Mahler measure of multivariable polynomials

Matilde N. Lalín

Université de Montréal
mlalin@dms.umontreal.ca
http://www.dms.umontreal.ca/~mlalin

Oregon Number Theory Days
University of Oregon
November 23, 2019

Université de Montréal

Extending Mersenne's construction

Pierce (1918): A construction for finding large prime numbers.

$P(x) \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - r_i)$$

$$\Delta_n = \prod_i (r_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Mersenne sequence!

Université de Montréal

What are the best polynomials?

D. H. Lehmer (1933): To improve the chances of finding a prime, we need n big, or Δ_n that grows slowly.

$$\lim_{n\to\infty} \frac{|\Delta_{n+1}|}{|\Delta_n|} > 1, \text{ but close to } 1.$$

$$\lim_{n\to\infty} \frac{|r^{n+1}-1|}{|r^n-1|} = \begin{cases} |r| & \text{if } |r|>1, \\ 1 & \text{if } |r|<1. \end{cases}$$

Université de Montréal

Mahler measure

For

$$P(x) = a\prod_i (x - r_i)$$

$$M(P) = |a| \prod_{|r_i|>1} |r_i|, \qquad \mathrm{m}\,(P) = \log|a| + \sum_{|r_i|>1} \log|r_i|.$$

Thus, we want,

$$M(P) > 1 \text{ but close, or } \mathrm{m}(P) > 0 \text{ but close.}$$

Université de Montréal

Kronecker's Lemma

Kronecker (1857)

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$\mathrm{m}(P) = 0 \Longleftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

where Φ_{n_i} are cyclotomic polynomials.

Université de Montréal

Lehmer's Question

Lehmer (1933)

Given $\varepsilon > 0$, can we find a polynomial $P(x) \in \mathbb{Z}[x]$ such that $0 < \mathrm{m}(P) < \varepsilon$?

Conjecture: No.

$$\mathrm{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 0.162357612\ldots$$

Conjecture: This polynomial is the best possible.

With this polynomial, Lehmer found

$$\sqrt{\Delta_{379}} = 1,794,327,140,357.$$

Université de Montréal

Lehmer's Question - particular families

- $P \in \mathbb{C}[x]$ reciprocal iff

$$P(x) = \pm x^{\deg P} P\left(x^{-1}\right).$$

Breusch (1951), Smyth (1971)

$P \in \mathbb{Z}[x]$ nonreciprocal,

$$\mathrm{m}(P) \geq \mathrm{m}(x^3 - x - 1) = 0.2811995743\ldots.$$

$$\Delta_{127} = 3,233,514,251,032,733$$

- Borwein, Drobrowolski, Mossinghoff (2007) $P \in \mathbb{Z}[x]$ with no cyclotomic factors and odd coefficients,

$$\mathrm{m}(P) \geq \frac{\log 5}{4}\left(1 - \frac{1}{\deg(P)+1}\right).$$

Université de Montréal

Lehmer's Question - degree dependent bounds

Dobrowolski (1979)

If $P \in \mathbb{Z}[x]$ is monic, irreducible and noncyclotomic of degree d, then

$$M(P) \geq 1 + c \left(\frac{\log \log d}{\log d} \right)^3,$$

where c is an absolute positive constant.

Université de Montréal

Mahler measure of multivariable polynomials

$P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} \mathrm{m}(P) &= \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})| d\theta_1 \cdots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}, \end{aligned}$$

where $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : |x_i| = 1\}$.

Jensen's formula implies

$$\mathrm{m}(P) = \log |a| + \sum_{|r_i|>1} \log |r_i| \quad \text{for} \quad P(x) = a \prod_i (x - r_i)$$

$$M(P) := \exp(\mathrm{m}(P)).$$

Université de Montréal

Mahler measure is ubiquitous!

- Interesting questions about distribution of values
- Heights
- Special values of L-functions
- Volumes in hyperbolic space
- Entropy of certain arithmetic dynamical systems

Université de Montréal

Some simple properties

- $\mathrm{m}(P) \geq 0$ if P has integral coefficients.
- For $P, Q \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$$\mathrm{m}(P \cdot Q) = \mathrm{m}(P) + \mathrm{m}(Q)$$

- For $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the length is given by

$$L(P) = \sum |\text{coefficient}|$$

It measures the complexity of the polynomial.

$$M(P) \leq L(P) \leq 2^{d_1 + \cdots + d_n} M(P)$$

Similarly with the height $H(P) = \max |\text{coefficient}|$.

Used in transcendence theory.

Université de Montréal

Boyd–Lawton Theorem

Boyd (1981), Lawton (1983)

For $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$,

$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} \mathrm{m}(P(x, x^{k_2}, \ldots, x^{k_n})) = \mathrm{m}(P(x_1, \ldots, x_n))$$

With $k_2, \ldots, k_n \to \infty$ independently from each other.

The Mahler measure of several variable polynomials does not say much new about Lehmer's Question.

Université de Montréal

Examples in several variables

Smyth (1981)

- $$\mathrm{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3, \\ -1 & n \equiv -1 \bmod 3, \\ 0 & n \equiv 0 \bmod 3. \end{cases}$$

- $$\mathrm{m}(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Université de Montréal

More examples in several variables

- L. (2006)

$$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=\frac{93}{\pi^4}\zeta(5)$$

- Known formulas for

$$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

Université de Montréal

How do we compute this?

$$\begin{aligned}
\mathrm{m}(1+x+y) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|1+e^{it}+e^{is}|dtds \\
&= \int_{-\pi}^{\pi} \log\max\{|1+e^{it}|,1\}dt \\
&= \frac{1}{2\pi}\int_{-2\pi/3}^{2\pi/3} \log|1+e^{it}|dt.
\end{aligned}$$

We use

$$\log|1+e^{it}| = \mathrm{Re}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{int},$$

and

$$\int_{-2\pi/3}^{2\pi/3} e^{int}dt = \frac{2}{n}\sin\frac{2n\pi}{3} = \frac{\sqrt{3}}{n}\chi_{-3}(n).$$

Université de Montréal

How do we compute this?

$$\begin{aligned} \mathrm{m}(1+x+y) &= \frac{\sqrt{3}}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\chi_{-3}(n)}{n^2} \\ &= \frac{\sqrt{3}}{2\pi}\left(\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^2} - 2\sum_{n=1}^{\infty}\frac{\chi_{-3}(2n)}{(2n)^2}\right) \end{aligned}$$

Use $\chi_{-3}(2n) = \chi_{-3}(2)\chi_{-3}(n) = -\chi_{-3}(n)$.

$$\mathrm{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) = L'(\chi_{-3},-1).$$

Université de Montréal

Elliptic curves

$$E : Y^2 = X^3 + aX + b$$

Example:

$$x + \frac{1}{x} + y + \frac{1}{y} + \alpha = 0$$

$$x = \frac{\alpha X - 2Y}{2X(X-1)} \qquad y = \frac{\alpha X + 2Y}{2X(X-1)}.$$

$$E_{N(\alpha)} : Y^2 = X\left(X^2 + \left(\frac{\alpha^2}{4} - 2\right)X + 1\right).$$

Université de Montréal

L-function

$$L(E,s) = \prod_{\text{good } p} (1 - a_p p^{-s} + p^{1-2s})^{-1} \prod_{\text{bad } p} (1 - a_p p^{-s})^{-1}$$

$$a_p = 1 + p - \#E(\mathbb{F}_p)$$

$$m(\alpha) := \mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + \alpha\right)$$

Conjecture (Boyd (1998))

$$m(\alpha) \stackrel{?}{=} \frac{L'(E_{N(\alpha)}, 0)}{s_\alpha} \quad \alpha \in \mathbb{N} \neq 0, 4$$

$s_\alpha \in \mathbb{Q}$ of low height (often in $\mathbb{Z}$)

de Montréal

Boyd's conjectures

$$m(\alpha) = \mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + \alpha\right) \stackrel{?}{=} \frac{L'(E_{N(\alpha)}, 0)}{s_\alpha}$$

α	s_α	$N(\alpha)$	α	s_α	$N(\alpha)$
1	1	15	11	-8	1155
2	1	24	12	1/2	48
3	1/2	21	13	-4	663
4	*	*	14	8	840
5	1/6	15	15	-24	3135
6	2	120	16	1/11	15
7	2	231	17	-24	4641
8	1/4	24	18	-16	1848
9	2	195	19	-40	6555
10	-8	840	20	2	240

Red cases proven by Brunault, L., Rogers, Zudilin.

Université de Montréal

Why do we get nice numbers?

In many cases, the Mahler measure is a special period coming from Beĭlinson's conjectures!

$$L'(X,0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

Deninger (1997), Rodriguez-Villegas (1997)

In many cases, the Mahler measure can be related to the right side of the above equation.

Université de Montréal

Philosophy of Beĭlinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($L(F,s) = \zeta_F(s)$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \text{ rank } 1) \qquad L'(X,0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta_F'(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

Université de Montréal

Mahler measure of genus-one curves

$$xy\left(x+\frac{1}{x}+y+\frac{1}{y}+\alpha\right)$$

$$P(x,y)=a_2(x)(y-y_1(x))(y-y_2(x)) \qquad E: P(x,y)=0 \quad \text{elliptic curve}$$

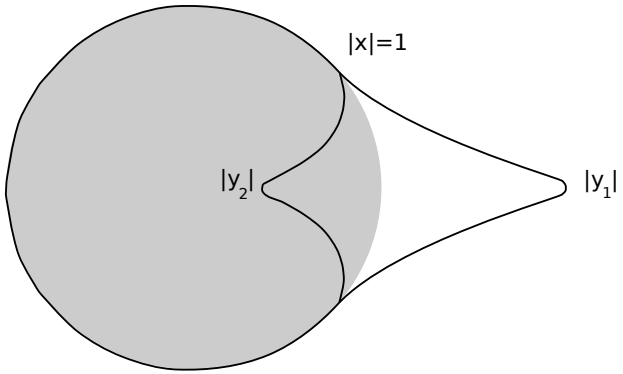
$$\mathrm{m}(P)-\mathrm{m}(a_2(x))=\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\left(\log|y-y_1(x)|+\log|y-y_2(x)|\right)\frac{dx}{x}\frac{dy}{y}$$

Université de Montréal

Mahler measure of genus-one curves

$$\mathrm{m}(P) - \mathrm{m}(a_2(x)) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} (\log|y - y_1(x)| + \log|y - y_2(x)|) \frac{dx}{x} \frac{dy}{y}$$

Suppose that $|y_1(x)| \geq 1$ and $|y_2(x)| \leq 1$.



By Jensen's formula,

$$= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log \max\{1, |y_1(x)|\} \frac{dx}{x} = \frac{1}{2\pi i} \int_{\gamma} \log|y| \frac{dx}{x}$$

$$\gamma = \{(x, y) \in E, |x| = 1, |y| \geq 1\}.$$

Université de Montréal

The regulator

$$\mathrm{m}(P) - \mathrm{m}(a_2(x)) = \frac{1}{2\pi i}\int_\gamma \log|y|\frac{dx}{x} = -\frac{1}{2\pi i}\int_\gamma \eta(x,y),$$

where

$$\eta(x,y) = \log|x|\, di \arg y - \log|y|\, di \arg x$$

and

$$d\arg x = \mathrm{Im}\left(\frac{dx}{x}\right).$$

$\eta(x,y)$ is a closed differential form defined on $P=0$ minus the set S of zeros and poles of x, y.

Université de Montréal

Some properties of $\eta(x, y)$

- $\eta(x, y) = -\eta(y, x)$
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$
- $\eta(x, 1-x)$ exact.

$$\eta(x, 1-x) = diD(x),$$

the Bloch-Wigner Dilogarithm

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad |x| < 1$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-w)}{w} dw.$$

Université de Montréal

When can we compute $\int_\gamma \eta(x,y)$?

- $\eta(x,y)$ is exact and γ has non-trivial boundary,

$$x \wedge y = \sum r_i x_i \wedge (1 - x_i).$$

Example:

$$\mathrm{m}(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2),$$

and extensions to more variables, formulas involving $\zeta(3)$, $\zeta(5)$, etc.

- $\eta(x,y)$ is non-exact and γ has trivial boundary? The case of many elliptic curves formulas.
After Bloch,

$$\int_\gamma \eta(x,y) = D^E\big((x) \diamond (y)\big),$$

D^E is the elliptic dilogarithm and $(x) \diamond (y)$ is a divisor.

Université de Montréal

The Beĭlinson Conjectures

For $E/\mathbb{Q}$ an elliptic modular curve/a CM elliptic curve, Beĭlinson/Bloch proved

$$L(E,2) = \frac{\pi}{N} D^E(\xi), \quad \xi \in \mathbb{Z}[E(\bar{\mathbb{Q}})_{\mathrm{tors}}].$$

For $E/\mathbb{Q}$ an elliptic curve, Zagier conjectured

$$L(E,2) \stackrel{?}{=} \frac{\pi}{N} D^E(\xi), \quad \xi \in \mathbb{Z}[E(\bar{\mathbb{Q}})]^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}.$$

Université de Montréal

How are the elliptic curve formulas proven?

- Rodriguez-Villegas (1997): When the curve has complex multiplication.
- Rogers & Zudilin (2012): By decomposing the cusp form given by the modularity theorem into products of Eisenstein series that can be related to the integral.
- Brunault, Mellit, & Zudilin (2014): If there is a modular unit parametrization.

L. & Rogers (2007): We may relate the Mahler measures of different polynomials with same N by working with the regulator on $K(E)$.

$$m(8) = 4m(2), \quad m(5) = 6m(1).$$

For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(ih + \frac{1}{ih}\right)\right) = m\left(\frac{4}{h^2}\right).$$

Université de Montréal

Higher genus identities

Boyd (1998) studied

$$P_k(x_1, y_1) = (x_1 + 1)y_1^2 + (x_1^2 + kx_1 + 1)y + (x_1^2 + x_1)$$

$$E_k : Y_1^2 + (k-2)X_1Y_1 + kY_1 = X_1^3 \qquad g = 1,\ k \in \mathbb{Z} \setminus \{-6, 2, 3\}$$

and

$$Q_k(x, y) = y^2 + (x^4 + kx^3 + 2kx^2 + kx + 1)y + x^4.$$

Its Jacobian splits into two elliptic curves.

$$Y^2 = f(X^2) \qquad g = 2,\ k \in \mathbb{Z} \setminus \{-1, 0, 4, 8\}$$

where $f(Z) = (k^2 + k)Z^3 + (-2k^2 + 5k + 4)Z^2 + (k^2 - 5k + 8)Z - k + 4.$

Université de Montréal

Higher genus

Boyd conjectured

$$\mathrm{m}(Q_k) = \begin{cases} 2\mathrm{m}(P_{2-k}) & 0 \leq k \leq 4, \\ \mathrm{m}(P_{2-k}) & k \leq -1. \end{cases}$$

- This was proved by Bertin and Zudilin (2016) by relating the derivatives of the Mahler measures viewing them as solutions of a certain Picard–Fuchs differential equation.
- L and Wu (2019) recovered the result by proving the identity at the level of the regulator of $Y^2 = f(X)$.

Université de Montréal

Other cases

- Liu & Qin (2019+) higher genus! ($g = 3$)

$$\mathrm{m}(y^2 + (x^6 + \alpha x^5 - x^4 + (2 - 2\alpha)x^3 - x^2 + \alpha x + 1)y + x^6) \stackrel{?}{=} \frac{L'(E_{N(\alpha)}, 0)}{s_\alpha}$$

L.& Wu (2019+) The left hand side equals

$$\mathrm{m}(xy^2 + (\alpha x - 1)y - x^2 + x).$$

- Boyd (2005), L. (2015) negative L-values!

$$\mathrm{m}(z + (x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15}, -1)$$

Université de Montréal

Higher Mahler measure

Let $k \in \mathbb{N}$, $P \in \mathbb{C}[x^{\pm}]$, the k-high (logarithmic) *Mahler measure* is :

$$\mathrm{m}_k(P) \;=\; \frac{1}{2\pi i}\int_{\mathbb{T}^1} \log^k |P(x)| \frac{dx}{x}$$

Special formulas for the multivariable case (apparently) give new examples of Beĭlinson's conjectures.

Kurokawa, L., Ochiai (2008)

$$\mathrm{m}_6(1-x) = \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6.$$

Université de Montréal

Mahler measure for arbitrary tori

Let $a_1, \ldots, a_n \in \mathbb{R}_{>0}$. The $(a_1, \ldots, a_n)$-Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)$ is defined by

$$\mathrm{m}_{a_1,\ldots,a_n}(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}_{a_1} \times \cdots \times \mathbb{T}_{a_n}} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}_a = \{x \in \mathbb{C} : |x| = a\}$.

Université de Montréal

Some cases of Mahler measure for arbitrary tori

- L. & Mittal (2018)

$$\mathrm{m}_{a,a}(y^2+2xy-x^3+x) = 2\log a + 2L'(E_{20},0) \qquad \frac{\sqrt{5}-1}{2} \leq a \leq \frac{1+\sqrt{5}}{2}$$

- Roy (2019++)

$$\mathrm{m}_{a,\sqrt{a}}\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right) = -\frac{1}{2}\log a + 4L'(E_{24},0) \quad \text{for certain } a$$

Université de Montréal

More Mahler measure with non-trivial coefficients

- L., Samart, & Zudilin (2016), Meemark & Samart (2019+)

$$\mathrm{m}\left(a\left(x+\frac{1}{x}\right)+y+\frac{1}{y}+c\right)$$

Several cases, including $a=2, c=4, N=30$.

- Boyd (1998)

$$\mathrm{m}(y^2+kxy+\beta y-x^3) \stackrel{?}{=} \frac{1}{3}\log|\beta| + s_{k,b}L'(E_{N(k,\beta)},0)$$

Nice formulas when $\beta \mid k!$ Giard (2019+): $k=4, \beta=2$.

These formulas stretch the application of Beĭlinson's conjectures to cases of "relative K-theory".

Université de Montréal

Looking ahead

We would like to understand...

- ... the formulas for Boyd's families in a systematic way,
- ... the Mahler measure of higher genus curves,
- .... more formulas yielding $L'(E, -k)$,
- ... the role of varying the integration torus and other cases with non-trivial coefficients.

We hope that this will yield light to Beĭlinson's conjectures and the nature of special values of L-functions.

Université de Montréal

Thanks for your attention!

Université de Montréal