Mahler measure of multivariable polynomials

Matilde N. Lalín

Université de Montréal
mlalin@dms.umontreal.ca
http://www.dms.umontreal.ca/~mlalin

Oregon Number Theory Days
University of Oregon
November 23, 2019
Extending Mersenne’s construction

Pierce (1918): A construction for finding large prime numbers.

\[ P(x) \in \mathbb{Z}[x] \text{ monic,} \]

\[ P(x) = \prod_{i}(x - r_{i}) \]

\[ \Delta_{n} = \prod_{i}(r_{i}^{n} - 1) \]

\[ P(x) = x - 2 \Rightarrow \Delta_{n} = 2^{n} - 1 \]

Mersenne sequence!
What are the best polynomials?

D. H. Lehmer (1933): To improve the chances of finding a prime, we need $n$ big, or $\Delta_n$ that grows slowly.

$$\lim_{n \to \infty} \frac{\left| \Delta_{n+1} \right|}{\left| \Delta_n \right|} > 1, \text{ but close to } 1.$$

$$\lim_{n \to \infty} \frac{\left| r^{n+1} - 1 \right|}{\left| r^n - 1 \right|} = \begin{cases} |r| & \text{if } |r| > 1, \\ 1 & \text{if } |r| < 1. \end{cases}$$
Mahler measure

For

\[ P(x) = a \prod_{i} (x - r_i) \]

\[ M(P) = |a| \prod_{|r_i| > 1} |r_i|, \quad m(P) = \log |a| + \sum_{|r_i| > 1} \log |r_i|. \]

Thus, we want,

\[ M(P) > 1 \text{ but close, or } m(P) > 0 \text{ but close}. \]
Kronecker’s Lemma

Kronecker (1857)

\[ P \in \mathbb{Z}[x], \ P \neq 0, \]

\[ m(P) = 0 \iff P(x) = x^k \prod \Phi_{n_i}(x) \]

where \( \Phi_{n_i} \) are cyclotomic polynomials.
Lehmer’s Question

Lehmer (1933)

Given \( \varepsilon > 0 \), can we find a polynomial \( P(x) \in \mathbb{Z}[x] \) such that \( 0 < m(P) < \varepsilon \)?

**Conjecture:** No.

\[
m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 0.162357612\ldots
\]

**Conjecture:** This polynomial is the best possible.

With this polynomial, Lehmer found

\[
\sqrt{\Delta_{379}} = 1,794,327,140,357.
\]
Lehmer’s Question - particular families

- $P \in \mathbb{C}[x]$ reciprocal iff
  \[ P(x) = \pm x^{\deg P} P(x^{-1}). \]

Breusch (1951), Smyth (1971)

- $P \in \mathbb{Z}[x]$ nonreciprocal,
  \[ m(P) \geq m(x^3 - x - 1) = 0.2811995743\ldots \]

- \[ \Delta_{127} = 3, 233, 514, 251, 032, 733 \]

- Borwein, Drobrowolski, Mossinghoff (2007) $P \in \mathbb{Z}[x]$ with no cyclotomic factors and odd coefficients,
  \[ m(P) \geq \frac{\log 5}{4} \left( 1 - \frac{1}{\deg(P) + 1} \right). \]
Dobrowolski (1979)

If $P \in \mathbb{Z}[x]$ is monic, irreducible and noncyclotomic of degree $d$, then

$$M(P) \geq 1 + c \left( \frac{\log \log d}{\log d} \right)^3,$$

where $c$ is an absolute positive constant.
Mahler measure of multivariable polynomials

$P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the (logarithmic) **Mahler measure** is:

$$m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| d\theta_1 \cdots d\theta_n$$

$$= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : |x_i| = 1\}$.

Jensen’s formula implies

$$m(P) = \log |a| + \sum_{|r_i| > 1} \log |r_i| \quad \text{for} \quad P(x) = a \prod_i (x - r_i)$$

$$M(P) := \exp(m(P)).$$
Mahler measure is ubiquitous!

- Interesting questions about distribution of values
- Heights
- Special values of $L$-functions
- Volumes in hyperbolic space
- Entropy of certain arithmetic dynamical systems
Some simple properties

- $m(P) \geq 0$ if $P$ has integral coefficients.
- For $P, Q \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$
  
  $$m(P \cdot Q) = m(P) + m(Q)$$

- For $P \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$, the length is given by
  
  $$L(P) = \sum |\text{coefficient}|$$

  It measures the complexity of the polynomial.

  $$M(P) \leq L(P) \leq 2^{d_1 + \cdots + d_n} M(P)$$

  Similarly with the height $H(P) = \max |\text{coefficient}|$.

  Used in transcendence theory.
Boyd–Lawton Theorem


For $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$,

$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots, x^{k_n})) = m(P(x_1, \ldots, x_n))$$

With $k_2, \ldots, k_n \to \infty$ independently from each other.

The Mahler measure of several variable polynomials does not say much new about Lehmer’s Question.
Examples in several variables

Smyth (1981)

\[ m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \]

\[ L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \]

\[ \chi_{-3}(n) = \begin{cases} 
1 & n \equiv 1 \text{ mod } 3, \\
-1 & n \equiv -1 \text{ mod } 3, \\
0 & n \equiv 0 \text{ mod } 3.
\end{cases} \]

\[ m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \]

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]
More examples in several variables


\[
 m\left(1 + x + \left(\frac{1 - x_1}{1 + x_1}\right) \left(\frac{1 - x_2}{1 + x_2}\right) (1 + y)z\right) = \frac{93}{\pi^4} \zeta(5)
\]

- Known formulas for

\[
 m\left(1 + x + \left(\frac{1 - x_1}{1 + x_1}\right) \ldots \left(\frac{1 - x_n}{1 + x_n}\right) (1 + y)z\right)
\]
How do we compute this?

\[ m(1 + x + y) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log |1 + e^{it} + e^{is}| dt ds \]

\[ = \int_{-\pi}^{\pi} \log \max\{|1 + e^{it}|, 1\} \, dt \]

\[ = \frac{1}{2\pi} \int_{-2\pi/3}^{2\pi/3} \log |1 + e^{it}| \, dt. \]

We use

\[ \log |1 + e^{it}| = \Re \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{int}, \]

and

\[ \int_{-2\pi/3}^{2\pi/3} e^{int} \, dt = \frac{2}{n} \sin \frac{2n\pi}{3} = \frac{\sqrt{3}}{n} \chi_{-3}(n). \]
How do we compute this?

\[ m(1 + x + y) = \frac{\sqrt{3}}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \chi_3(n)}{n^2} \]

\[ = \frac{\sqrt{3}}{2\pi} \left( \sum_{n=1}^{\infty} \frac{\chi_3(n)}{n^2} - 2 \sum_{n=1}^{\infty} \frac{\chi_3(2n)}{(2n)^2} \right) \]

Use \( \chi_3(2n) = \chi_3(2)\chi_3(n) = -\chi_3(n) \).

\[ m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_3, 2) = L'(\chi_3, -1). \]
Elliptic curves

\[ E : Y^2 = X^3 + aX + b \]

Example:

\[ x + \frac{1}{x} + y + \frac{1}{y} + \alpha = 0 \]

\[ x = \frac{\alpha X - 2Y}{2X(X - 1)} \quad \quad y = \frac{\alpha X + 2Y}{2X(X - 1)}. \]

\[ E_{N(\alpha)} : Y^2 = X \left( X^2 + \left( \frac{\alpha^2}{4} - 2 \right) X + 1 \right). \]
\begin{align*}
L(E, s) &= \prod_{\text{good } p} (1 - a_p p^{-s} + p^{1-2s})^{-1} \prod_{\text{bad } p} (1 - a_p p^{-s})^{-1} \\
&= 1 + p - \#E(\mathbb{F}_p) \\
\text{Conjecture (Boyd (1998))} \\
m(\alpha) &= m \left( x + \frac{1}{x} + y + \frac{1}{y} + \alpha \right) \\
m(\alpha) \overset{?}{=} \frac{L'(E_{N(\alpha)}, 0)}{s_\alpha} \quad \alpha \in \mathbb{N} \neq 0, 4 \\
s_\alpha \in \mathbb{Q} \text{ of low height (often in } \mathbb{Z})
\end{align*}
Boyd’s conjectures

\[ m(\alpha) = m \left( \frac{1}{x} + \frac{1}{y} + \alpha \right) \]

\[ \frac{L'(E_{N(\alpha)}, 0)}{s_\alpha} \]

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Red cases proven by Brunault, L., Rogers, Zudilin.
Why do we get nice numbers?

In many cases, the Mahler measure is a special period coming from Beĭlinson’s conjectures!

\[ L'(X, 0) \sim_{\mathbb{Q}^*} \text{reg}(\xi) \]


In many cases, the Mahler measure can be related to the right side of the above equation.
Global information from local information through $L$-functions

- **Arithmetic-geometric object** $X$ (for instance, $X = \mathcal{O}_F$, $F$ a number field)
- **$L$-function** ($L(F, s) = \zeta_F(s)$)
- **Finitely-generated abelian group** $K$ ($K = \mathcal{O}_F^*$)
- **Regulator map** $\text{reg} : K \rightarrow \mathbb{R}$ ($\text{reg} = \log | \cdot |$)

\[
(K \text{ rank } 1) \quad L'(X, 0) \sim_{\mathbb{Q}^*} \text{reg}(\xi)
\]

(Dirichlet class number formula, for $F$ real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log |\epsilon|, \epsilon \in \mathcal{O}_F^*$)
Mahler measure of genus-one curves

\[
xy \left( x + \frac{1}{x} + y + \frac{1}{y} + \alpha \right)
\]

\[P(x, y) = a_2(x)(y - y_1(x))(y - y_2(x)) \quad E : P(x, y) = 0 \quad \text{elliptic curve}
\]

\[
m(P) - m(a_2(x)) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} (\log |y - y_1(x)| + \log |y - y_2(x)|) \frac{dx}{x} \frac{dy}{y}
\]
Suppose that \( |y_1(x)| \geq 1 \) and \( |y_2(x)| \leq 1 \).

By Jensen’s formula,

\[
= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log \max\{1, |y_1(x)|\} \frac{dx}{x} = \frac{1}{2\pi i} \int_{\gamma} \log |y| \frac{dx}{x}
\]

\( \gamma = \{(x, y) \in E, |x| = 1, |y| \geq 1\} \).
The regulator

\[ m(P) - m(a_2(x)) = \frac{1}{2\pi i} \int_{\gamma} \log |y| \frac{dx}{x} = -\frac{1}{2\pi i} \int_{\gamma} \eta(x, y), \]

where

\[ \eta(x, y) = \log |x| \text{d} \arg y - \log |y| \text{d} \arg x \]

and

\[ \text{d} \arg x = \text{Im} \left( \frac{dx}{x} \right). \]

\( \eta(x, y) \) is a closed differential form defined on \( P = 0 \) minus the set \( S \) of zeros and poles of \( x, y \).
Some properties of $\eta(x, y)$

- $\eta(x, y) = -\eta(y, x)$
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$
- $\eta(x, 1 - x)$ exact.

$$
\eta(x, 1 - x) = d\text{i}D(x),
$$

the Bloch-Wigner Dilogarithm

$$
D(x) := \text{Im}(\text{Li}_2(x)) + \text{arg}(1 - x) \log |x|
$$

$$
\text{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad |x| < 1
$$

$$
\text{Li}_2(x) = -\int_{0}^{x} \frac{\log(1 - w)}{w} dw.
$$
When can we compute $\int_\gamma \eta(x, y)$?

- $\eta(x, y)$ is **exact** and $\gamma$ has **non-trivial boundary**,

  $$x \wedge y = \sum r_i x_i \wedge (1 - x_i).$$

  Example:

  $$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2),$$

  and extensions to more variables, formulas involving $\zeta(3)$, $\zeta(5)$, etc.

- $\eta(x, y)$ is **non-exact** and $\gamma$ has **trivial boundary**? The case of many elliptic curves formulas.

  After Bloch,

  $$\int_\gamma \eta(x, y) = D^E ((x) \diamond (y)),$$

  $D^E$ is the elliptic dilogarithm and $(x) \diamond (y)$ is a divisor.
The Beĭlinson Conjectures

For $E/\mathbb{Q}$ an elliptic modular curve/a CM elliptic curve, Beĭlinson/Bloch proved

$$L(E, 2) = \frac{\pi}{N} D^E(\xi), \quad \xi \in \mathbb{Z}[E(\bar{\mathbb{Q}})_{\text{tors}}].$$

For $E/\mathbb{Q}$ an elliptic curve, Zagier conjectured

$$L(E, 2) \overset{?}{=} \frac{\pi}{N} D^E(\xi), \quad \xi \in \mathbb{Z}[E(\bar{\mathbb{Q}})]^{\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}.$$
How are the elliptic curve formulas proven?

- Rodriguez-Villegas (1997): When the curve has complex multiplication.
- Rogers & Zudilin (2012): By decomposing the cusp form given by the modularity theorem into products of Eisenstein series that can be related to the integral.
- Brunault, Mellit, & Zudilin (2014): If there is a modular unit parametrization.
- L. & Rogers (2007): We may relate the Mahler measures of different polynomials with same $N$ by working with the regulator on $K(E)$.

\[
m(8) = 4m(2), \quad m(5) = 6m(1).
\]

For $|h| < 1$, $h \neq 0$, \[
m\left(2 \left( h + \frac{1}{h} \right) \right) + m\left(2 \left( ih + \frac{1}{ih} \right) \right) = m\left(\frac{4}{h^2}\right).
\]
Boyd (1998) studied

\[ P_k(x_1, y_1) = (x_1 + 1)y_1^2 + (x_1^2 + kx_1 + 1)y + (x_1^2 + x_1) \]

\[ E_k : Y_1^2 + (k - 2)X_1 Y_1 + kY_1 = X_1^3 \quad g = 1, \quad k \in \mathbb{Z} \setminus \{-6, 2, 3\} \]

and

\[ Q_k(x, y) = y^2 + (x^4 + kx^3 + 2kx^2 + kx + 1)y + x^4. \]

Its Jacobian splits into two elliptic curves.

\[ Y^2 = f(X^2) \quad g = 2, \quad k \in \mathbb{Z} \setminus \{-1, 0, 4, 8\} \]

where \( f(Z) = (k^2 + k)Z^3 + (-2k^2 + 5k + 4)Z^2 + (k^2 - 5k + 8)Z - k + 4. \)
Boyd conjectured

\[ m(Q_k) = \begin{cases} 
2m(P_{2-k}) & 0 \leq k \leq 4, \\
m(P_{2-k}) & k \leq -1. 
\end{cases} \]

- This was proved by Bertin and Zudilin (2016) by relating the derivatives of the Mahler measures viewing them as solutions of a certain Picard–Fuchs differential equation.
- \( L \) and Wu (2019) recovered the result by proving the identity at the level of the regulator of \( Y^2 = f(X) \).
Other cases

- Liu & Qin (2019+) higher genus! \((g = 3)\)

\[
m(y^2 + (x^6 + \alpha x^5 - x^4 + (2 - 2\alpha)x^3 - x^2 + \alpha x + 1)y + x^6) = \frac{L'(E_{N(\alpha)}, 0)}{s_{\alpha}}
\]

L. & Wu (2019+) The left hand side equals

\[
m(xy^2 + (\alpha x - 1)y - x^2 + x).
\]

- Boyd (2005), L. (2015) negative \(L\)-values!

\[
m(z + (x + 1)(y + 1)) = 2L'(E_{15}, -1)
\]
Higher Mahler measure

Let $k \in \mathbb{N}$, $P \in \mathbb{C}[x^\pm]$, the $k$-high (logarithmic) Mahler measure is:

$$m_k(P) = \frac{1}{2\pi i} \int_{\mathbb{T}} \log^k |P(x)| \frac{dx}{x}$$

Special formulas for the multivariable case (apparently) give new examples of Beĭlinson’s conjectures.

Kurokawa, L., Ochiai (2008)

$$m_6(1 - x) = \frac{45}{2} \zeta(3)^2 + \frac{275}{1344} \pi^6.$$
Let $a_1, \ldots, a_n \in \mathbb{R}_{>0}$. The $(a_1, \ldots, a_n)$-Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)$ is defined by

$$m_{a_1, \ldots, a_n}(P) := \frac{1}{(2\pi i)^n} \int_{T_{a_1} \times \cdots \times T_{a_n}} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $T_a = \{ x \in \mathbb{C} : |x| = a \}$. 
Some cases of Mahler measure for arbitrary tori

- L. & Mittal (2018)

\[ m_{a,a}(y^2 + 2xy - x^3 + x) = 2 \log a + 2L'(E_{20}, 0) \quad \frac{\sqrt{5} - 1}{2} \leq a \leq \frac{1 + \sqrt{5}}{2} \]

- Roy (2019++)

\[ m_{a,\sqrt{a}} \left( x + \frac{1}{x} + y + \frac{1}{y} + 8 \right) = -\frac{1}{2} \log a + 4L'(E_{24}, 0) \quad \text{for certain } a \]
More Mahler measure with non-trivial coefficients


\[ m\left( a \left( x + \frac{1}{x} \right) + y + \frac{1}{y} + c \right) \]

Several cases, including \( a = 2, c = 4, N = 30 \).

- Boyd (1998)

\[ m(y^2 + kxy + \beta y - x^3) \equiv \frac{1}{3} \log |\beta| + s_{k,b}L'(E_{N(k,\beta)}, 0) \]

Nice formulas when \( \beta \mid k! \) Giard (2019+): \( k = 4, \beta = 2 \).

These formulas stretch the application of Beilinson’s conjectures to cases of “relative K-theory”.

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Looking ahead

We would like to understand...

- ... the formulas for Boyd’s families in a systematic way,
- ... the Mahler measure of higher genus curves,
- .... more formulas yielding $L'(E, -k)$,
- ... the role of varying the integration torus and other cases with non-trivial coefficients.

We hope that this will yield light to Beilinson’s conjectures and the nature of special values of $L$-functions.
Thanks for your attention!