The University of Oregon

May 16, 2015

Oregon Invitational Mathematics Tournament: Geometry Exam

Closed book examination Time: 90 minutes

Last Name ______________________ **First Name** ______________________

School ____

Grade (please circle one): 7 8 9 10 11 12

I would like my exam score to be posted online with my first initial and last name
(please circle one): yes no

Special Instructions:

- Problems 1-8 require only a correct answer. There is space to work through the problem, but only the answer will be evaluated. In other words, there is no partial credit. These problems are **10 points each**.
- Problems 9-16 require thorough and complete justifications. You can receive partial credit for these problems. These problems are **25 points each**.

Rules governing examinations

- Participants are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No participants shall be permitted to re-enter once the exam has begun, or leave before 30 minutes has passed.
- Participants suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination.
  (a) Having at the place of writing any books, papers or memoranda, CALCULATORS, computers, sound, or image players/recorders/transmitters (including phones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

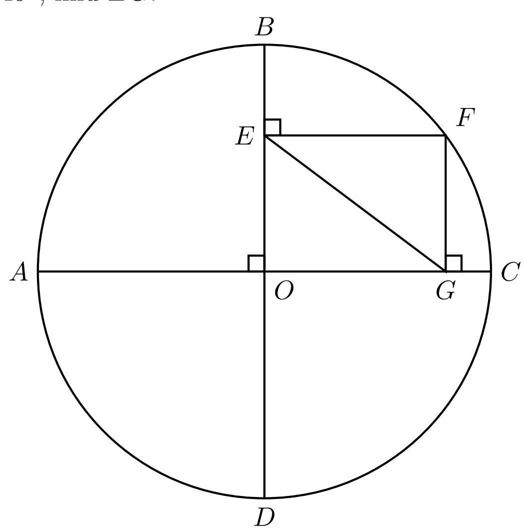
Question	Points Possible	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	25	
10	25	
11	25	
12	25	
13	25	
14	25	
15	25	
16	25	
Total	280	

1. (10 points) Find in terms of n the measure of the interior angle of a regular n-gon (n sided polygon with all sides and all angles congruent).
   $180° - \frac{360°}{n} = 180°\left(1 - \frac{2}{n}\right) = \frac{n-2}{n} \cdot 180°$

2. (10 points) In a right triangle the hypotenuse is twice as long as one of the legs. Find the angles of the triangle.

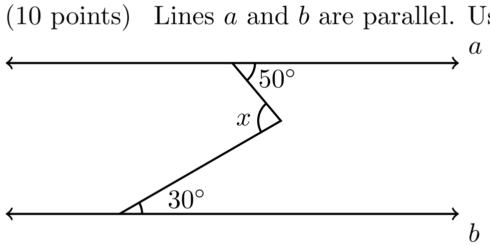
   $30°, 60°, 90°$

3. (10 points) $\overline{AC}$ and $\overline{BD}$ are perpendicular diameters in a circle with center O. Point F is on the circle. $\overline{FG}$ and $\overline{EF}$ are perpendicular to the diameters. If the radius of the circle is 10 cm and $\angle GOF$ measures $40°$, find EG.



   $EG = 10$ cm. Deduct 2 points if correct units are not included.

4. (10 points) Lines a and b are parallel. Use the figure to find x.



   $80°$

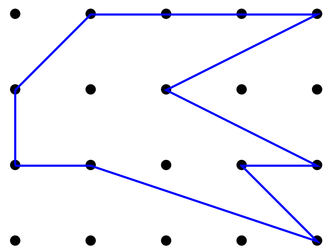
5. (10 points) $ABCD$ is a rectangle in which $AB < BC$, E is on BC and F is on AD. If $ABEF$ is a square and the rectangles $ECDF$ and $ABCD$ are similar, find the numerical value of BC/AB (leave your answer in simplest radical form).

   $\frac{1+\sqrt{5}}{2}$

6. (10 points) Complete the following sentence. A quadrilateral is a rhombus if and only if its diagonals are

   perpendicular bisectors of each other.

7. (10 points) If a smallest square on the dot paper has an area of one square unit, find the area of the polygon shown.

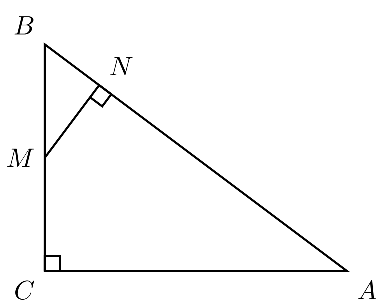


6.5 square units

8. (10 points) A boat starts at point A, moves 3 km due north, then 2 km due east, then 1 km due south, and then 4 km due east to point B, find the distance AB. Leave your answer in radical form.

$\sqrt{40}, 2\sqrt{10}$. Consider either form correct. Deduct 2 points if correct units are not included.

9. (25 points) In the right $\triangle ABC$ shown below, $BC = 3$, $AC = 4$, M is the midpoint of $\overline{BC}$ and $\overline{MN}$ is perpendicular to $\overline{AB}$. Find the ratio of the area of $\triangle BMN$ to the area of $\triangle ABC$. Justify your answer.

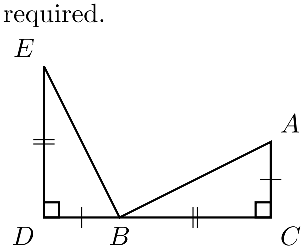


10 points: $\triangle BMN \sim \triangle BAC$ by AA.

10 points: $\dfrac{\text{area}(\triangle BMN)}{\text{area}(\triangle BAC)} = \left(\dfrac{3/2}{5}\right)^2$

5 points: $\dfrac{9}{100}$ or 0.09

10. (25 points) $\triangle ABC$ and $\triangle BED$ are right congruent triangles with congruent sides and the right angles as shown. Describe a sequence of isometries that will take $\triangle ABC$ onto $\triangle BED$. No justification required.



Infinitely many correct answers. Here are three:

1. Rotation about B by 90° counterclockwise followed by half turn about the midpoint of EB.

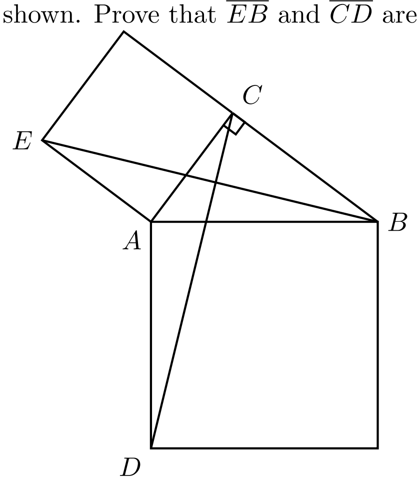
2. Rotation about B clockwise by 90° followed by a translation by vector $\vec{BE}$.
3. Rotate 90° clockwise about C then translate by vector $\vec{CB}$ and then by vector $\vec{BD}$ (or just by one vector $\vec{CD}$).

If a rotation by 90° about B counterclockwise or clockwise, or rotation by 90° clockwise about C are mentioned award 15 points. Then additional 10 points for correct shifting.

11. (25 points) Find the equation of the image of the parabola $y = x^2$ under the size transformation that takes (x, y) to $\left(\frac{3x}{2}, \frac{3y}{2}\right)$. Justify your answer.

$y = \frac{2}{3}x^2$. 15 points for correct answer without justification.

12. (25 points) On the side $\overline{AC}$ and the hypotenuse $\overline{AB}$ of the right $\triangle ABC$ squares were constructed as shown. Prove that $\overline{EB}$ and $\overline{CD}$ are perpendicular.

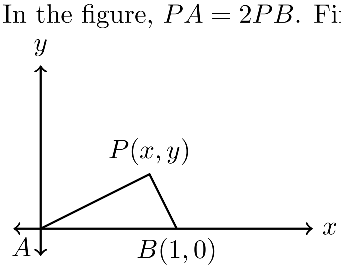


Full credit for any correct proof. 15 points for proof with missing details. Here's one possible proof: The image of $\overline{DC}$ under rotation by 90° counterclockwise about A is $\overline{BE}$ (since the image of D is B, the image of C is E and the image of A is A). Because under rotation by 90° the image of a line is perpendicular to the line, $\overline{EB}$ is perpendicular to $\overline{CD}$.

13. (25 points) The coordinates of three vertices of a parallelogram are $A(1, 1)$, $B(2, 4)$, $C(-2, 3)$. Find the coordinates of all the possible locations of the fourth vertex. Justify your answer.

$D(5, 2), D(-3, 0), D(-1, 6)$. 15 points for one correct solution plus 5 points for each additional correct solution.

14. (25 points) Prove that the set of all points that are twice as far from $A(0, 0)$ as from $B(1, 0)$ is a circle. In the figure, $PA = 2PB$. Find the coordinates of the center of the circle and its radius.

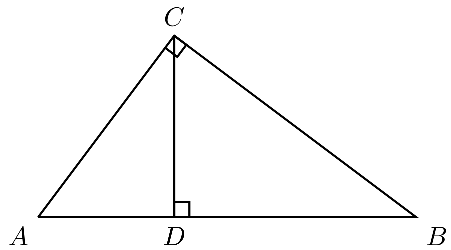


5 points: $\sqrt{x^2 + y^2} = 2\sqrt{(x-1)^2 + y^2}$

10 points: $\left(x - \frac{4}{3}\right)^2 + y^2 = \left(\frac{2}{3}\right)^2$

10 points: center $= \left(\frac{4}{3}, 0\right)$, radius $= \frac{2}{3}$

15. (25 points) $\triangle ACB$ is a right triangle in which $\overline{CD}$ is the altitude to the hypotenuse $\overline{AB}$.



(a) Prove that each of the triangles created by the altitude is similar to $\triangle ABC$.

(i) $\triangle ADC \sim \triangle ACB$, by AA since they share $\angle A$ and each has right angle.
(ii) $\triangle BDC \sim \triangle BCA$, by AA since they share $\angle B$ and each has right angle.

10 points

(b) Use similarity of triangles from part (a) to prove the Pythagorean theorem.

$$\frac{AC}{AB} = \frac{AD}{AC}$$
$$(AC)^2 = AD \cdot AB$$

Similarly $(BC)^2 = BC \cdot AB$. Adding we get $(AC)^2 + (BC)^2 = (AB)^2$.
15 points.

16. (25 points) Half of the air is let out of a spherical balloon. If the balloon remains in the shape of a sphere, how does the radius of the smaller sphere compare to the original radius? (You need to find the ratio of the radii. (The volume of a sphere with radius R is $\frac{4}{3}\pi R^3$.) Leave your answer in radical form and justify your answer.

Let the smaller radius be r and the larger R.
10 points: $\frac{4}{3}\pi r^3 = \frac{1}{2} \cdot \frac{4}{3}\pi R^3$
10 points: $\left(\frac{r}{R}\right)^3 = \frac{1}{2}$
5 points: $\frac{r}{R} = \sqrt[3]{\frac{1}{2}}$

The End