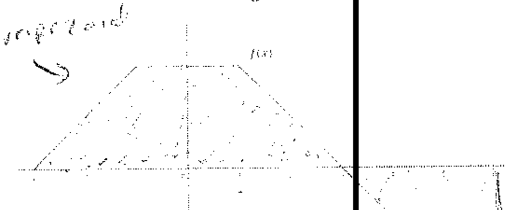
3600 ft

WHAT IS AN INTEGRAL???

The **AREA** under the curve!!!

EXAMPLE – FIND $\int_{-3}^{6} f(x)$

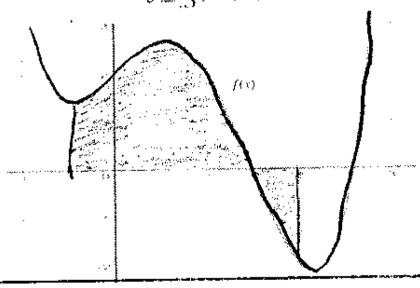
trapezoid



$\frac{1}{2}(2)(6+2) - \frac{1}{2}(1)(2+3)$

$8 - \frac{5}{2} = \frac{11}{2}$

EXAMPLE – FIND $\int_{-.5}^{2} f(x)$



Not easily calculated
Must use Riemann's
Sums.

RIEMANN SUMS

$$\int_a^b f(x)\,dx$$

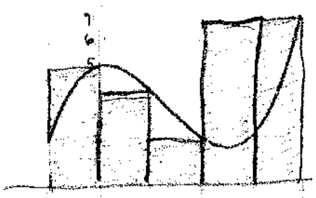
Right Riemann Sum (RRS)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)]$$

Left Riemann Sum (LRS)

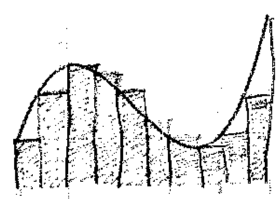
$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}[f(x_0) + f(x_1) + \cdots + f(x_n)]$$

APPROXIMATE THE INTEGRAL USING THE RIGHT ENDPOINT METHOD – 5 SUBINTERVALS



$1(6+4+2+7+7) = 25$

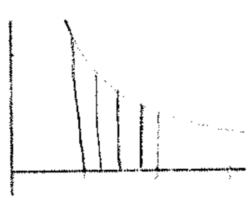
APPROXIMATE THE INTEGRAL USING THE LEFT ENDPOINT METHOD – 10 SUBINTERVALS



$\frac{1}{2}(2+4+5+4.8+4.2+3+2.2+2+2.2+3.8)$

16.6

USE THE LEFT ENDPOINT METHOD TO APPROXIMATE THE INTEGRAL - 4 SUBINTERVALS

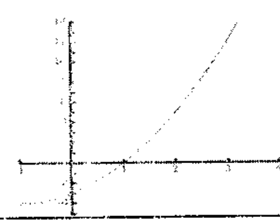


$$\int_1^2 \frac{1}{x}\,dx \qquad \frac{1}{4} = \Delta x$$

$\frac{1}{4}(4+2+\frac{4}{3}+1)$

$\frac{1}{4}(7+\frac{4}{3}) = \frac{1}{4}\cdot\frac{25}{3} = \frac{25}{12}$

USE THE RIGHT ENDPOINT METHOD TO APPROXIMATE THE INTEGRAL – 6 SUBINTERVALS



$$\int_0^3 x^2 + 2x - 3\,dx$$

$\Delta x = \frac{3-0}{6} = \frac{1}{2}$

$\frac{1}{2}(f(\frac{1}{2}) + f(\frac{1}{1}) + f(\frac{3}{2}) + f(2) + f(\frac{5}{2}) + f(3))$

$\frac{1}{2}(-\frac{7}{4} + 0 + \frac{9}{4} + 5 + \frac{33}{4} + 12)$

$\frac{103}{8} = 12.875$

RIEMANN SUMS

$$\int_a^b f(x)\,dx$$

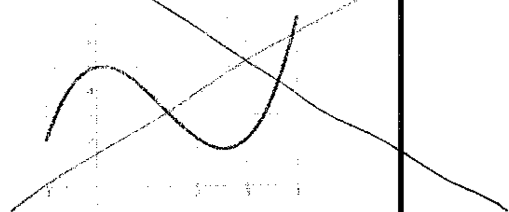
Midpoint Riemann Sum (MRS)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

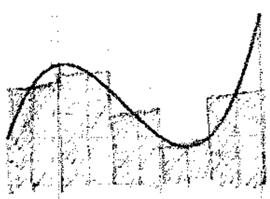
Trapezoid Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\cdot\frac{b-a}{n}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$$

~~APPROXIMATE THE INTEGRAL USING THE RIGHT ENDPOINT METHOD – 5 SUBINTERVALS~~



APPROXIMATE THE INTEGRAL USING THE MIDPOINT METHOD – 5 SUBINTERVALS



$1(4+5+3+1.8+4)$

17.8

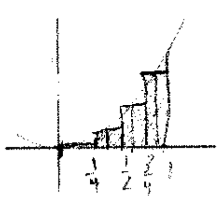
APPROXIMATE THE INTEGRAL USING THE TRAPEZOID METHOD – 5 SUBINTERVALS

$\frac{1}{2}\cdot 1(0+2(5)+2(4)+2(2)+2(2)+7)$

$\frac{1}{2}(10+8+4+4+7)$

$33/2$

USE THE MIDPOINT METHOD TO APPROXIMATE THE INTEGRAL – 4 SUBINTERVALS



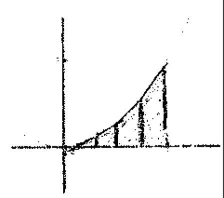
$$\int_0^1 x^2$$

$\frac{1}{4}\left(f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right)$

$\frac{1}{4}\left(\frac{1}{64}+\frac{9}{64}+\frac{25}{64}+\frac{49}{64}\right)$

$\frac{21}{64}$

USE THE TRAPEZOID METHOD TO APPROXIMATE THE INTEGRAL – 4 SUBINTERVALS



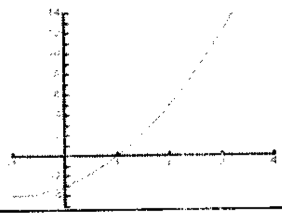
$$\int_0^1 x^2$$

$\frac{1}{2}\left(\frac{1}{4}\right)\left(f(0)+2f(1/4)+2f(1/2)+2f(3/4)+f(1)\right)$

$\frac{1}{8}\left(0+\frac{1}{8}+\frac{1}{2}+\frac{9}{8}+1\right)$

$11/32$

USE THE TRAPEZOID METHOD TO APPROXIMATE THE INTEGRAL – 6 SUBINTERVALS



$$\int_0^3 x^2 + 2x - 3\,dx$$

$\Delta x = \frac{3-0}{6} = \frac{1}{2}$

$\frac{1}{2}\left(\frac{1}{2}\right)\left[f(0) + 2f(1/2) + 2f(1) + 2f(3/2) + 2f(2) + 2f(5/2) + f(3)\right]$

$\frac{1}{4}\left(-3 + -7/2 + 0 + 9/2 + 10 + 33/2 + 12\right)$

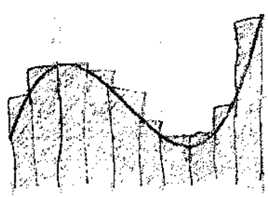
$\frac{73}{8}$

RIEMANN SUMS – 2 MORE METHODS

Circumscribed Method – Highest point in interval is used to create the rectangle.

Inscribed Method – Lowest Point in the interval is used to create the rectangle.

APPROXIMATE THE INTEGRAL USING THE CIRCUMSCRIBED METHOD – 10 SUBINTERVALS

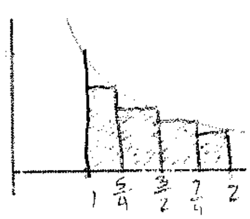


$\frac{1}{2}(h_1 + h_2 + \dots + h_{10})$

APPROXIMATE THE INTEGRAL USING THE INSCRIBED METHOD – 10 SUBINTERVALS

USE THE INSCRIBED METHOD TO APPROXIMATE THE INTEGRAL

4 sub intervals



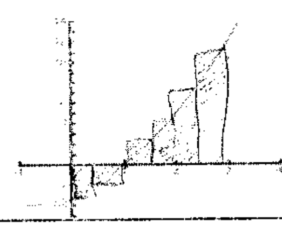
$$\int_1^2 \frac{1}{x}\,dx$$

$\frac{1}{4}\left(f(5/4) + f(3/2) + f(7/4) + f(2)\right)$

$\frac{1}{4}\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right)$

$\frac{533}{840} = .6345$

USE THE CIRCUMSCRIBED METHOD TO APPROXIMATE THE INTEGRAL – 6 SUBINTERVALS



$$\int_0^3 x^2 + 2x - 3\,dx$$

$\frac{1}{2}\left(2(0) + f(1/2) + f(3/2) + f(2) + f(5/2) + f(3)\right)$

$\frac{1}{2}\left(-3 - \frac{7}{4} + 0 + \frac{9}{4} + 5 + \frac{33}{4} + 12\right)$

$\frac{91}{8}$

DEFINITE INTEGRALS

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STEPS FOR EVALUATING DEFINITE INTEGRALS

To find the area under the graph of a nonnegative, continuous function f over the interval $[a, b]$:

1. Find any antiderivative $F(x)$ of $f(x)$.
2. Evaluate $F(x)$ using b and a, and compute $F(b) - F(a)$. The result is the area under the graph over the interval $[a, b]$.

EXAMPLE 1

Find the area under the graph of $y = x^2 + 1$ over the interval $[-1, 2]$.

$\int_{-1}^{2} x^2 + 1$

$\frac{x^3}{3} + 1x \Big|_{-1}^{2}$

$\frac{2^3}{3} + 2 - \left(\frac{-1}{3} - 1\right) = 6$

EXAMPLE 1

Find the area under the graph of $y = x^2 + 1$ over the interval $[-0,5]$.

$\int_{0}^{5} x^2 + 1$

$\frac{x^3}{3} + x \Big|_{0}^{5}$

$\frac{5^3}{3} + 5 = \frac{140}{3}$

DEFINITE INTEGRAL DEFINITION

Let f be any continuous function over the interval $[a, b]$ and F be any antiderivative of f. Then, the **definite integral** of f from a to b is

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

EVALUATE $\int_a^b x^2 dx.$

$\frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}$

EVALUATE $\int_{-1}^{4} (x^2 - x)\, dx$

$\frac{x^3}{3} - \frac{x^2}{2} \Big|_{-1}^{4} = \frac{85}{6}$

EVALUATE $\int_0^3 e^x \, dx$

$e^x \Big|_0^3 = e^3 - 1$

EVALUATE $\int_1^e \left(1 + 2x - \frac{1}{x}\right) dx$ (assume $x > 0$).

$x + x^2 - \ln|x| \Big|_1^e = 7{,}107$

EVALUATE $\int_2^4 \left(2x^3 - 3x\right) dx$

$\frac{2x^4}{4} - \frac{3x^2}{2} \Big|_2^4 = 102$

EVALUATE $\int_0^{\ln 4} 2e^x dx$

$2e^x \Big|_0^{\ln 4} = 8 - 2 = 6$

EVALUATE $\int_1^5 \frac{x-1}{x} dx$

$x - \ln|x| \Big|_1^5 =$

$5 - \ln 5 - 1$

$4 - \ln 5$

THE FUNDAMENTAL THEOREM OF CALCULUS

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THE FUNDAMENTAL THEOREM OF CALCULUS

Part 1:
If a continuous function f has an antiderivative F over $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

EXAMPLE

$$\int_1^2 (x^2 - 3)dx$$

$$\frac{x^3}{3} - 3x \Big|_1^2 = -\frac{2}{3}$$

EXAMPLE

$$\int_1^4 3\sqrt{x}\,dx$$

$$\frac{3x^{3/2}}{3/2} \Big|_1^4 = 14$$

EXAMPLE

$$\int_0^{\frac{\pi}{4}} (\sec^2 x)dx$$

$$\tan x \Big|_0^{\pi/4} = 1$$

EXAMPLE

$$\int_0^2 (2x^2 - 3x + 2)dx$$

$$\frac{2x^3}{3} - \frac{3x^2}{2} + 2x \Big|_0^2 = \frac{10}{3}$$

THE FUNDAMENTAL THEOREM OF CALCULUS

Part 2:
If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

EXAMPLE

$$\frac{d}{dx}\int_0^x \sqrt{t^2+1}\,dt$$

$\sqrt{x^2+1}$

EXAMPLE

$$\frac{d}{dx}\int_2^x \csc^2 t\,dt$$

$\csc^2 x$

EXAMPLE

$$\frac{d}{dx}\int_x^2 \sin t^2\,dt$$

$-\sin x^2$

EXAMPLE

$$\frac{d}{dx}\int_{\frac{\pi}{2}}^{x^3} \cos t\,dt$$

$3x^2 \cos x^3$

EXAMPLE

$$\frac{d}{dx}\int_3^{x^2} \sqrt{t^2 - 4\sin t}\,\,dt$$

$2x\sqrt{x^4 - 4\sin x^2}$

EXAMPLE

$$\frac{d}{dx}\int_{3x}^{4x^2} \frac{4t}{1+t^2}\,dt$$

$$8x \cdot \frac{4 \cdot 4x^2}{1+(4x^2)^2} - 3 \cdot \frac{4(3x)}{1+(3x)^2}$$

$$\boxed{\frac{128x^3}{1+16x^4} - \frac{36x}{1+9x^2}}$$