Desirability of Nominal GDP Targeting under Adaptive Learning
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Desirability of Nominal GDP Targeting under Adaptive Learning

Nominal GDP targeting has been advocated by a number of authors since it produces relative stability of inflation and output. However, all of the papers assume rational expectations on the part of private agents. In this paper I provide an analysis of this assumption. I use stability under recursive learning as a criterion for evaluating nominal GDP targeting in the context of a model with explicit micro-foundations which is currently the workhorse for the analysis of monetary policy.

MONETARY POLICY RULES that utilize as their principal target variable the level or growth rate of some aggregate measure of nominal spending, such as nominal GDP, have had considerable academic support since the early 1980s. More recently, arguments in favor of nominal GDP targeting have also been made by, among others, Hall and Mankiw (1994), McCallum (1997c), and McCallum and Nelson (1999a) (see Hall and Mankiw, 1994, and McCallum and Nelson, 1999a, for a more extensive discussion of this literature). Nominal output targeting has two desirable features as a strategy for monetary policy. First, it automatically takes into account movements in both prices and real output, which in practice are the two variables central banks care about most. Second, nominal GDP can serve as a long-run nominal anchor for monetary policy, given the common belief that monetary policy cannot affect the real economy in the long-run.

Rudebusch (2002) pointed out that two distinct developments have also boosted an interest in nominal GDP targeting in recent years. The formation of the European Central Bank (ECB) in Europe has encouraged a lively debate about the appropriate strategy for European monetary policy. The announced ECB strategy contains an element of monetary targeting which is closely related to nominal output targeting if there are no large shifts in monetary velocity. In fact, the ECB (1999) has explicitly

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derived its 4.5% reference value for M3 growth from a desired growth rate for nominal output. The ECB’s announced monetary strategy, therefore, provides support for the consideration of nominal output targeting. The second development that has increased interest in nominal GDP targeting has been the behavior of the U.S. economy in recent years. A number of macroeconomic forecasters have been making forecasting errors – both overpredicting inflation and underpredicting output growth (see, for instance, Figure 1 of Brayton, Roberts, and Williams 1999). In light of this uncertainty about the level of potential output and the dynamics of the U.S. economy, several authors like McCallum (1999) and Orphanides (2000) suggest that monetary policy should focus on nominal GDP growth.

In this paper I study the desirability of nominal GDP targeting in the context of a standard forward-looking model that is currently the workhorse in the analysis of monetary policy. I use the New Phillips curve model, which has explicit microeconomic foundations and has been derived in a number of papers and reviewed, for instance, in Clarida, Gali, and Gertler (1999) and Woodford (1999). I first show that a monetary policy of targeting the growth rate of nominal income every period yields a locally unique stationary equilibrium when the private sector has rational expectations (RE) of inflation and output. Such a question is of utmost importance, because it is well known since the paper of Sargent and Wallace (1975) that uniqueness of equilibrium cannot be taken for granted in such models. This question takes added importance in the context of nominal GDP targeting since Ball (1999) has argued that this policy would be destabilizing for output and inflation.

The novel contribution of this paper, however, comes from a different angle. The entire literature of targeting nominal GDP has, to the best of my knowledge, assumed RE on the part of economic agents. By now it is well known that this assumption need neither be innocuous nor realistic. Agents are somehow assumed to be able to coordinate on a particular rational expectations equilibrium (REE). However, it is not obvious whether or how such coordination may arise. In order to complete such an argument, one needs to show the potential for agents to learn the equilibrium of the model being analyzed. An early instance where this was emphasized was in the work by Howitt (1992) in which the instability under learning of interest rate pegging and related rules in flexible price and ad hoc IS-LM type models were shown. In his conclusion he explicitly warned that, in general, any RE analysis of monetary policy should be supplemented with an investigation of its stability under learning. He emphasized that the assumption of RE can be quite misleading in the context of a fixed monetary regime; if the regime is not conducive to learnability, then the consequences can be quite different from those predicted under RE.

Surely, expectational errors may arise in practice from changes in the economic structure or in the practices of policymakers. The assumption that agents somehow have RE immediately after such changes is clearly very strong and need not be correct empirically. The learning analysis, on the other hand, allows for the possibility that expectations might not be initially fully rational, and that, if agents make forecast errors and try to correct them over time, the economy may or may not converge to the REE asymptotically. The purpose of this paper is to conduct such an analysis
in the context of nominal GDP targeting. This has the potential to determine whether or not the particular REE can ever be observed. Bullard and Mitra (2000, 2002) recently used learning to evaluate monetary policy rules where the interest rate responds to measures of inflation and output gap—rules that have been popularized since Taylor’s (1993) seminal contribution. They show that if agents are assumed to follow adaptive learning rules, then the stability of these Taylor-type monetary policy rules cannot be taken for granted even in cases when the economy has a determinate equilibrium under RE.

The central result of the paper is that a particular policy of targeting nominal GDP growth every period is stable under learning dynamics for all possible parametrizations of the model. This is a policy where the central bank bases the nominal interest rate directly on the expectations of private agents. In such a scenario, agents using adaptive learning mechanisms are able to coordinate on the particular equilibrium under consideration.

The paper is organized as follows. Section 1 introduces the basic model which is analyzed for determinacy and learnability in Section 2 under a monetary policy of targeting the growth rate of nominal GDP every period. An important weakness of the basic model presented in Section 1 is that there are no backward looking elements. Section 3 rectifies this defect and analyzes GDP targeting in a model with endogenous output and inflation persistence. Some analysts interpret GDP targeting in the sense of keeping expected next period GDP growth fixed. This policy is briefly analyzed in Section 4. Finally, Section 5 presents some concluding remarks.

1. THE MODEL

I study a small forward-looking macroeconomic model recently analyzed in Clarida, Gali, and Gertler (1999), which is currently the workhorse for the theoretical analysis of monetary policy (see, for example, McCallum and Nelson, 1999b, and Rotemberg and Woodford, 1999). It is a dynamic general equilibrium model with temporary nominal price rigidities. Within the model, monetary policy affects the real economy in the short run, as in the traditional IS/LM model. A key difference from the traditional IS/LM model, however, is that the aggregate behavioral equations evolve explicitly from optimization by households and firms.

The basic model analyzed in Section 2 of Clarida, Gali, and Gertler (1999) consists of two structural equations:

\[ x_t = -\phi(i_t - \hat{E}_t\pi_{t+1}) + \hat{E}_t\pi_{t+1} + g_t, \]  

(1)

and

\[ \pi_t = \lambda x_t + \beta \hat{E}_t\pi_{t+1} + u_t, \]  

(2)

where \( x_t \) is the output gap, i.e., the difference between actual and potential output, \( \pi_t \) is the period \( t \) inflation rate defined as the percentage change in the price level from \( t-1 \) to \( t \), and \( i_t \) is the nominal interest rate; each variable is expressed as a
deviation from its long run level. Since our main focus will be on an analysis of learning, we use the notation $\hat{E}_t\pi_{t+1}$ and $\hat{E}_t x_{t+1}$ to denote the possibly (subjective) private sector expectations of inflation and output gap next period, whereas the same notation without the “^” superscript will denote the RE values.

Equation (1) is the intertemporal IS equation, whereas Equation (2) is the aggregate supply equation. The IS Equation (1) can be derived from log-linearizing the Euler equation associated with the household’s saving decision. The aggregate supply equation (the New Phillips curve) (2) can be derived from optimal pricing decisions of monopolistically competitive firms facing constraints on the frequency of future price changes. The parameters $\varphi$, $\lambda$, and $\beta$ are structural and are assumed to be positive on economic grounds. In particular, $\beta \in (0,1)$ is the discount factor and the interest elasticity of the IS curve, $\varphi$, corresponds to the intertemporal elasticity of substitution of consumption. The slope of the Phillips curve, $\lambda$, depends on the average frequency of price changes and the elasticity of demand faced by suppliers of goods. Prices are more nearly flexible the higher is $\lambda$. The demand shock $g_t$ in Equation (1) may be easily rationalized as a preference shock or as expected changes in government purchases relative to expected changes in potential output. However, the nature of the “cost push” shock $u_t$ in Equation (2), which captures features that might affect expected marginal costs, is a matter of considerable dispute.

The shocks $g_t$ and $u_t$ are assumed to follow first order autoregressive processes:

$g_t = \mu g_{t-1} + \hat{g}_t,$ \hspace{1cm} (3)

and

$u_t = \rho u_{t-1} + \hat{u}_t,$ \hspace{1cm} (4)

where $0 < \mu, \rho < 1$ and both $\hat{g}_t$ and $\hat{u}_t$ are i.i.d noise with zero means and variances $\sigma_g^2$ and $\sigma_u^2$, respectively. The model is closed by assuming that the nominal interest rate $i_t$ is the instrument of monetary policy.

2. GROWTH TARGETING

We assume that the objective of the central bank is to keep nominal income growth equal to a (constant) target value $\Delta z$. Thus the central bank sets the interest rate $i_t$ every period so as to make $\Delta i_t = \Delta z$, where $z_t = x_t + p_t$, with $p_t$ denoting (log) of the price level and $z_t$ denoting (log) nominal GDP. In terms of inflation this implies

$\pi_t + x_t - x_{t-1} = \Delta z.$ \hspace{1cm} (5)

Consequently, Equation (5) will be satisfied if the central bank targets a constant growth of GDP every period. Substitution of Equations (1) and (2) into Equation (5) yields

$(1 + \lambda)[-\varphi(i_t - \hat{E}_t\pi_{t+1}) + \hat{E}_t x_{t+1} + g_t] + \beta \hat{E}_t \pi_{t+1} + u_t - x_{t-1} = \Delta z,$
and solving this gives the following rule for setting the interest rate that will stabilize $\Delta z_t$ at $\Delta z$:

$$i_t = [1 + \beta \varphi^{-1}(1 + \lambda)^{-1}]\hat{\pi}_t\pi_{t+1} + \varphi^{-1}\hat{\pi}_t\pi_{t+1} - \varphi^{-1}(1 + \lambda)^{-1}x_t$$

$$+ \varphi^{-1}(1 + \lambda)^{-1}u_t + \varphi^{-1}g_t - \varphi^{-1}(1 + \lambda)^{-1}\Delta z. \quad (6)$$

Plugging this rule (Equation 6) into Equation (1) yields

$$(1 + \lambda)x_t = -\beta\hat{\pi}_t\pi_{t+1} + x_{t-1} - u_t + \Delta z. \quad (7)$$

Note that the interest elasticity of the IS Equation (1), $\varphi$, does not appear in this reduced form IS curve (Equation 7). We provide some intuition for this in Section 3. For the time being, we note that our complete system, under a policy of targeting a constant growth of nominal GDP every period, is given by Equations (7) and (2), representing the evolution of the endogenous variables $x_t$ and $\pi_t$, respectively.

2.1 Determinacy

I first ask whether a monetary policy of nominal income targeting leads to the existence of a (locally) unique stationary REE. This question is of utmost importance since it is well known that determinacy of equilibrium cannot be taken for granted in monetary models. Sargent and Wallace (1975) argued against interest rate rules on the grounds that they resulted in price-level indeterminacy. Since the central bank uses the interest rate in targeting nominal GDP, this question is a valid one to ask for our model. In addition, Ball (1999) and Svensson (1999) have argued that there fails to exist a stationary REE under GDP targeting. However, as McCallum (1997a) has shown, this is a consequence of the backward-looking model these authors use. I now show that under nominal income targeting, REE is unique for generic parameter values.

In the model represented by Equations (7) and (2), there is only one predetermined endogenous variable $x_{t-1}$. We can rewrite our system in matrix form as

$$\begin{bmatrix}
1 & 0 & -(1 + \lambda)^{-1} \\
-\lambda & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_t \\
\pi_t \\
x_{t-1}
\end{bmatrix}
= \begin{bmatrix}
(1 + \lambda)^{-1} \Delta z \\
0 \\
0
\end{bmatrix}$$

$$+ \begin{bmatrix}
0 & -\beta(1 + \lambda)^{-1} & 0 \\
0 & \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
E_t\pi_{t+1} \\
E_t\pi_{t+1} \\
x_t
\end{bmatrix}
+ \begin{bmatrix}
-(1 + \lambda)^{-1} \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
E_t x_{t+1} \\
x_t \\
0
\end{bmatrix}
+ u_t.$$
The matrix which is relevant for uniqueness is obtained by pre-multiplying the matrix associated with the expectational variables on the right hand side with the inverse of the left hand matrix. This matrix is given by

\[
B_0 = \begin{bmatrix}
0 & 0 & 1 \\
0 & \beta & \lambda \\
0 & \beta & 1 + \lambda
\end{bmatrix}.
\] (8)

We have two free endogenous variables, \(x_t\) and \(\pi_t\), and one predetermined endogenous variable, \(x_{t-1}\). Consequently, following Farmer (1999), we need exactly two of the three eigenvalues of \(B_0\) to be inside the unit circle for uniqueness.

**PROPOSITION 1:** Under nominal income growth targeting, equilibrium is locally unique.

**PROOF:** See Appendix A.

The problem of indeterminacy is particularly acute for Taylor type rules which respond to future forecasts of output and inflation (see Bernanke and Woodford, 1997, and Bullard and Mitra, 2002). However, Proposition 1 shows that this need not be a problem with nominal income targeting even though such a policy makes the interest rate rule (Equation 6) forward looking. Ignoring the response of the interest rate to the lagged output gap in Equation (6), this can be understood as follows. Proposition 4 of Bullard and Mitra (2002) provides necessary and sufficient conditions for determinacy of equilibrium for rules of the form \(i_t = \phi_x \hat{E}_t x_{t+1} + \phi_\pi \hat{E}_t \pi_{t+1}\). They show that equilibrium is determinate in a similar model if \(\phi_x\) is small enough and \(\phi_\pi\) is more than one, corresponding to a rule satisfying the Taylor principle (see Woodford 2001). Intuitively, a value of \(\phi_\pi\) more than one means that an increase in inflationary pressures causes the nominal interest rate to rise enough to also raise the real interest rate, thereby, reducing the output gap via the IS curve (Equation 1) and inflation via the AS curve (Equation 2). However, even with a small \(\phi_\pi\), one may have indeterminacy if either \(\phi_x\) is less than one, or if it is too large. It is interesting to observe that the response to \(\hat{E}_t \pi_{t+1}\) and \(\hat{E}_t x_{t+1}\) in rule (Equation 6) satisfies the determinacy conditions of Proposition 4 of Bullard and Mitra (2002)—the response to \(\hat{E}_t x_{t+1}\) is small enough and that to \(\hat{E}_t \pi_{t+1}\) is between 1 and 2 (i.e., it is aggressive but not overly aggressive). This shows that while the problem of indeterminacy may be acute for ad hoc Taylor type forward-looking rules, the situation is different for similar rules geared towards maintaining a constant growth rate of GDP every period.

### 2.2 Learning

I now adapt methods developed by Marcet and Sargent (1989) and Evans and Honkapohja (2001) to understand how learning affects these systems.³ I assume that the agents in the model no longer have RE at the outset. Instead, I replace expected values with adaptive rules, in which the agents form expectations using the data
generated by the system in which they operate. We can imagine the agents to use versions of recursive least squares updating. I use theorems developed by Evans and Honkapohja (2001) and calculate the conditions for expectational stability (E-stability). Evans and Honkapohja (2001) have shown that expectational stability, a notional time concept, corresponds to stability under real-time adaptive learning under quite general conditions. In particular, under E-stability of an REE, recursive least squares learning is locally convergent to that equilibrium. We may assume that the fundamental disturbances have bounded (small) support since Equations (1) and (2) arise out of local linearization of the original nonlinear model. Under this assumption, if an REE is not E-stable, then the probability of convergence of the recursive least squares algorithm to it is zero. Due to this one to one correspondence between the expectational stability of a stationary REE and the stability under real-time adaptive learning, I focus only on expectational stability conditions throughout the paper and the terms “learnability”, “expectational stability”, and “stability in the learning dynamics” are all used interchangeably.

The analysis of learning is conducted under two different formulations, which corresponds to different assumptions made about the behavior of private agents by the central bank. In the first formulation, the central bank sets the interest rate in accordance with the policy rule (Equation 6) recognizing that the private sector does not have RE. In other words, the interest rate rule is based directly on the subjective expectations of agents and is called the contemporaneous expectations based policy rule. This seems to me to be the most realistic assumption to make on the part of the central bank. I also discuss below ways in which such a policy can be implemented.

In the second formulation, the central bank (mistakenly) assumes that the private sector has perfect RE at every point of time. This means that the policy rule (Equation 6) is now based directly on the actual conditional expectations under RE (i.e., on \( E_t \pi_t+1 \) and \( E_t x_t+1 \) instead of \( \hat{E}_t \pi_t+1 \) and \( \hat{E}_t x_t+1 \)) and is called the RE based policy rule. While I believe this assumption is not entirely realistic, it does serve as a useful benchmark to judge how useful the assumption of RE is as a guide for monetary policy in the long run. Furthermore, Evans and Honkapohja (2000) have recently shown that the use of such an interest rate rule by the central bank in the conduct of optimal monetary policy leads to instability of the REE for all structural parameter values. One would like to see whether this instability result continues to be true when the central bank targets a given growth rate of GDP every period.

Contemporaneous Expectations Based Policy Rule. I first turn to a discussion of contemporaneous expectations based policy rule. As mentioned above, in this formulation, I assume that the policy maker sets the nominal interest rate in accordance with the rule (Equation 6). This means that the central bank bases its policy on the lagged output gap, the contemporaneous demand, and cost push shocks as well as the subjective expectations of private agents. In particular, we are assuming that the bank is able to observe private sector expectations. Before turning to a formal discussion, I first discuss different ways of implementing this rule. One may view
the commercial forecasts published by various agencies as being the expectations of the private sector. This is discussed extensively in Romer and Romer (2000) and Hall and Mankiw (1994). These commercial forecasts are often created by firms managing large portfolios so that in a sense these are indeed the forecasts of market participants. It is also the case that market participants often pay for these commercial forecasts; this suggests that they view information processing as difficult and commercial forecasts as valuable. Given this scenario, it is plausible to assume that the private sector will just adopt the commercial forecasts for their own use. Consequently, one way to implement the proposal would be for the central bank to target the predictions of private sector forecasts. In the United States, there is a published consensus of respected private forecasters like the Blue Chip Economic Indicators. Romer and Romer (2000) also discuss commercial forecasts prepared by the Survey of Professional Forecasters (SPF) and by Data Resources, Inc. (DRI).

In summary, this shows that the interest rate rule (Equation 6) represents a feasible way of implementing monetary policy.

I consider learning by private agents of the minimum state variable (MSV) solution (see McCallum 1983). Before starting the analysis, I must warn the reader that even though Section 2.1 showed that equilibrium is unique, this does not automatically guarantee learnability of this equilibrium (see Bullard and Mitra 2000, 2002). One has to verify this in each case.

For the model given by Equations (7) and (2) the MSV solution takes the form

\[ x_t = a_1 + b_1 x_{t-1} + c_1 u_t + d_1 g_t, \]

and

\[ \pi_t = a_2 + b_2 x_{t-1} + c_2 u_t + d_2 g_t, \]

where \((a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2)\) are to be determined by the method of undetermined coefficients. The MSV parameter values will be denoted by \(\hat{a}_1, \hat{b}_1\) etc. Appendix B develops the analysis of learning step by step. Using this we are in a position to prove the following proposition.

**Proposition 2:** Suppose the time \(t\) information set is \((1, y_t', w_t')\). The MSV solution under nominal income growth targeting is \(E\)-stable for all parameter values.

**Proof:** See Appendix C.

Proposition 2 shows that for all admissible values of the structural parameters the relevant REE in this economy can be learnt by agents using simple adaptive learning rules. Some intuition can be provided for this result. In the context of ad hoc Taylor type rules, Proposition 5 of Bullard and Mitra (2000, 2002) shows that expectation stability of rules which respond to \(E_t \pi_{t+1}\) and \(E_t x_{t+1}\) holds if and only if these rules satisfy the Taylor principle. Since the rule (Equation 6) for targeting nominal GDP calls for the interest rate to respond to \(E_t \pi_{t+1}\) with a coefficient bigger than one, it fulfills the Taylor principle and contributes to stability under learning dynamics. This means that a deviation of \(E_t \pi_{t+1}\) above its RE value calls
for a rise in the real interest rate which reduces \( x_t \) via the IS curve (1) and \( \pi_t \) via the inflation Equation (2). This in turn reduces \( \hat{E}_t \pi_{t+1} \) driving the economy towards the initial equilibrium. Similarly, the rule (Equation 6) tightens monetary policy when \( \hat{E}_t \pi_{t+1} \) rises above its RE value.

In Proposition 2, we have assumed that the private sector and the central bank have information on contemporaneous dated variables (like the output gap) in formulating their forecasts. However, several authors, like McCallum (1993, 1997b, 1999), are especially critical of this assumption, since information on the current period output gap is rarely available when the bank (or the private sector) makes a decision. An alternative, suggested by McCallum (1993, 1997b, 1999), would be to assume that the private sector and the bank base their actions only on the last period’s information, i.e., on the last quarter’s output gap. Even with this more realistic information structure, it is possible to show that the REE continues to be learnable so that this result is quite robust to different assumptions about the information structure.

Before concluding this section, we point out some assumptions made (implicitly) in the analysis of stability. The forecasts of agents under learning, given by Equations (B1) and (B2), were directly incorporated in the model; therefore, it is implicitly assumed that the Euler equations represent behavioral rules which describe how private agents respond to their forecasts. In particular, agents respond only to expectations about next period variables and not to expectations further out in the future. This enables us to write the model in a linear setup and use the results of Evans and Honkapohja (2001). In addition, the private sector, being monopolistically competitive, is assumed to be populated by large number of “small” agents – strategic behavior in expectations formation and learning is absent. Finally, the central bank is implicitly assumed to have knowledge of the true structure of the economy, (Equations 1 and 2), as well as the key structural parameter values \( \beta, \lambda, \) and \( \phi \) since it makes use of rule (Equation 6) in setting the interest rate. I believe this serves as a reasonable first approximation and, as we have seen, yields a very strong result on the learnability of the REE solution. In addition, if the bank follows the rule (Equation 6) with parameters deviating from the specified values by small amounts, then the economy will converge over time to an equilibrium that deviates from the REE by small amounts. I also conjecture that the results on E-stability will be unaltered if the central bank does not have knowledge of the key structural parameters \( \lambda \) and \( \phi \) and is instead learning about them using least squares, as do the private agents.\(^4\)

It is, however, important that the central bank recognizes that expectations of the private sector need not be fully rational during the transition to REE and set the interest rate accordingly. If the central bank mistakenly assumes that agents have RE during the transition and sets the interest rate accordingly then it no longer follows that the MSV solution will be expectationally stable, as we now show.

*Rational Expectations (RE) Based Policy Rule.* We now assume that the central bank continues to have knowledge of the true structure of the economy as well as the structural parameters. However, it mistakenly assumes that the private sector has RE at every point of time. Recall that the interest rule given by Equation (6)
responds directly to the expectations of the private sector. If the central bank assumes that the private sector has RE at every point of time, then it will use the actual conditional expectations, $E_t \pi_{t+1}$ and $E_t x_{t+1}$, in the rule (Equation 6) instead of the subjective expectations, $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$. Such an interest rate rule is called the RE-based policy rule.

The structure of the economy is still given by Equations (1), (2), and (6) so that the MSV solution continues to take the form (9) and (10). One computes the subjective expectations (B1)–(B2). Inserting into Equations (1), (2), and (6) one obtains, as before, the MSV parameter values $(a_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2)$. The corresponding interest rate under RE can be computed from Equation (6) as

$$
i_t = [1 + \beta \varphi^{-1}(1 + \lambda)^{-1}]E_t \pi_{t+1} + \varphi^{-1}E_t x_{t+1} - \varphi^{-1}(1 + \lambda)^{-1}x_{t-1}$$

$$+ \varphi^{-1}(1 + \lambda)^{-1}u_t + \varphi^{-1}g_t - \varphi^{-1}(1 + \lambda)^{-1}\Delta z,$$

$$= [1 + \beta \varphi^{-1}(1 + \lambda)^{-1}](\hat{a}_2 + \hat{b}_2 x_t + \hat{c}_2 p u_t + \hat{d}_2 \mu g_t)$$

$$+ \varphi^{-1}(\hat{a}_1 + \hat{b}_1 x_t + \hat{c}_1 p u_t + \hat{d}_1 \mu g_t) - \varphi^{-1}(1 + \lambda)^{-1}x_{t-1}$$

$$+ \varphi^{-1}(1 + \lambda)^{-1}u_t + \varphi^{-1}g_t - \varphi^{-1}(1 + \lambda)^{-1}\Delta z .$$

(11)

Note that unlike the contemporaneous expectations based policy rule, which was based on the subjective expectations of inflation and output, we have now plugged the RE parameter values to get the actual conditional expectations in deriving the RE-based policy rule. This is the key difference between the two policy rules.

Appendix D fills in the details of the analysis of learning and provides E-stability conditions of the MSV solution. Unfortunately, the actual E-stability conditions are quite complicated and analytical results do not seem possible. However, we can at least check E-stability for plausible values of structural parameters. The calibrated parameter values for the U.S. economy in Clarida, Gali, and Gertler (2000) are $\varphi = 1$, $\beta = 0.99$, $\lambda = 0.3$, $\mu = \rho = 0.9$. With these values, the REE is expectation unstable. A similar situation prevails if we use the structural values in Woodford (1999) for the U.S. economy, namely $\varphi = (0.157)^{-1} = 6.37$, $\beta = 0.99$, $\lambda = 0.024$. We get instability even with some other values of structural parameters so that the result seems quite robust.

This result is similar in flavor to the one obtained in Evans and Honkapohja (2000), where it was shown that the optimal monetary policy rule was unstable for all parameter values when the central bank (mistakenly) assumes RE at every point of time for the private sector. In this model too, if the central bank targets a nominal variable like GDP and assumes that agents have RE, then it is quite likely that the REE outcome will not emerge as the long-run outcome of the economy. Some intuition for this is as follows. A deviation of $\hat{E}_t \pi_{t+1}$ above its RE value leads, through the reduced form IS curve, (D2), to an increase in $x_t$ (recall from Equation D1, $\psi_x > 0$ since $\hat{b}_1$ and $\hat{b}_2 > 0$), and through the aggregate supply equation, to an
increase in \( \pi_t \). Over time this leads to upward revisions of both \( \hat{E}_t \pi_{t+1} \) and \( \hat{E}_t x_{t+1} \).

There is nothing in the interest rate rule (Equation D1) to offset this tendency and, over time, the economy moves further away from the REE. It is also quite instructive here to compare the reduced form IS curves under the contemporaneous expectations based policy rule (Equation 7) with that under the RE based policy rule (Equation D2). In the former case, a rise in \( \hat{E}_t \pi_{t+1} \) above its RE value causes a decrease in \( x_t \) (and hence \( \pi_t \)) pushing the economy back towards the REE whereas in the latter case, the same deviation causes an increase in \( x_t \) and \( \pi_t \), pushing the economy further away from the REE. I think this is the key to understanding stability under the contemporaneous expectations based policy rule and instability under the RE based policy rule. The instability result is reminiscent of Howitt’s (1992) warning about the assumption of RE leading to misleading results in monetary models. It also points to the importance of targeting the observable expectations of the private sector on the part of the central bank in formulating its interest rate rule.

3. ENDOGENOUS INFLATION AND OUTPUT PERSISTENCE

A basic problem with the model given by Equations (1) and (2) is that it is entirely forward looking. The only backward looking element that enters the model is through the monetary policy of targeting GDP, i.e., via the lagged output gap. As a result, this specification has difficulty capturing the inertia in output and inflation evident in the data (see Fuhrer and Moore, 1995a,b, and Rudebusch and Svensson, 1999). Consequently, we now look at a model considered in Clarida, Gali, and Gertler (1999), Section 5, which has important backward-looking elements. The model consists of the structural equations

\[
x_t = -\varphi(i_t - \hat{E}_t \pi_{t+1}) + (1 - \theta)\hat{E}_t x_{t+1} + \theta x_{t-1} + g_t,
\]

and

\[
\pi_t = \lambda x_t + (1 - \chi)\hat{E}_t \pi_{t+1} + \chi \pi_{t-1} + u_t.
\]

The parameters \( \theta \) and \( \chi \) capture the inertia in output and inflation and are assumed to be between 0 and 1. The shocks are still assumed to follow the processes (3) and (4). The central bank continues to target the growth rate of GDP so that Equation (5) applies. Substituting Equations (12) and (13) into Equation (5) yields the implied interest rate \( i_t \) that will stabilize \( \Delta z_t \) at \( \Delta z \). This rule is given by

\[
i_t = [1 + \beta \varphi^{-1}(1 + \lambda)^{-1}(1 - \chi)]\hat{E}_t \pi_{t+1} + \varphi^{-1}(1 - \theta)\hat{E}_t x_{t+1} + \chi \varphi^{-1}(1 + \lambda)^{-1} \pi_{t-1} + \varphi^{-1}(1 + \lambda)^{-1} x_{t-1} + \varphi^{-1} g_t - \varphi^{-1}(1 + \lambda)^{-1} \Delta z_t.
\]

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Note that we are now assuming that the central bank uses the contemporaneous expectations based policy rule. Plugging the above rule into Equation (12) yields

$$x_t = -\beta(1 + \lambda)^{-1}(1 - \chi)\hat{E}_t\pi_{t+1} + (1 + \lambda)^{-1}x_{t-1}$$

$$-\chi(1 + \lambda)^{-1}\pi_{t-1} - (1 + \lambda)^{-1}u_t + (1 + \lambda)^{-1}\Delta x.$$ \hspace{1cm} (15)

We can then get the reduced form of the price adjustment equation by substituting Equation (15) into Equation (13). This gives us

$$\pi_t = \beta(1 - \chi)[1 - \lambda(1 + \lambda)^{-1}]\hat{E}_t\pi_{t+1} + \lambda(1 + \lambda)^{-1}x_{t-1}$$

$$+ \chi[1 - \lambda(1 + \lambda)^{-1}]\pi_{t-1} + [1 - \lambda(1 + \lambda)^{-1}]u_t$$

$$+ \lambda(1 + \lambda)^{-1}\Delta x.$$ \hspace{1cm} (16)

Thus, our complete system is now given by Equations (15) and (16), representing the evolution of the endogenous variables $x_t$ and $\pi_t$. We write this in matrix form as

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} (1 + \lambda)^{-1}\Delta x \\ \lambda(1 + \lambda)^{-1}\Delta x \end{bmatrix} + \begin{bmatrix} 0 & -\beta(1 + \lambda)^{-1}(1 - \chi) \\ 0 & \beta(1 + \lambda)^{-1}(1 - \chi) \end{bmatrix} \begin{bmatrix} \hat{E}_t\pi_{t+1} \\ \hat{E}_t\pi_{t+1} \end{bmatrix}$$

$$+ \begin{bmatrix} (1 + \lambda)^{-1} \\ \lambda(1 + \lambda)^{-1} \end{bmatrix} \begin{bmatrix} -\chi(1 + \lambda)^{-1} \\ \chi(1 + \lambda)^{-1} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} - (1 + \lambda)^{-1} \\ (1 + \lambda)^{-1} \end{bmatrix} u_t.$$ \hspace{1cm} (17)

To shorten notation again, we can write Equation (17) in the form

$$y_t = \alpha_2 + B_2\hat{E}_t\gamma_{t+1} + \delta_2\gamma_{t-1} + \kappa_2\nu_t,$$

where $y_t = [x_t, \pi_t]'$, $\nu_t = [u_t, g_t]'$, and $\alpha_2$, $B_2$, $\delta_2$, and $\kappa_2$ denote the matrices on the right hand side of Equation (17).

We can immediately make one observation from the system (17). The IS equation parameters, $\phi$ and $\theta$ do not enter Equation (17). This generalizes a similar conclusion obtained in Section 2. One can understand this phenomenon by getting the reduced form IS equation in a slightly different manner. Equation (5) may be rewritten as

$$\pi_t + x_t = x_{t-1} + \Delta x.$$ \hspace{1cm} (18)

If we now substitute the equation for inflation (Equation 16) into Equation (18) we get

$$(1 + \lambda)x_t + \beta(1 - \chi)\hat{E}_t\pi_{t+1} + \chi\pi_{t+1} + u_t = x_{t-1} + \Delta x.$$ \hspace{1cm} (19)

After rearranging it can be seen that this is exactly the reduced form IS Equation (15) obtained above, after substituting the interest rate rule (Equation 14) into the original IS Equation (12). The central bank does not have direct control over expected inflation $\hat{E}_t\pi_{t+1}$ and inflation $\pi_t$. The interest rate only allows the bank to control
directly the demand \( x_t \) and through it \( \pi_t \). The evolution of the growth rate of nominal GDP, on the other hand, depends directly on the evolution of \( \dot{E}_t \pi_{t+1} \) as well as the cost push shock \( u_t \) and \( \pi_t \) as must be clear from Equation (18) or (19). Consequently, even though the bank is able to offset the influence of the IS equation parameters and the demand shock \( g_t \) from the growth rate of GDP, through suitable control of the interest rate (namely, via the rule (14)), it is unable to do the same as far as the parameters of the aggregate supply Equation (16) are concerned. In particular, the discount factor of the suppliers, \( \beta \), the persistence in inflation, \( \chi \), and the degree of price stickiness, \( \lambda \), continue to affect the growth rate of GDP under this policy.

We have seen that even though output inertia does not matter for uniqueness and learnability of equilibrium, inflation inertia could still potentially matter. For uniqueness of equilibrium, we have to obtain the matrix corresponding to Equation (8) in Section 2.1. One can check that this matrix (denoted \( J \)) is given by

\[
J = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & \beta(1 - \chi^{-1}) & -\lambda\chi^{-1} & \chi^{-1}
\end{bmatrix}.
\]

Since there are now two free \( \{x_t, \pi_t\} \) and two pre-determined \( \{x_{t-1}, \pi_{t-1}\} \) endogenous variables, we need exactly 2 eigenvalues of \( J \) to be inside the unit circle for the existence of a unique stationary equilibrium. One can prove that determinacy is always obtained if \( \chi < 0.5 \) and also for \( \chi \geq 0.5 \) provided \( \beta + \lambda \geq 1 \). The latter condition is satisfied for values used in both Woodford (1999) and Clarida, Gali, and Gertler (2000) so that determinacy obtains for plausible values.

For analyzing the E-stability of equilibrium, we proceed as before. The MSV solutions of Equation (17) take the form

\[
y_t = a + by_{t-1} + cu_t,
\]

where \( y_t = (x_t, \pi_t)' \). The MSV solution for \( b, \tilde{b}, \) is given by solving the matrix quadratic

\[
B_2b^2 - b + \delta_2 = 0,
\]

where \( B_2 \) and \( \delta_2 \) are given in Equation (17). For E-stability, we assume that agents have a PLM of the form (20), which leads to an ALM of a similar form. Solving the fixed points of this map will give us the REE solutions, which can then be checked for E-stability. Unfortunately, analytical conditions for E-stability are not available.\(^5\)

Nevertheless, we can check E-stability numerically for plausible values. We use the calibrated values for the U.S. economy in Clarida, Gali, and Gertler (2000)
(see Section 2). We also set \( \chi = 0.71 \), which is the value estimated in Rudebusch (2000). The unique stationary MSV solution for \( \beta \) is given by

\[
\hat{\beta} = \begin{bmatrix} 0.68 \\ -0.58 \\ 0.32 \\ 0.58 \end{bmatrix},
\]

and using Proposition 10.3 of Evans and Honkapohja (2001, chapter 10) it can be checked that the MSV solution is E-stable since all the matrices required for checking this Proposition have real eigenvalues, with the maximal one being 0.23. Experimenting with some other values of \( \chi \), we continue to find that the unique stationary MSV solution is E-stable. We get the same conclusions if we use Woodford’s (1999) values for the U.S. economy. Note that these results are to be expected given our previous experience with the basic model in Section 2 since the interest rate (Equation 14) continues to satisfy the Taylor principle – it reacts to \( \hat{E}_{t}^{\pi_{t+1}} \) with a coefficient bigger than one and to \( \hat{E}_{t}x_{t+1} \) with a positive coefficient. Observe also that there continues to be a negative relationship between \( \hat{E}_{t}^{\pi_{t+1}} \) and \( x_{t} \) in the reduced form IS curve (15).

4. HOW TO PRACTICE GDP TARGETING?

The entire analysis so far was conducted under the assumption that the central bank kept nominal GDP growth equal to a target level every period. Some analysts, however, interpret GDP targeting as setting expected next period nominal GDP growth equal to a fixed target (see Ball, 2000, and Dennis, 2001). This is equivalent to setting \( \hat{E}_{t}^{\Delta z_{t+1}} = \Delta z \). A natural question to ask is whether this policy can be defended on the grounds of determinacy and learnability. We revert to the basic model presented in Section 1 to answer this question.

A policy which sets \( \hat{E}_{t}^{\Delta z_{t+1}} = \Delta z \), i.e., \( \hat{E}_{t}[\pi_{t+1} + x_{t+1} - x_{t}] = \Delta z \), implies an interest rate rule of the form

\[
i_{t} = (1 - \varphi^{-1})\hat{E}_{t}^{\pi_{t+1}} + \varphi^{-1}g_{t} + \varphi^{-1}\Delta z.
\]

(21)

With this policy, the model given by Equations (1) and (2) reduces to

\[
\left[\begin{array}{c}
x_{t} \\
\pi_{t}
\end{array}\right] = \left[\begin{array}{c}
-\Delta z \\
-\lambda \Delta z
\end{array}\right] + \left[\begin{array}{c}
1 \\
1
\end{array}\right] \hat{E}_{t}^{x_{t+1}} + \left[\begin{array}{c}
0 \\
1
\end{array}\right] u_{t},
\]

(22)

or in shortened notation

\[
y_{t} = \alpha_{3} + B_{3}\hat{E}_{t}^{y_{t+1}} + \kappa_{3}u_{t},
\]

(23)

where \( y_{t} = [x_{t}, \pi_{t}]' \), and \( \alpha_{3}, B_{3}, \) and \( \kappa_{3} \), for instance, denote the right hand matrices in Equation (22).

We first check whether the model is determinate. Since both variables \( x_{t} \) and \( \pi_{t} \) are free, determinacy requires both eigenvalues of \( B_{3} \) to be inside the unit circle. However, it is easy to check that the eigenvalues \( \{\gamma_{1}, \gamma_{2}\} \) of \( B_{3} \) satisfy \( 0 < \gamma_{1} < 1 < \gamma_{2} \).
Hence, this policy rule does not imply a determinate REE and so may permit fluctuations arising purely from self-fulfilling expectations. The intuition behind this result is simple: the interest rule (Equation 21) does not satisfy the Taylor principle (see Bullard and Mitra 2002, Proposition 4).

One can also check whether E-stability of the MSV solution holds. The MSV solution of Equation (23) takes the form $y_t = a + cu_t$. Assuming a PLM of agents of this form generates an ALM of the same form, the fixed points of which give the MSV solution. A necessary condition for E-stability, from Proposition 10.3 of Evans and Honkapohja (2001), is that both eigenvalues of $B_1$ have real parts less than 1. However, since $\gamma_2 > 1$, the MSV solution is E-unstable. The reason is again that the interest rule (Equation 21) violates the Taylor principle (see Bullard and Mitra 2002, Proposition 5).

In fact, the results of Honkapohja and Mitra (2001) are applicable to the model (Equation 23) and it is shown there that all the (known) forms of indeterminate (sunspot) equilibria are E-unstable. This provides a novel explanation for avoiding GDP targeting of this form – the policy leads to persistent learning dynamics as agents try (unsuccessfully) to find some equilibrium when in fact no learnable RE solution exists.

This analysis, therefore, supports a monetary policy of keeping nominal GDP growth equal to a target level every period vis-a-vis a policy of targeting expected (next period) GDP growth. As shown before, the former policy is also operational since the nominal interest rate set by the bank ultimately depends only on past data.

5. CONCLUSIONS

A monetary policy of targeting nominal GDP has been advocated by economists for almost two decades now. However, most of the theoretical and empirical defence has taken place in the context of ad hoc macroeconomic models. Not much is known about the theoretical behavior of such a policy in models with explicit micro foundations, which are currently being used to give advice to policy makers. In this paper, I have shown that a policy that targets the GDP growth rate every period leads to a determinate equilibrium in the context of such a model.

I have gone further than analyzing determinacy. It is recognized by a number of economists now that the assumption of rational expectations on the part of private agents is very strong. Consequently, I have studied the stability of these macroeconomic systems under learning. In general, determinacy alone is insufficient to induce learnability of an REE as shown in Bullard and Mitra (2000, 2002).

However, I find that the determinate equilibrium is indeed learnable as long as the bank targets the growth rate of GDP every period. This is important since policy rules which lead to unlearnable equilibria are to be avoided. This is because we have already endowed agents with quite a bit of information about the economy in the formulation of adaptive learning in the sense that the perceived law of motion (PLM) of the agents corresponds to the MSV solution. The agents have
the right variables and the right relationship between the variables, as well as initial conditions in the neighborhood of the equilibrium. If agents are unable to learn the MSV solution even under this favorable assumption, then they are unlikely to learn the equilibrium under more general assumptions. Consequently, learnability of MSV solutions under a particular policy rule should be taken as a minimal requirement before being advocated to policy makers.

We have also seen that it may be dangerous for central banks to assume RE on the part of the private agents at every point of time – the economy may diverge from the REE in this case. The central bank should instead base the interest rate directly on the expectations of private agents. This type of policy rule is conducive to agents being able to coordinate on the unique equilibrium of the economy. This positive result provides an additional argument in favor of nominal GDP targeting and some support to the announced ECB strategy of monetary targeting. Nevertheless, we have also shown that a policy which targets expected GDP growth rate is to be avoided since it leads to problems of indeterminacy and unlearnability.

APPENDIX A: PROOF OF PROPOSITION 1

From the structure of $B_0$ (given by Equation 8) it is evident that one of the eigenvalues is zero. The remaining two eigenvalues are given by those of the matrix

$$
\begin{bmatrix}
\beta & \lambda \\
\beta & 1 + \lambda
\end{bmatrix},
$$

with the following characteristic polynomial:

$$p(\gamma) = \gamma^2 - (1 + \beta + \lambda)\gamma + \beta.$$

Note that $p(0) = \beta > 0$ and $p(1) = -\lambda < 0$, so that one of these eigenvalues is between 0 and 1 and the other is more than 1 (by exploiting the continuity of $p(\gamma)$ in $\gamma$). This shows that exactly 2 eigenvalues of $B_0$ are inside the unit circle. □

APPENDIX B: DETAILS FOR THE CONTEMPORANEOUS EXPECTATIONS BASED POLICY RULE

Given the MSV solution of the form (9) and (10), one computes the (subjective) expectations

$$
\hat{E}_t x_{t+1} = a_1 + b_1 x_t + c_1 p u_t + d_1 \mu g_t, \quad (B1)
$$

$$
\hat{E}_t \pi_{t+1} = a_2 + b_2 x_t + c_2 p u_t + d_2 \mu g_t, \quad (B2)
$$
where we assume that the private sector is able to observe the contemporaneous output gap (and shocks) in forming its forecasts. Inserting Equation (B2) into the reduced form IS Equation (7), one obtains

\[ (1 + \lambda)\Delta x_t = -\beta a_2 + \beta b_2 x_t - (\beta c_2 \rho + 1) u_t - \beta d_2 \mu g_t + x_{t-1}. \]

Solving for \( x_t \) from above finally yields the actual law of motion (ALM) for output as

\[ x_t = \hat{a}_1 + \hat{b}_1 x_{t-1} + \hat{c}_1 u_t + \hat{d}_1 g_t, \tag{B3} \]

where

\[ \hat{a}_1 = (1 + \lambda + \beta b_2)^{-1}(\Delta z - \beta a_2), \tag{B4} \]
\[ \hat{b}_1 = (1 + \lambda + \beta b_2)^{-1}, \tag{B5} \]
\[ \hat{c}_1 = -(1 + \lambda + \beta b_2)^{-1}(\beta c_2 \rho + 1), \tag{B6} \]

and

\[ \hat{d}_1 = -(1 + \lambda + \beta b_2)^{-1} \beta d_2 \mu. \tag{B7} \]

Similarly inserting Equation (B2) into the price adjustment Equation (2), one can obtain the ALM for inflation as

\[ \pi_t = (\lambda + \beta b_2)(\hat{a}_1 + \hat{b}_1 x_{t-1} + \hat{c}_1 u_t + \hat{d}_1 g_t) + \beta a_2 + (\beta c_2 \rho + 1) u_t + \beta d_2 \mu g_t. \]

Collecting terms finally gives the ALM for \( \pi_t \) as

\[ \pi_t = \hat{a}_2 + \hat{b}_2 x_{t-1} + \hat{c}_2 u_t + \hat{d}_2 g_t, \tag{B8} \]

where

\[ \hat{a}_2 = (\lambda + \beta b_2)\hat{a}_1 + \beta a_2, \tag{B9} \]
\[ \hat{b}_2 = (\lambda + \beta b_2)\hat{b}_1, \tag{B10} \]
\[ \hat{c}_2 = (\lambda + \beta b_2)\hat{c}_1 + \beta c_2 \rho + 1, \tag{B11} \]

and

\[ \hat{d}_2 = (\lambda + \beta b_2)\hat{d}_1 + \beta d_2 \mu. \tag{B12} \]

The MSV solution is obtained by solving the set of equations \( a_1 = \hat{a}_1, b_1 = \hat{b}_1, c_1 = \hat{c}_1, d_1 = \hat{d}_1, a_2 = \hat{a}_2, b_2 = \hat{b}_2, c_2 = \hat{c}_2, d_2 = \hat{d}_2 \). For the analysis of expectational stability, we only need the MSV solutions for \( b_1 \) and \( b_2 \). The two equations involving \( b_1 \) and \( b_2 \), i.e., Equations (B5) and (B10), are independent from the rest of the system and they yield two solutions for \( b_1 \) and \( b_2 \). However, given our assumptions
on the structural parameters, one can check that only one solution is stationary (see Mitra 2000 for the details) and this is given by (denoted $\tilde{b}_1$ and $\tilde{b}_2$)

$$\tilde{b}_1 = \frac{2}{1 + \beta + \lambda + \sqrt{(1 + \lambda - \beta)^2 + 4\beta\lambda}},$$  \hfill (B13)

and

$$\tilde{b}_2 = 1 - \tilde{b}_1.$$  \hfill (B14)

Observe that $0 < \tilde{b}_1 < 1$ and $0 < \tilde{b}_2 < 1$. For the analysis of learning we regard Equations (9) and (10) as a PLM for the agents. Computing expectations, as before in Equations (B1) and (B2), we obtain the corresponding ALM for output and inflation in Equations (B3) and (B8), respectively. We can then define a mapping from the PLM to the ALM in the parameter space as

$$T(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) = (\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2).$$  \hfill (B15)

The $T$ mapping gives rise to the differential equation defining E-stability, namely

$$\frac{d}{dt} (a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) = T(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) - (\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2).$$

The MSV solution corresponds to a fixed point of the $T$ mapping and, hence, an equilibrium of the differential equation. The MSV solution is said to be expectationally stable (E-stable) if the MSV fixed point of the differential equation is locally asymptotically stable at that point.

If we put the model in matrix form, we can directly apply the results in Evans and Honkapohja (2001, chapter 10). Consequently, we first put the (reduced form) model given by Equations (8) and (2) in matrix form as

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} (1 + \lambda)^{-1} \Delta c \\ \lambda(1 + \lambda)^{-1} \Delta \pi \end{bmatrix} + \begin{bmatrix} 0 & -\beta(1 + \lambda)^{-1} \\ 0 & \beta(1 + \lambda)^{-1} \end{bmatrix} \begin{bmatrix} \hat{E}x_{t+1} \\ \hat{E}\pi_{t+1} \end{bmatrix}$$

$$+ \begin{bmatrix} (1 + \lambda)^{-1} \\ \lambda(1 + \lambda)^{-1} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} - (1 + \lambda)^{-1} & 0 \\ (1 + \lambda)^{-1} & 0 \end{bmatrix} \begin{bmatrix} u_t \\ g_t \end{bmatrix}. \hfill (B16)$$

To shorten notation, we write Equation (B16) in the form

$$y_t = \alpha + B\hat{E}_t y_{t+1} + \delta y_{t-1} + \kappa w_t, \hfill (B17)$$

where $y_t = [x_t, \pi_t]'$, $w_t = [u_t, g_t]'$, $B = \begin{bmatrix} 0 & -\beta(1 + \lambda)^{-1} \\ 0 & \beta(1 + \lambda)^{-1} \end{bmatrix}$. \hfill (B18)
and
\[
\delta = \begin{bmatrix}
(1 + \lambda)^{-1} & 0 \\
\lambda(1 + \lambda)^{-1} & 0
\end{bmatrix},
\]  
\hspace{1cm} (B19)

and the forms of \( \alpha, \nu \) are omitted since they are not needed in what follows. We also write \( w_t = \Phi w_{t-1} + \epsilon_t \) where \( \epsilon_t = [\hat{a}_t, \hat{g}_t]' \) and
\[
\Phi = \begin{bmatrix}
\rho & 0 \\
0 & \mu
\end{bmatrix},
\]  
\hspace{1cm} (B20)

The PLM of the agents takes the same form as the MSV solution given in Equations (9) and (10) and we write this in matrix form as
\[
y_t = a + by_{t-1} + cw_t,
\]  
\hspace{1cm} (B21)

where
\[
a = [a_1, a_2]',
\]  
\hspace{1cm} (B22)

\[
b = \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix},
\]  
\hspace{1cm} (B23)

and
\[
c = \begin{bmatrix}
c_1 & d_1 \\
c_2 & d_2
\end{bmatrix}.
\]  
\hspace{1cm} (B24)

The corresponding ALM is given by Equations (B3) and (B8) and we write this in matrix form as
\[
y_t = \hat{a} + \hat{b}y_{t-1} + \hat{c}u_t
\]  
\hspace{1cm} (B25)

where \( \hat{a}_i, \hat{b}_i, \hat{c}_i, \hat{d}_i \) replace \( a_i, b_i, c_i, d_i \) in Equations (B22), (B23), and (B24). Note that \( \hat{a}_i, \hat{b}_i, \hat{c}_i, \hat{d}_i \) are given by Equations (B4)–(B7) and (B9)–(B12). The (unique) stationary MSV solution for \( b, \hat{b} \) is given by the solution to the system \( b = \hat{b} \) and this matrix \( \hat{b} \) takes the form (B23) with \( \hat{b}_1 \) and \( \hat{b}_2 \) given by Equations (B13) and (B14) replacing \( b_1 \) and \( b_2 \), respectively.

APPENDIX C: PROOF OF PROPOSITION 2

Appendix B spells out the PLM, the ALM, and the \( T \) map from the PLM to the ALM in Equations (B21), (B25), and (B15), respectively. Proposition 10.3 of Evans and Honkapohja (2001, chapter 10) is directly applicable to this scenario. As shown there, the MSV solution \( (\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2) \) is E-stable if the eigenvalues of all
of the following matrices \((I - Bb)^{-1}B\), \([(I - Bb)^{-1}b]' \otimes [(I - Bb)^{-1}B]\), and \(\Phi' \otimes (I - Bb)^{-1}B\) have real parts less than one, where the required matrices have been defined in Appendix B. The eigenvalues of \((I - Bb)^{-1}B\) can be shown to be 0 and

\[
\gamma = \beta[1 + \beta(1 - \hat{b}_1) + \lambda]^{-1} = \beta \hat{b}_1 < 1 ,
\]

where \(\hat{b}_1\) is given by Equation (B13), is between 0 and 1. The eigenvalues of \(\Phi' \otimes (I - Bb)^{-1}B\) are given by 0,0,\(\gamma\) by the properties of kronecker products which are again less than one by our assumptions on \(\rho\) and \(\mu\). Finally, it can be shown that three of the eigenvalues of \([(I - Bb)^{-1}b]' \otimes [(I - Bb)^{-1}B]\), are 0 and the only non-zero eigenvalue is \(\beta[1 + \beta(1 - \hat{b}_1) + \lambda]^{-2} = \beta \hat{b}_1^2 < 1\) (see Mitra 2000 for a Mathematica routine which computes these eigenvalues). \(\square\)

**APPENDIX D: DETAILS FOR THE RE BASED POLICY RULE**

On simplification, the RE based policy rule (11) becomes

\[
i_t = \psi_0 + \psi_x x_t - \varphi^{-1} (1 + \lambda)^{-1}x_{t-1} + \psi_u \mu_t + \psi_g g_t ,
\]

where

\[
\psi_0 = [1 + \beta \varphi^{-1} (1 + \lambda)^{-1}] a_2 + \varphi^{-1} a_1 - \varphi^{-1} (1 + \lambda)^{-1} \Delta z \, ,
\]

\[
\psi_x = [1 + \beta \varphi^{-1} (1 + \lambda)^{-1}] \hat{b}_2 + \varphi^{-1} \hat{b}_1 ,
\]

\[
\psi_u = [1 + \beta \varphi^{-1} (1 + \lambda)^{-1}] \varepsilon_2 \rho + \varphi^{-1} \varepsilon_1 \rho + \varphi^{-1} (1 + \lambda)^{-1} ,
\]

and

\[
\psi_g = [1 + \beta \varphi^{-1} (1 + \lambda)^{-1}] \varepsilon_2 \mu + \varphi^{-1} \varepsilon_1 \mu + \varphi^{-1} .
\]

We now plug the rule (Equation D1) into Equation (1) and after some rearrangement get the evolution of output as

\[
(1 + \varphi \psi_x) x_t = - \varphi \psi_0 - \varphi^{-1} (1 + \lambda)^{-1} x_{t-1} + \psi_u \mu_t + \varphi \hat{E}_t \pi_{t+1} + \hat{E}_t x_{t+1} + (1 - \varphi \psi_g) g_t ,
\]

We can then get the evolution of inflation from Equation (2) as

\[
\pi_t = \beta + \lambda \varphi (1 + \varphi \psi_x)^{-1} \hat{E}_t \pi_{t+1} + \lambda (1 + \varphi \psi_x)^{-1} \hat{E}_t x_{t+1} + \lambda (1 + \varphi \psi_u (1 + \varphi \psi_x)^{-1}) \mu_t + \lambda (1 + \varphi \psi_g )^{-1} (1 - \varphi \psi_g ) g_t .
\]

Note that the actual evolution of output and inflation during the transition to rational expectations (obviously) still depends on the subjective expectations of
agents as shown by Equations (D2) and (D3). We put Equations (D2) and (D3) in matrix form as
\[
y_t = \alpha_1 + \mathcal{E}y_{t-1} + \delta_1y_{t-1} + \chi_1w_t,
\]
where \(y_t = [x_t, \pi_t]'\), \(w_t = [u_t, \pi_t]'\) and
\[
\mathcal{E}_1 = \begin{bmatrix}
(1 + \varphi \psi_y)^{-1} & \varphi(1 + \varphi \psi_y)^{-1} \\
\lambda(1 + \varphi \psi_y)^{-1} & \beta + \lambda \varphi(1 + \varphi \psi_y)^{-1}
\end{bmatrix},
\]
and
\[
\delta_1 = \begin{bmatrix}
(1 + \lambda)^{-1}(1 + \varphi \psi_y)^{-1} & 0 \\
\lambda(1 + \lambda)^{-1}(1 + \varphi \psi_y)^{-1} & 0
\end{bmatrix}.
\]

For the analysis of learning, we regard Equations (9) and (10) as the PLM of the agents with corresponding forecasts given by Equations (B1)–(B2). Plugging the interest rate rule (D1) into Equations (1) and (2) and simplifying yields
\[
x_t = (1 + \varphi \psi_y - \varphi b_2 - b_1)^{-1}[\alpha_1 + \varphi \alpha_2 - \varphi \psi_0 + (1 + \lambda)^{-1}x_{t-1}
+ (\varphi \rho_c + c_1 \rho - \varphi \psi_0)u_t + (\varphi \rho_d + \mu_d + 1 - \varphi \psi_y)g_t],
\]
and
\[
\pi_t = (\lambda + \beta b_2)x_t + \beta \alpha_2 + (\beta \rho_c + 1)u_t + \beta \rho_d g_t.
\]

After plugging in the value of \(x_t\) from Equation (D7) into Equation (D8), we can define a map from the PLM, (Equations 9–10), to the ALM, (Equations D7–D8), and the fixed points of this map give us the MSV values. For the analysis of E-stability we need \(b_1\) and \(b_2\) which can be shown to be given by Equations (B13) and (B14). As shown in Appendix C, the MSV solution \((\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{g})\) will be E-stable if the eigenvalues of all of the following matrices \((I - B_1\tilde{b})^{-1}B_1, [(I - B_1\tilde{b})^{-1}\delta] \otimes [(I - B_1\tilde{b})^{-1}B_1], \) and \(\Phi \otimes (I - B_1\tilde{b})^{-1}B_1\) have real parts less than 1, with \(\tilde{b}\) as in Appendix B.

NOTES

1. McCallum’s (1997a) model, which assumes rational expectations, is a special case of the model presented in Section 1 since he normalizes the discount factor \(\beta\) to 1. In addition, I assume that the nominal interest rate is the policy instrument instead of the real interest rate (as assumed by McCallum 1997a) since in practice this is a more realistic description of actual policy and it also facilitates the analysis of learning later on.

2. I should clarify in what sense the terms “determinacy” and “indeterminacy” are being used here. “Indeterminacy” here pertains to multiplicity of equilibria, not the phenomena that McCallum (1999) refers to as “nominal determinacy”.

4. Evans and Honkapohja (2000) show that the results on E-stability are unaffected by such simultaneous learning by the central bank and the private sector for optimal monetary policy. Replicating the arguments in the appendix of their paper should generate a similar positive result here since the analysis is essentially similar.

5. When $\chi = 1$, one can show that there exists a determinate equilibrium which is E-stable. By continuity, the same must be true for values of $\phi$ close to 1, i.e., for high levels of inflation (and output) inertia.

6. To economize on space, I have only presented the case of the bank targeting a given growth rate of nominal GDP. Some authors like Hall and Mankiw (1994) have, however, advocated a policy of targeting the level of GDP every period. It can be shown that the stationary equilibrium is unique and E-stable for all possible parametrizations of the model for this policy too.

7. An analogy I have in mind are the notions of weak and strong E-stability used in the learning literature. If a certain equilibrium is not weakly E-stable, then it cannot be strongly E-stable (see Evans and Honkapohja 2001, chapter 8).

LITERATURE CITED


