Warm-up:
Given the following solutions, find the factors of the quadratic. Then, find the standard form.

1. $x = -3, 5$
2. $x = 2/3, 0$
3. $x = -3/5, -1$
4. $x = \pm \sqrt{3}$
5. $x = \pm 2i$
6. $x = 3 \pm \sqrt{2}$
7. $x = -2 \pm 5i$
8. $x = 1 \pm 2i\sqrt{3}$

Vocabulary

**Multiplicity** - If a polynomial $P(x)$ has a multiple root at $r$, the multiplicity of $r$ is the number of times $(x - r)$ appears as a factor in $P(x)$.

**Roots** - Any value of the variable that makes the equation true. Also known as $x$-intercepts, zeros, or solutions.
Fundamental Theorem of Algebra

MGSE9-12.N.CN.9 Use the Fundamental Theorem of Algebra to find all roots of a polynomial equation.

What am I learning today?
How to use the Fundamental Theorem of Algebra to solve polynomial equations

How will I show that I learned it?
Apply the Fundamental Theorem of Algebra to find all roots of a polynomial equation
Fundamental Theorem of Algebra

Every polynomial of degree \( n \geq 1 \) has at least one zero, where a zero may be a complex number.

Corollary: Every polynomial function of degree \( n \geq 1 \) has exactly \( n \) zeros including multiplicities and imaginary roots.

Solving for all possible solutions using the Fundamental Theorem of Algebra

1. Set the function equal to 0.

2. Factor fully.

3. Set each factor equal to zero and solve using the quadratic formula for unfactorable quadratics.

4. You should have the same number of answers as the degree of the polynomial when including the multiplicities.
Example 1: Solve for all solutions.
\[ P(x) = 8x^3 - 4x^2 - 50x + 25 \]
\[ O = (8x^3 - 4x^2 - 50x + 25) \]
\[ O = 4x^2(2x-1) - 25(2x-1) \]
\[ O = (4x^2 - 25)(2x-1) \]
\[ O = (2x+5)(2x-5)(2x-1) \]
\[ 2x + 5 = 0 \Rightarrow x = -\frac{5}{2} \]
\[ 2x - 5 = 0 \Rightarrow x = \frac{5}{2} \]
\[ 2x - 1 = 0 \Rightarrow x = \frac{1}{2} \]

Example 2: Solve for all solutions.
\[ P(x) = 9x^9 - 16x^7 + 9x^6 - 16x^4 \]
\[ O = 9x^9 - 16x^7 + 9x^6 - 16x^4 \]
\[ O = x^4(9x^5 - 16x^3 + 9x^2 - 16) \]
\[ O = x^4[(x^3(9x^2 - 16) + 1)(9x^2 - 16)] \]
\[ O = x^4(x^3 + 1)(9x^2 - 16) \]
\[ O = x^4(x+1)(x^2 - x + 1)(3x + 4)(3x - 4) \]
\[ x^4 = 0 \Rightarrow x = 0 \]
\[ x^4 = 0 \Rightarrow x = 0 \]
\[ x + 1 = 0 \Rightarrow x = -1 \]
\[ x^2 - x + 1 = 0 \]
\[ x = \frac{1 \pm \sqrt{1 - 4}}{2} \]
\[ x = \frac{1 \pm i\sqrt{3}}{2} \]
\[ 3x + 4 = 0 \Rightarrow x = -\frac{4}{3} \]
\[ 3x - 4 = 0 \Rightarrow x = \frac{4}{3} \]
Example 3: Solve for all solutions.

\[ P(x) = 6x^3 - 29x^2 - 45x + 18 \text{ if } x = 6 \text{ is a root} \]

\[
\begin{array}{c|cccc}
6 & 6 & -29 & -45 & 18 \\
36 & 42 & -18 & \\
-3 & 0 & \\
\end{array}
\]

\( (x-6) \text{ is a factor} \)

\[
(6x^2 + 7x - 3)(x - 6) = 0
\]

\[
(6x^2 + 9x - 2x - 3)(x - 6) = 0
\]

\[
(3x - 1)(2x + 3)(x - 6) = 0
\]

\[
\begin{align*}
3x - 1 &= 0 & \Rightarrow & x = \frac{1}{3} \\
2x + 3 &= 0 & \Rightarrow & x = -\frac{3}{2} \\
x - 6 &= 0 & \Rightarrow & x = 6
\end{align*}
\]

Example 4: Solve for all solutions.

\[ P(x) = 4x^3 - 13x^2 + 11x - 2 \text{ if } x = 1/4 \text{ is a root} \]

\[
\begin{array}{c|cccc}
\frac{1}{4} & 4 & -13 & 11 & -2 \\
1 & -3 & 2 & \\
\frac{1}{4} & -\frac{12}{4} & \frac{8}{4} & \frac{10}{4} \\
\end{array}
\]

\( (4x - 1) \text{ is a factor} \)

\[
(4x - 1)(x^2 - 3x + 2) = 0
\]

\[
(4x - 1)(x - 2)(x - 1) = 0
\]

\[
4x - 1 = 0 \quad x - 2 = 0 \quad x - 1 = 0
\]

\[
\begin{align*}
x &= \frac{1}{4} & x &= 2 & x &= 1
\end{align*}
\]
Rational Roots Theorem

If the polynomial $P(x)$ has integer coefficients, then every rational root of the polynomial equation $P(x) = 0$ can be written in the form $\frac{p}{q}$ where:

- $p$ is a factor of the CONSTANT TERM of $P(x)$
- $q$ is a factor of the LEADING COEFFICIENT of $P(x)$

*must remove GCF before evaluating possible rational roots*

Example 1: Find all possible rational roots

$P(x) = 8x^3 - 4x^2 - 40x + 20$

$\frac{4}{4} \quad \frac{4}{4} \quad \frac{4}{4} \quad \frac{4}{4}$

$P(x) = 2x^3 - x^2 - 10x + 5$

$\pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{5}{1}, \pm \frac{5}{2}$

$\boxed{\pm 1, \pm \frac{1}{2}, \pm 5, \pm \frac{5}{2}}$
Example 2: Find all possible rational roots

\[ P(x) = 3x^3 - 18x^2 - x + 6 \]

\[
\begin{align*}
P &= \frac{6}{1, 2, 3, 6} \\
q &= \frac{3}{1, 3}
\end{align*}
\]

\[ \pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}, \pm 3, \pm \frac{6}{3}, \pm 6, \pm \frac{6}{3} \]

\[ \pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}, \pm 3, \pm 6 \]

You Try: Find all possible rational roots

1. \[ P(x) = 16x^6 - 1 \]
2. \[ P(x) = 24x^2 + 3x^5 \]
3. \[ P(x) = 9x^9 - 16x^7 + 9x^6 - 16x^4 \]
Steps for solving a polynomial:
1. List all possible rational roots of a polynomial.

2. Begin testing roots using synthetic division. Remember, the same root might be used multiple times.

3. Keep factoring remaining polynomial until unfactorable or a degree of 2.

4. Set all factors equal to 0 and solve.

Ex. 1  \[ P(x) = x^4 + x^3 + 2x^2 + 4x - 8 \]
\[ p = 8 \]
\[ q = 1 \]
\[ \pm 1, \pm 2, \pm 4, \pm 8 \]

\[
\begin{array}{c|cccc}
1 & 1 & 1 & 2 & 4 \\
\hline \\
 & 1 & 2 & 4 & 8
\end{array}
\]

\[ (x-1)(x^3+2x^2+4x+8) = 0 \]
\[ (x-1)(x^2+4(x+2)) = 0 \]
\[ (x-1)(x^2+4)(x+2) = 0 \]

\[ x-1=0 \quad x^2+4=0 \quad x+2=0 \]
\[ x=1 \quad x=\pm 2i \quad x=-2 \]
Ex. 2  \[ P(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3 \]

\[
\begin{array}{cccc}
p & = & \frac{3}{1,3} & \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2} \\
q & = & \frac{2}{1,2} & -5 & -5 & 5 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{x} & = & 2 & -3 & -8 & -3 \\
\text{q} & = & 2 & -3 & -8 & -3 \\
\end{array}
\]

\[
(x-1)(x+1)(2x^2-5x-3) = 0 \]

\[
(x-1)(x+1)(2x+1)(x-3) = 0
\]

\[
\begin{array}{c}
x-1=0 \\
x=1 \\
\hline
x+1=0 \\
x=-1 \\
\hline
2x+1=0 \\
x=-\frac{1}{2} \\
\hline
x-3=0 \\
x=3
\end{array}
\]

Ex. 3  \[ P(x) = 2x^4 - x^3 - 15x^2 + 8x + 20 \]

\[
\begin{array}{cccc}
p & = & \frac{20}{1,2,4,5,10,20} & \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20 \\
q & = & \frac{2}{1,2} & -1 & -15 & 8 & -20 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{x} & = & 2 & -3 & -12 & 20 \\
\text{q} & = & 1 & -10 & -10 & 0 \\
\end{array}
\]

\[
(x+1)(x-2)(2x^2+x-10) = 0 \]

\[
(x+1)(x-2)(2x^3+5x-4x-10) = 0 \\
(x+1)(x-2)(2x+5) = 0
\]

\[
\begin{array}{c}
x+1=0 \\
x=-1 \\
\hline
x-2=0 \\
x=2 \\
\hline
2x+5=0 \\
x=-\frac{5}{2}
\end{array}
\]