Oregon Invitational Mathematics Tournament

Algebra 2 level exam

May 16, 2015

Instructions:

There are two parts to this exam.

- Part I consists of 15 problems with numerical answers. Each is worth 4 points. Only your final answer will be graded. There will be NO partial credit given. You must write your answer in the place provided.
- Part II consists of 6 problems. Each is worth 10 points. Partial credit may be given for these questions. Each question requires a fully written-up solution. You should clearly write out each of your steps and provide justification when necessary. A correct answer without explanation will receive no credit.

Calculators are not allowed.

PART I: 4-Point Problems:

(1) To convert from degrees Celsius to Fahrenheit, you multiply by $9/5$ and then add 32. If T degrees Celsius is the same as T degrees Fahrenheit, what is the value of T?

Answer: -40

(2) The product of the ages of two friends is 1140. If they are both less than 40 years of age, how old are they?

Answer: 30 and 38

(3) The sum of two numbers is 165 and their product is 2015. Find the sum of their squares.

Answer: $23,195$

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(4) To number pages in a book 1000 digits were used. How many pages are in the book if we start numbering from page three and number it as page 3?

Answer: 370 pages

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(5) Find all real solutions of the equation $(x^2+3)^2 - 4x^2 - 24 = 0$.

Answer: $\pm\sqrt{3}$

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(6) Find the number of integers in $1, 2, \cdots, 300$ that are divisible by 3,5, or 7.

Answer: 162

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(7) For what value(s) of c will the equation $\frac{1}{x(x-c)} = \frac{1}{1-c}$ have exactly one solution?

Answer: 2

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(8) Given $f(x) = \frac{x+1}{x+2}$, find the x value in terms of a for which $f(x) = 1 - \frac{1}{a}$.

Answer: $a - 2$

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(9) Find the point (a, b) on the line $y = 2x - 1$ which is closest to the point $(0, 1)$.

Answer: $\left(\frac{4}{5}, \frac{3}{5}\right)$

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(10) A polynomial $P(x) = x^3 + ax^2 + bx + c$ satisfies $P(1) = 1^2$, $P(2) = 2^2$ and $P(3) = 3^2$. What is $P(4)$?

Answer: 22

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(11) A new operation $\otimes$ satisfies the following equations.

- $0 \otimes b = b + 1$
- $(a + 1) \otimes b = a \otimes (a \otimes b)$ for all $a \geq 0$.

What is the value of $4 \otimes 5$?

Answer: 21

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(12) Find the smallest 4 digit number $abcd$ (for example 2375), where a, b, c, d are all distinct integers between 1 to 9 (inclusive) such that the two digit numbers ab, ac, ad, bc and cd are all prime.

Answer: 1379

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(13) It takes Tim 4 hours to get from Seattle to Eugene. It takes Mary 6 hours to get from Seattle to Eugene. Aaron's speed is an average of Tim's and Mary's speeds. How long will it take Aaron to get from Seattle to Eugene?

Answer: 4.8 hours

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(14) Simplify:

$$z^{\frac{p-3}{p^2+3p}} \div z^{\frac{12}{9-p^2}} \cdot z^{\frac{3}{3p-p^2}}$$

Answer: $z^{\frac{1}{p-3}}$

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(15) Numbers a, b, and c are not equal to each other and $a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}$. Find $a^2b^2c^2$.

Answer: 1

PART II: 10-Point Problems:

(1) For any numbers x and y such that $x + y \neq 0$, define $x \oplus y$ as

$$x \oplus y = \frac{xy}{x+y}.$$

Show that

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

whenever both the expressions make sense.

Solution:

$$(x \oplus y) \oplus z = \frac{xy}{x+y} \oplus z = \frac{\frac{xyz}{x+y}}{\frac{xy}{x+y} + z} = \frac{\frac{xyz}{x+y}}{\frac{xy+xz+yz}{x+y}} = \frac{xyz}{xy+xz+yz}$$

and

$$x \oplus (y \oplus z) = x \oplus \frac{yz}{y+z} = \frac{\frac{xyz}{y+z}}{x + \frac{yz}{y+z}} = \frac{\frac{xyz}{y+z}}{\frac{xy+xz+yz}{y+z}} = \frac{xyz}{xy+xz+yz}$$

Grading guidelines:

- 10 points - complete solution
- 6 points - mostly correct computations on both sides
- 3 points - computing either left or right side of the desired equality in terms of usual * and +
- 2 points - mistakes were made in the beginning steps of working with the expressions leading to wrong/no solution

(2) We know that for any numbers a and b, $f(a+b) + f(a-b) = 2f(a) + 2f(b)$.
Is it true that $f(x)$ is even (i.e. $f(x) = f(-x)$ for all values of x from the domain of $f(x)$)?

Solution:

$$f(0+x) + f(0-x) = 2f(0) + 2f(x)$$
$$f(x+0) + f(x-0) = 2f(x) + 2f(0)$$

From the second equation we get

$$f(0) = 0$$

And from the first equation:

$$f(x) + f(-x) = 2f(x)$$
$$f(x) = f(-x)$$

so $f(x)$ is even.

Grading guidelines:

- 10 points - complete solution
- 4 points - Writing down any two (different) equations involving "x" (or some other variable) coming from f's property is 3 pts
- 2 points - some correct steps were made with mistakes leading to wrong/no solution

(3) For which real values of the parameter t does the equation

$$t^2x^2 + 2(t^2 - 2)x + 1 = 0$$

have a unique real solution x?

Solution:

If $t = 0$, this equation is $-4x + 1 = 0$, which certainly has a unique real solution. If $t \neq 0$, this equation is a quadratic equation with discriminant

$$\begin{aligned} 4(t^2-2)^2 - 4t^2 &= 4[t^4 - 4t^2 + 4 - t^2] \\ &= 4[t^4 - 5t^2 + 4] = 4(t-1)(t+1)(t-2)(t+2), \end{aligned}$$

so it has a unique real solution if and only if $t \in \{-2, -1, 1, 2\}$. Thus the equation in the problem has a unique real solution exactly for $t \in \{-2, -1, 0, 1, 2\}$.

Grading guidelines:

- 10 points - complete solution
- 8 points - small mistake at the end of the solution/ not listing all of the solutions in the answer
- 6 points - discriminant found correctly, equation with the discriminant was not solved correctly
- 2 points - mistakes were made in the beginning steps of working with the equation leading to wrong/no solution

(4) A function from the non-negative integers to the non-negative integers satisfies

- $f(mn) = mf(n) + nf(m)$
- $f(15) = 60$
- $f(10) = 15$
- $f(12) = 12$

Find $f(9)$.

Solution:
The answer is $f(9) = 36$.
From the relations, we get the system of equations

$$\begin{array}{rrrrl} & 5f(3) & +3f(5) & = 60 & \dots(1) \\ 5f(2) & & +2f(5) & = 15 & \dots(2) \\ 12f(2) & +4f(3) & & = 12 & \dots(3) \end{array}$$

For example, to get equation 3, we have

$$f(12) = 3f(4) + 4f(3) = 3(2f(2) + 2f(2)) + 4f(3) = 12f(2) + 4f(3) = 12$$

is how you get equation 3.

The system of equations can be solved in various ways. After solving, one should get $f(2) = -1$, $f(3) = 6$ and $f(5) = 10$. Since we only need to know $f(3)$, it is possible that students stop after finding $f(3)$.

Then to get $f(9)$, we have

$$f(9) = 3f(3) + 3f(3) = 18 + 18 = 36$$

Grading guidelines:

- 10 points - complete solution
- 8 points - small mistake at the end of the solution e.g. $f(9)$ is not found correctly or no explanation on how to break $f(9)$
- 6 points - the system was solved correctly/ or $f(3)$ was found correctly
- 2 points - mistakes were made in the beginning steps of working with the system of equations leading to wrong/no solution

(5) Prove that for any integers $a_1, a_2, \cdots, a_7$, there exist a pair of them a_i and a_j $(1 \leq i, j \leq 7)$ such that $a_i + a_j$ or $a_i - a_j$ is divisible by 10.

Proof: Use the Pigeon-hole Principle. If no described pair exists, then of the six sets $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}$ and $\{0\}$, we need to avoid having two numbers in $a_1, a_2, \cdots, a_7$ whose remainders when divided by 10 fall in the same set. But there are 7 numbers, so this cannot be done by the Pigeonhole Principle.

Grading guidelines:

- 10 points - complete solution
- 8 points - small mistake at the end of the solution
- 6 points - some meaningful explanations needed to complete the proof were given
- 2 points - some intermediate explanations were given, mistake was made in the beginning of the process

(6) Prove the following for any $a, b > 0$:

$$\frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

Proof: The first inequality follows from the fact that

$$a + b - 2\sqrt{ab} = (\sqrt{a} - \sqrt{b})^2 \geq 0.$$

For the second inequality, use the left to get

$$\sqrt{ab} = \frac{1}{\sqrt{\frac{1}{a} \cdot \frac{1}{b}}} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

Grading guidelines:

- 10 points - complete proof
- 6 points - proving only one inequality
- 2 points -mistakes were made in the beginning steps of working with the expressions leading to wrong/no solution; or both sides of the inequality were verified in the case of some example