OIMT 2015 - Open Exam II Solutions

Name:____

School: ____

Instructions: The exam lasts TWO AND A HALF HOURS. Calculators are NOT allowed. There are three parts to this exam.

- PART I consists of 12 multiple choice questions.
  Scoring: 4 points for each correct answer.
- PART II consists of 8 problems with single number answers.
  Scoring: 6 points for each correct answer.
- PART III consists of 4 longer problems.
  Scoring: 12 points for each correct answer. Partial credit may be given, so you should show your work.

Each part starts with easier problems and ends in harder ones, so consider moving on to the next part if you find you are getting stuck on the later questions. We recommend you spend no more than 50 minutes on each part.

	Score
Part I	
Part II	
Part III	
Total	

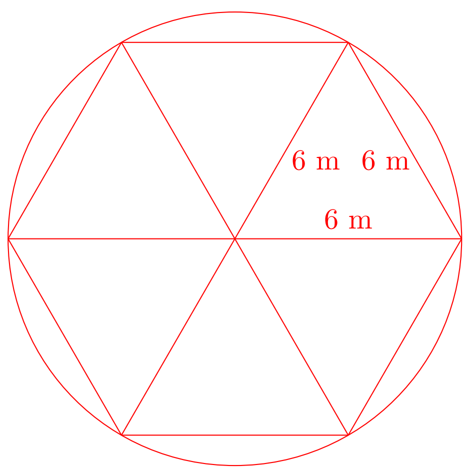
PART I.

1. Find the perimeter of a regular hexagon that is inscribed in a circle of radius 6m.

   (a) 24 m
   (b) 36 m
   (c) $24\sqrt{3}$ m
   (d) $36\sqrt{2}$ m
   (e) $36\sqrt{3}$ m
   (f) None of the above

   Answer: (b)

   Solution. A hexagon inscribed in a circle can be divided into six equilateral triangles, each with one vertex at the center of the circle and the other two vertices being consecutive vertices of the hexagon. See the picture:



   Thus two sides of each of these triangles are radii of the circle, and the other side is one of the 6 sides of the hexagon. So the perimeter of the hexagon is 6 times the radius of the circle, or $6 \cdot 6 = 36$. □

2. Express $\dfrac{1\cdot2\cdot3+2\cdot4\cdot6+4\cdot8\cdot12+7\cdot14\cdot21}{1\cdot3\cdot5+2\cdot6\cdot10+4\cdot12\cdot20+7\cdot21\cdot35}$ as a simple fraction in lowest terms.

   (a) $\frac{2}{3}$
   (b) $\frac{3}{5}$
   (c) $\frac{1}{2}$
   (d) $\frac{2}{5}$

(e) None of the above

Answer: (d)

Solution. We factor the numerator and denominator, getting

$$\frac{1\cdot 2\cdot 3+2\cdot 4\cdot 6+4\cdot 8\cdot 12+7\cdot 14\cdot 21}{1\cdot 3\cdot 5+2\cdot 6\cdot 10+4\cdot 12\cdot 20+7\cdot 21\cdot 35}=\frac{(1\cdot 2\cdot 3)(1+2^3+4^3+7^3)}{(1\cdot 3\cdot 5)(1+2^3+4^3+7^3)}=\frac{6}{15}=\frac{2}{5}.$$

□

3. If $f(x) = 3x - 7$ and $g(x) = x^2 - 4$, what is $f(g(f^{-1}(5)))$?

(a) 27
(b) 42
(c) 54
(d) 173
(e) None of the above

Answer: (e)

Solution. We can solve

$$f^{-1}(y) = \frac{y+7}{3},$$

so $f^{-1}(5) = 4$. (We can also get $f^{-1}(5) = 4$ by easy trial and error, assuming it should be an integer because the proposed answers are all integers.) Now

$$f(g(f^{-1}(5))) = f(g(4)) = f(12) = 36 - 7 = 29.$$

So the answer is "none of the above". □

4. A 180 m long train is traveling at a constant speed. Suppose it takes the train 90 seconds to pass by a fixed signal. Traveling at the same constant speed, the train passes through a tunnel that is 360 m long. How much time passes from the instant the front end of the train enters the tunnel to the instant the back end of the train exits the tunnel?

(a) 180 seconds
(b) 270 seconds
(c) 360 seconds

(d) 450 seconds
(e) None of the above

Answer: (b)

Solution. It takes the train 90 seconds to go 180 meters. Therefore it takes the train $2 \cdot 90 = 180$ seconds to go $360 = 2 \cdot 180$ meters. This is the length of time from when the front of the train enters the tunnel to when the *front* of the train leaves the tunnel. It takes another 90 seconds for the whole train to pass any single point, so the total time is $180 + 90 = 270$ seconds. □

5. Which answer correctly describes all of the solutions to the following equation?

$$\log(x+2) + \log(x-3) = \log(14x) + \log(1/x)$$

(a) -4
(b) 4
(c) 20
(d) 5
(e) Both -4 and 5
(f) None of the above

Answer: (d)

Solution. Use the properties of logarithms to combine terms:

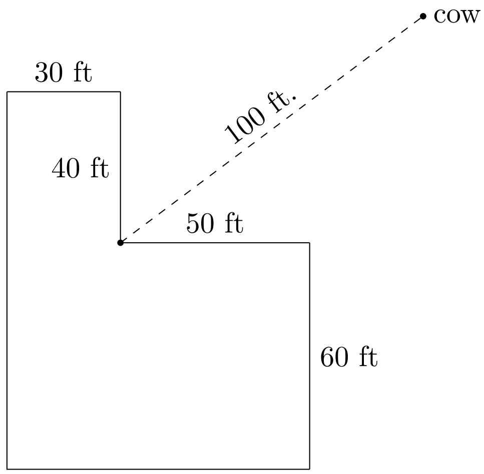
$$\log\big((x+2)(x-3)\big) = \log\left(\frac{14x}{x}\right) = \log(14).$$

Exponentiate, expand, rearrange, and solve by factoring:

$$\begin{aligned}(x+2)(x-3) &= 14\\ x^2 - x - 6 &= 14\\ 0 = x^2 - x - 20 &= (x+4)(x-5)\\ x = 5 \quad \text{or} \quad x &= -4.\end{aligned}$$

Substitute back in the original equation. The result is not defined if $x = -4$, since we are dealing with real numbers. (It would also not work with complex numbers, since the conventional choice of branch of the logarithm defined for negative real numbers would give right hand side $\log(14 \cdot 4) + \pi i + \log\left(\frac{1}{x}\right) + \pi i$, which is not real, while the left hand side is real.) At $x = 5$, the left hand side is $\log(7) + \log(2) = \log(14)$ and the right hand side is $\log(14 \cdot 5) - \log(5) = \log(14)$, so $x = 5$ *is* a solution. Thus there is exactly one solution, namely $x = 5$. □

6. A cow is tethered on a grass field by a 100 ft rope attached to the inside corner of an L-shaped building, as shown below. Find the area of grass that the cow can graze.



(a) 4250π ft^2

(b) 17000π ft^2

(c) 240π ft^2

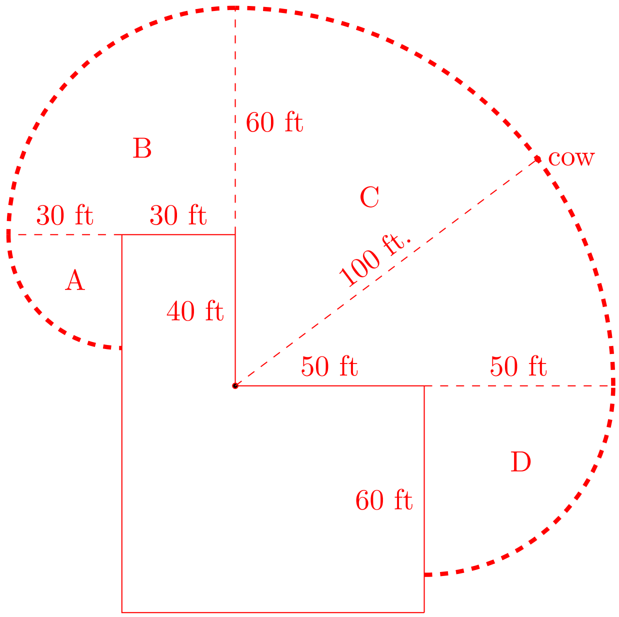
(d) 3650π ft^2

(e) 2150π ft^2

(f) None of the above

Answer: (a)

Solution. The region accessible to the cow is the area bounded by the walls of the building and the thick dashed lines in the picture below. It has four parts, labelled A, B, C, and D, each of which is a quarter circle. Region A has center at the top left corner of the building, and radius 30 feet. Region B has center at the corner of the building above the interior corner, and radius 60 feet. Region C has center at the interior corner of the building, and radius 100 feet. Region D has center at the corner of the building to the right of the interior corner, and radius 50 feet.



The total area, in square feet, is

$$\frac{1}{4}\left[\pi(30)^2 + \pi(60)^2 + \pi(100)^2 + \pi(50)^2\right] = \frac{100\pi}{4}(3^2 + 6^2 + 10^2 + 5^2)$$
$$= \frac{100\pi}{4}(170) = 4250\pi.$$

□

7. Arrange the numbers 2^{300}, 3^{200} and 5^{120} in increasing order.

   (a) $5^{120} < 3^{200} < 2^{300}$
   (b) $2^{300} < 5^{120} < 3^{200}$
   (c) $5^{120} < 2^{300} < 3^{200}$
   (d) $2^{300} < 3^{200} < 5^{120}$
   (e) $3^{200} < 5^{120} < 2^{300}$
   (f) None of the above

   Answer: (c)

   Solution. The numbers 2^{300} and 3^{200} are in the same order as

$$\sqrt[100]{2^{300}} = 2^3 = 8 \quad \text{and} \quad \sqrt[100]{3^{200}} = 3^2 = 9.$$

The numbers 5^{120} and 2^{300} are in the same order as

$$\sqrt[40]{5^{120}} = 5^3 = 125 \quad \text{and} \quad \sqrt[40]{2^{300}} = 2^{15/2} = 2^7\sqrt{2} = 128\sqrt{2}.$$

So $5^{120} < 2^{300} < 3^{200}$. □

8. Three snails, Alice, Bobby and Cindy, were racing down a road. Whenever one snail passed another, it waved at the snail it passed. During the race, Alice waved 3 times and was waved at twice. Bobby waved 4 times and was waved at 3 times. Cindy waved 5 times. How many times was she waved at?

   (a) 7
   (b) 3
   (c) 6
   (d) 1
   (e) 2
   (f) None of the above

   Answer: (a)

   Solution. Let n be the number of times Cindy was waved at. The total number of times snails were waved at is the same as the total number of times snails waved. So $3+4+5 = 2+3+n$, giving $n = 7$. □

9. Ben leaves for school 5 minutes after his sister. He follows the same route as her but walks one and a half times faster. If Ben's sister is walking at a constant speed, how long will it take him to catch up to her?

   (a) 2 minutes and 30 seconds
   (b) 5 minutes
   (c) 7 minutes and 30 seconds
   (d) 10 minutes
   (e) None of the above

   Answer: (d)

   Solution. The relative speed at which Ben is catching up to his sister is equal to half of his sister's walking speed. Therefore, it will take Ben twice the amount of time his sister has been walking to make up the distance between them. Since Ben leaves 5 minutes after his sister, it will take him 10 minutes to catch up to her. □

10. If $\sqrt{15+\sqrt{216}} = x + \sqrt{y}$, where x and y are integers, what is $x+y$?

   (a) 7
   (b) 9
   (c) 10
   (d) 12
   (e) 15
   (f) None of the above

   Answer: (b)

   Solution. We give a complete solution first, then a much shorter argument which is sufficient for a contestant to get the problem right.

   Lemma 1. *Let $a, x \in \mathbb{Z}$ and let $b, y \in \mathbb{Z}_{\geq 0}$. Suppose $a + \sqrt{b} = x + \sqrt{y}$, and that one of b and y is not the square of an integer. Then $a = x$ and $b = y$.*

   Proof. Without loss of generality, b is not a perfect square. In particular, $b \neq 0$. It suffices to prove that $b = y$. We have $\sqrt{y} - \sqrt{b} \in \mathbb{Z}$, so

$$2\sqrt{by} = y + b - \left(\sqrt{y} - \sqrt{b}\right)^2 \in \mathbb{Z}.$$

   Therefore $\sqrt{by} \in \mathbb{Q}$. Since $by \in \mathbb{Z}$, this can only happen if there is an integer $n \geq 0$ such that $n^2 = by$. Since $b \neq 0$, we get

$$y = \frac{n^2}{b}.$$

   Since $n \geq 0$, it follows that

$$\frac{n}{\sqrt{b}} - \sqrt{b} = \sqrt{y} - \sqrt{b} \in \mathbb{Z}.$$

   Also

$$\sqrt{b}\left(\sqrt{y} - \sqrt{b}\right) = n - b \in \mathbb{Z}.$$

   If $b \neq y$, it follows that

$$\sqrt{b} = \frac{n-b}{\sqrt{y} - \sqrt{b}} \in \mathbb{Q}.$$

   But then we must have $\sqrt{b} \in \mathbb{Z}$, a contradiction. □

   By the lemma, it suffices to find some $x \in \mathbb{Z}$ and $y \in \mathbb{Z}_{\geq 0}$ such that y is not a perfect square and such that

$$\sqrt{15+\sqrt{216}} = x + \sqrt{y}.$$

   Squaring both sides gives

$$15 + \sqrt{216} = \left(x + \sqrt{y}\right)^2 = x^2 + y + 2x\sqrt{y} = x^2 + y + \sqrt{4x^2 y}.$$

Since $216 = 2^3 \cdot 3^3$ is not a perfect square, by the lemma we must have

$$x^2 + y = 15 \quad \text{and} \quad 4x^2 y = 216 = 4 \cdot 2 \cdot 3^3.$$

The only integer solutions to the second equation have $y = 6$ or $y = 54$. Substituting these in the first equation gives $x = \pm 3$ in the first case and no integer solutions for x in the second case. Taking $x = -3$ and $y = 6$ gives $x + \sqrt{y} < 0$, so $x + \sqrt{y}$ is not the square root of anything. Therefore $x = 3$ and $y = 6$ is the only possible solution. One easily checks that $\left(3 + \sqrt{6}\right)^2$ really is $15 + \sqrt{216}$. Since $3 + \sqrt{6} > 0$ and $15 + \sqrt{216} > 0$, this implies $3 + \sqrt{6} = \sqrt{15 + \sqrt{216}}$. So the answer is $x + y = 3 + 6 = 9$.

This is much more than what is needed to solve the problem as a contestant.

Fast "solution": The problem probably wouldn't be stated like this if there were more than one possible value of $x + y$. Start with the equation

$$\sqrt{15 + \sqrt{216}} = x + \sqrt{y}.$$

Square both sides to get

$$15 + \sqrt{216} = \left(x + \sqrt{y}\right)^2 = x^2 + y + 2x\sqrt{y}.$$

We implicitly assume the lemma is true, and guess that we must have

$$x^2 + y = 15 \quad \text{and} \quad 2x\sqrt{y} = \sqrt{216}.$$

The second equation simplifies to $x\sqrt{y} = \sqrt{54} = 3\sqrt{6}$. So we guess $x = 3$ and $y = 6$, and check that this gives a solution to the first equation. □

11. What are the last four digits of the number 5^{1000} ?

    (a) 3125
    (b) 5625
    (c) 0625
    (d) 8125
    (e) None of the above

Answer: (c)

Solution. For $n \in \mathbb{Z}_{\geq 0}$ define numbers $r_n, k_n \in \mathbb{Z}_{\geq 0}$ by

$$5^n = r_n + 10{,}000 \cdot k_n \quad \text{and} \quad 0 \leq r_n < 10{,}000.$$

We want to find r_{1000}.

For $n \geq 4$ set $s_n = r_n/5^4$. We have

$$5^{n-4} = s_n + \left(\frac{10^4}{5^4}\right) \cdot k_n = s_n + 16 \cdot k_n.$$

So $s_n \in \mathbb{Z}_{\geq 0}$, and is determined by the equation $5^{n-4} = s_n + 16 \cdot k_n$ subject to the conditions $s_n, k_n \in \mathbb{Z}_{\geq 0}$ and $0 \leq r_n < 16$. Clearly $s_4 = 1$. Repeatedly multiplying by 5 and reducing mod 16, we get

$$s_5 = 5, \quad s_6 = 9, \quad s_7 = 12, \quad \text{and} \quad s_8 = 1.$$

Clearly, then, for $j \in \mathbb{Z}_{\geq 0}$ we have

$$s_{4j+4} = 1, \quad s_{4j+5} = 5, \quad s_{4j+6} = 9, \quad \text{and} \quad s_{4j+7} = 12.$$

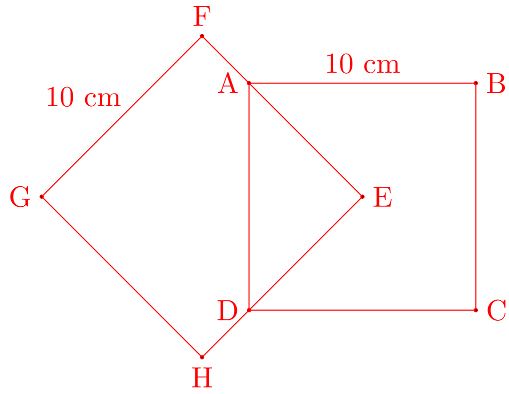
Since 1000 has the form $4j + 4$, we get $s_{1000} = 1$, so $r_{1000} = 5^4 = 625$. Therefore the correct answer is 0625. □

12. Two congruent squares ABCD and EFGH with sides of length 10 cm are placed on the plane so that the vertex E of the second square coincides with the center of square ABCD, and the side EF passes through A. Find the area of the part of the plane covered by these squares.

    (a) $12\sqrt{2}$ cm^2
    (b) 100 cm^2
    (c) 150 cm^2
    (d) $100 + 50\sqrt{2}$ cm^2
    (e) 175 cm^2
    (f) None of the above

    Answer: (e)

    Solution. The line EF contains part of a diagonal of the square ABCD. The line perpendicular to EF and through E passes through the center of the square ABCD and is perpendicular to one diagonal, so must contain the other. Therefore the side EH must pass through D or B. Assuming EH passes through D, the arrangement of squares is as shown in the diagram below. If EH passes through B instead, the diagram is congruent to this one via a reflection.



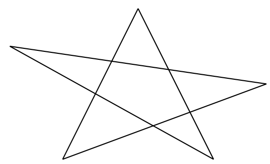
The squares ABCD and EFHG overlap in the triangle ADE. The square ABCD is exactly covered by four congruent copies of ADE, so the required area, in cm^2, is

$$10^2 + 10^2 - \left(\frac{1}{4}\right) 10^2 = 175.$$

□

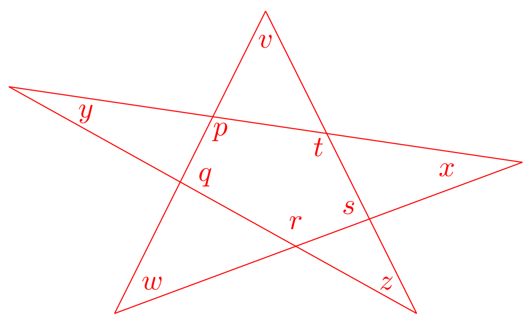
PART II.

1. What is sum of the angles (in degrees) at the five points of a pentagonal star?



Answer: $180°$

Solution. We label the angles in the following way



Notice that inside of the star sits a pentagon. Since the interior angles of an n-gon sum to $180(n-2)$ degrees, we have

$$p + q + r + s + t = 540.$$

By considering the 5 small triangles, which form the points of the star, we see that each interior angle of the pentagonal star is equal to the sum of the touching angles from the pentagon minus $180°$. For instance,

$$v + (180 - p) + (180 - t) = 180 \Longrightarrow v = p + t - 180.$$

Now each of the angles in the pentagon will appear in exactly two such sums. Therefore,

$$v + w + x + y + z = 2(p + q + r + s + t) - 5 \cdot 180 = 2 \cdot 540 - 900 = 180.$$

Hence, the sum of the interior angles of the pentagonal star is $180°$. □

2. For how many integers n between 1 and 200 (inclusive) is n^n a perfect square?

Answer: 107

Solution. We need

$$n^n = x^2$$

for some integer x, hence

$$n^{\frac{n}{2}} = x$$

is an integer. Then either n is even (there are 100 even numbers between 1 and 200) or n is a perfect square (there are 7 odd perfect squares; $14^2 = 196$). The answer is 107. □

3. Sam was rowing upstream in a boat and, while going under a bridge, dropped his water bottle. He noticed this 20 minutes later, and immediately turned around and rowed back to fetch it. He reached the bottle 2 miles downstream from the bridge. If Sam rows at a constant rate and the current moves at a constant speed, what is the speed of the current in miles per hour?

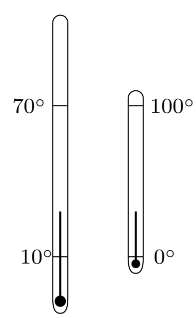
Answer: 3 miles per hour

Solution. Since Sam is always rowing at the same rate (and the current is always moving at the same speed), it will take 20 minutes for him to catch up to the bottle once he realizes that he dropped it. Hence, the bottle has been floating downstream for 40 minutes. If the river is flowing at x miles per minute, then

$$2 = 40x$$

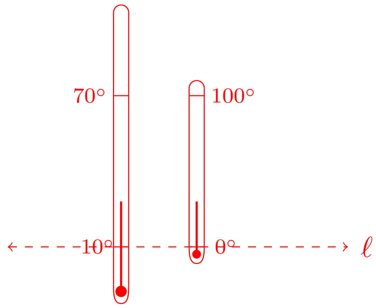
hence $x = \frac{1}{20}$. That is, the current is 3 miles per hour. □

4. Two thermometers are hanging next to each other. As shown in the picture, the 0° mark on the small thermometer is aligned with the 10° mark on the big thermometer, while the 100° mark on the small one is aligned with the 70° mark on the big one. What is the temperature when the mercury in the two thermometers is at exactly the same horizontal level?



Answer: 25°

Solution. Call the thermometer on the left A and the one on the right B. We'll denote the distance between the 10° mark and 70° mark on termometer A as 1 unit. Then, if the mercury is x units above the line ℓ:



the temperature read on thermometer A is $T_A = 10 + 60x$ and the temperature read on thermometer B is $T_B = 100x$. Setting $T_A = T_B$ gives $x = \frac{1}{4}$, hence at 25° the mercury in the two thermometers is at the same horizontal level. □

5. Find an integer n so that $\frac{1}{n} < \sqrt{2015} - \sqrt{2014} < \frac{1}{n-1}$.

Answer: 90

Solution. Notice that

$$\sqrt{2015} - \sqrt{2014} = \left(\sqrt{2015} - \sqrt{2014}\right) \frac{\sqrt{2015} + \sqrt{2014}}{\sqrt{2015} + \sqrt{2014}} = \frac{1}{\sqrt{2015} + \sqrt{2014}},$$

so we need to find an integer n such that

$$n - 1 < \sqrt{2015} + \sqrt{2014} < n.$$

We know that $45^2 = 2025$ and $44^2 = 1936$ so that

$$44 < \sqrt{2014} < \sqrt{2015} < 45.$$

Using $44.5^2 < 45 * 44 = 1980$, we improve the previous inequality:

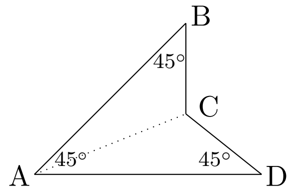
$$44.5 < \sqrt{2014} < \sqrt{2015} < 45.$$

Thus,

$$89 < \sqrt{2015} + \sqrt{2014} < 90,$$

and $n = 90$. □

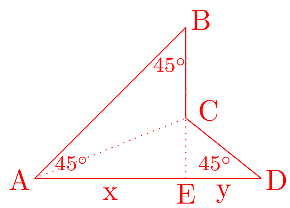
6. The length AC in the following quadrilateral ABCD is 16 cm.



What is the area of the quadrilateral?

Answer: 128

Solution. Extending the segment BC down to the segment AD at a point E divides AD into two segments: AE and ED. Let x be the length of AE and y be the length of ED.



Then the area of triangle AEB is $x^2/2$ and the area of triangle ECD is $y^2/2$. Since the length of EC equals y, we also have the relation $x^2 + y^2 = 16^2$ by Pythagoras. Therefore, the area of the quadrilateral is 128. □

7. Given that $a + a^{-1} = \frac{7}{3}$, find the value of $a^3 + a^{-3}$. Express your answer as a simple fraction in lowest terms.

Answer: $\dfrac{154}{27}$

Solution. In general, we have

$$(a + a^{-1})^3 = a^3 + a^{-3} + 3(a + a^{-1}).$$

Since $a + a^{-1} = \frac{7}{3}$, we see that

$$\frac{343}{27} = a^3 + a^{-3} + 7.$$

Therefore, $a^3 + a^{-3} = \frac{154}{27}$. □

8. Find the smallest positive integer n such that the number $111\cdots11$ (the digit 1 repeated n times) is a multiple of 999.

   Answer: 27

   Solution. We write $999 = 111 * 9$, and we set $N = 111\cdots11$ (the digit 1 repeated n times). Now, N is a multiple of 111 if and only if $n = 3k$ for some positive integer k. In this case $N/111 = 001001001\cdots001001$, where the sequence 001 is repeated k times. Set $M = 001001001\cdots001001$ (the sequence 001 repeated k times). Now M is a multiple of 9 if and only if the sum of the digits of M (which equals k) is a multiple of 9. Therefore N is a multiple of 999 if and only if $n = 3k$ where the positive integer k is a multiple of 9. The smallest such number n is 27. □

PART III.

1. How many 7-digit numbers have the property that each digit (except the units digit) is strictly greater than the digit to its right?

   Answer: 120

   Solution. Finding a 7-digit number whose digits are strictly decreasing is equivalent to choosing 7 distinct numbers between 0 and 9. Therefore, the answer is $\binom{10}{7} = 120$.

   □

2. Find the smallest possible perimeter of a triangle with one vertex on the x-axis, another on the line $y = x$, and the third at the point $(2, 1)$.

Answer: $\sqrt{10}$

Solution. Consider a triangle ABC with vertices $A = (a, 0)$, $B = (b, b)$, and $C = (2, 1)$. Also, let $D = (1, 2)$ be the reflection of C about the line $y = x$, and let $E = (2, -1)$ be the reflection of C about the x-axis. Then the perimeter $\mathcal{P}$ of triangle ABC satisfies:

$$\begin{aligned} \mathcal{P} &= \overline{CB} + \overline{BA} + \overline{AC} \\ &= \overline{DB} + \overline{BA} + \overline{AE} \\ &\geq \overline{DE} = \sqrt{10}, \end{aligned} \tag{1}$$

where equality holds if and only if the points A and B lie on the line segment DE. Therefore, the answer is $\sqrt{10}$. (This perimeter is achieved for the triangle ABC when $a = 3/5$ and $b = 4/5$.) □

3. Let $f(n)$ be a function that satisfies the following three conditions for all positive integers n:

   - $f(n)$ is a positive integer;
   - $f(n+1) > f(n)$;
   - $f(f(n)) = 3n$.

   Find $f(2015)$.

   Answer: 3858

   Solution. First, we will show that $f(n) > n$, for each positive integer n. Base case $n = 1$: To see that $f(1) > 1$ it suffices to show that $f(1) \neq 1$. Suppose to the contrary that $f(1) = 1$. Then $f(1) = f(f(1)) = 3$, which is a contradiction. Thus $f(1) > 1$. Inductive case: Suppose $f(n) > n$ for some $n \geq 1$. The first two conditions tell us that $f(n+1) \geq f(n)+1$, and it follows that $f(n+1) > n+1$. Therefore, by induction we conclude that $f(n) > n$, for each positive integer n.

   Next, we observe that $3 = f(f(1))$, $f(3) = f(f(f(1))) = 3f(1)$, and $9 = f(f(3)) = f(3f(1))$. Since $f(n) > n$, it follows that $9 > 3f(1)$, and we see that $f(1) < 3$. Therefore, the integer $f(1)$ satisfies $1 < f(1) < 3$, and we conclude that $f(1) = 2$.

   Another induction argument shows that $f(3^n) = 3^n \cdot 2$ and $f(3^n \cdot 2) = 3^{n+1}$ for each non-negative integer n. Base case $n = 0$: This follows from $f(1) = 2$ and $f(2) = f(f(1)) = 3$. Inductive case: Suppose $f(3^n) = 3^n \cdot 2$ and $f(3^n \cdot 2) = 3^{n+1}$ for some non-negative integer n. Then $f(3^{n+1}) = f(f(3^n \cdot 2)) = 3^{n+1} \cdot 2$ and $f(3^{n+1} \cdot 2) = f(f(3^{n+1})) = 3^{n+2}$.

   Notice that there are exactly $3^n - 1$ numbers between 3^n and $3^n \cdot 2$. Furthermore there are exactly $3^n - 1$ numbers between $f(3^n) = 3^n \cdot 2$ and $f(3^n \cdot 2) = 3^{n+1}$. Therefore, using the first two conditions, we see that for each positive integer k less than 3^n, we have $f(3^n + k) = 3^n \cdot 2 + k$.

   Finally, we will show that $f(3^n \cdot 2 + k) = 3^{n+1} + 3k$, for each positive integer k less than 3^n. The conclusion from the previous paragraph shows that $f(3^{n+1} + 3k) = 3^{n+1} \cdot 2 + 3k$ when k is a positive integer less than 3^n. We deduce that $f(f(3^n \cdot 2 + k)) = 3^{n+1} \cdot 2 + 3k = f(3^{n+1} + 3k)$. Since f is strictly increasing, we conclude that $f(3^n \cdot 2 + k) = 3^{n+1} + 3k$.

   Writing $2015 = 3^6 \cdot 2 + 557$, we have $f(2015) = 3^7 + 1671 = 3858$ □

4. Find integers m and n such that

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = \frac{m}{n}.$$

Answer: One of the answers is $m = -1$ and $n = 2$.

Solution 1. Consider the equation $z^7 - 1 = 0$, where z is a complex number. There are 7 complex solutions to this equation of the form $z_k = e^{2k\pi i/7}$, where $k = 0, \ldots, 6$. The sum of the roots of a polynomial $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is $-a_{n-1}/a_n$, so that

$$z_0 + z_1 + \cdots + z_6 = 0.$$

Looking at the real part of this equation, we have:

$$1 + \cos(2\pi/7) + \cos(4\pi/7) + \cdots \cos(12\pi/7) = 0.$$

Using the property that $\cos(\theta) = \cos(2\pi - \theta)$, we conclude that

$$\cos(2\pi/7) + \cos(4\pi/7) + \cos(6\pi/7) = -1/2,$$

and we may take, for instance, $m = -1$ and $n = 2$. □

Solution 2. Consider the 7 point masses $(\cos(2k\pi/7), \sin(2k\pi/7))$, $k = 0, \ldots 6$, which are evenly spaced around the unit circle. The center of mass of this system sits at the origin. Looking at the x-coordinate of the center of mass, we see that

$$1 + \cos(2\pi/7) + \cos(4\pi/7) + \cdots \cos(12\pi/7) = 0.$$

Using the property that $\cos(\theta) = \cos(2\pi - \theta)$, we conclude that

$$\cos(2\pi/7) + \cos(4\pi/7) + \cos(6\pi/7) = -1/2,$$

and we may take, for instance, $m = -1$ and $n = 2$. □