Public Economics

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CONTENTS

Chapter 1
COMPETITIVE EQUILIBRIUM AND WELFARE

In this section we shall examine

- The Concept of Pareto Efficiency.
- Pareto Efficiency in a Two Person Two Good Pure Exchange Economy
  - Efficiency in Consumption
  - Efficiency in Production
  - Efficiency in Production and Consumption.
- Efficiency and Social Welfare.
- The Fundamental Theorem of Welfare Economics.
- Failures of the Fundamental Theorem.
  - Imperfect Competition.
  - Public Goods.
  - Externalities.
  - Incomplete Markets.
  - Information Failures.

1.1 The Concept of Pareto Efficiency.

Definition 1 *An allocation of goods, either input or output goods, is said to be* Pareto Efficient *if we cannot find a reallocation of those goods such that we can produce more of something (utility or output) without producing less of something else.*

Definition 2 *A reallocation of goods that allows more of something to be produced without the sacrifice of something else is said to be* Pareto Improving.

It follows immediately from the definitions above that a Pareto Efficient allocation is one where all Pareto improvements have been exhausted. Pareto efficiency may be thought of as a minimum requirement for a "good" allocation of societies resources, one where all the opportunities to get something for nothing have been exploited. Pareto efficiency does not involve value judgements about what goods are produced or who receives them.

1.2 Pareto Efficiency in a Two Person, Two Firm, Two Good, Two Input, Economy.

We shall now define Pareto Efficiency for an economy consisting of two consumers who consume two goods that are produced using two factors of production.

1.2.1 Efficiency in Consumption.

We shall assume that there are two consumers Al (A) and Boris (B) each of whom may consume quantities of two goods Vodka (X) and Caviar (Y), we assume the total quantities of the two good to be given by $\{X, Y\}$ of which $\{X^A, Y^A\}$ and $\{X^B, Y^B\}$ are enjoyed by Al and Boris respectively. We write the *initial endowments* of the two goods as

$$\begin{aligned} X &= X_0^A + X_0^B \\ Y &= Y_0^A + Y_0^B \end{aligned}$$

The utilities that the two individuals derive from their endowments are described by the utility functions

$$\begin{aligned} U^A &= U^A(X^A, Y^A) \\ U^B &= U^B(X^B, Y^B) \end{aligned}$$

we assume the utility functions to be increasing and concave.

For an allocation of X and Y between the two individuals to be Pareto Efficient it is required that we cannot raise one individuals utility without lowering the utility of another, expressed another way this involves the problem

$$\begin{gathered}\underset{X^A,Y^A}{Max}\,U^A(X^A,Y^A)\\ s.t\;U^B(X^B,Y^B)\geq\overline{U}^B\\ X=X^A+X^B\\ Y=Y^A+Y^B.\end{gathered}$$

Where $\overline{U}^B$ is the level of utility that B must realize. By substitution this problem reduces to

$$\begin{gathered}\underset{X^A,Y^A}{Max}\,U^A(X^A,Y^A)\\ s.t\;U^B(X-X^A,Y-Y^A)\geq\overline{U}^B\end{gathered}$$

Forming the Lagrangian we obtain

$$\underset{X^A,Y^A}{Max}\,U^A(X^A,Y^A)+\lambda\left[\overline{U}^B-U^B(X-X^A,Y-Y^A)\right]$$

where λ is the Lagrange Multiplier associated with the utility constraint. Now maximizing yields

$$\begin{aligned}\frac{\partial U^A}{\partial X^A}-\lambda\frac{\partial U^B}{\partial X^B}\frac{\partial X^B}{\partial X^A}&=0\\ \frac{\partial U^A}{\partial Y^A}-\lambda\frac{\partial U^B}{\partial Y^B}\frac{\partial Y^B}{\partial Y^A}&=0\end{aligned}$$

utilizing $\frac{\partial X^B}{\partial X^A}=\frac{\partial Y^B}{\partial Y^A}=-1$ and rearranging the expressions gives the condition for Pareto Efficiency in consumption

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}}=\frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

The left hand side (LHS) of this expression is the ratio of marginal utilities for the two good for individual A, the RHS is the ratio of marginal utilities for the two good for individual B, alternatively expressed these are the *marginal rates of substitution.* The condition may be reexpressed as

$$MRS_{XY}^A=MRS_{XY}^B \qquad \text{(Pareto Efficiency in Consumption.)}$$

Remark 3 *Recall from intermediate micro that the marginal rate of substitution between two good is equal to the slope of the indifference curves, thus the condition for Pareto Efficiency in consumption tells us that the indifference curves for the two individuals must have equal slope at a pareto efficient allocation.*

The Contract Curve and Utility Transformation Frontier. The condition for Pareto Efficiency in consumption describes an efficient allocation of goods given that individual B is guaranteed some arbitrary level of utility $\overline{U}^B$. We can now think about varying this arbitrary utility level and examining the implications,.two things will occur

- For Pareto Efficient allocations as the utility of B rises (falls) the utility of A must necessarily fall (rise), hence a curve known as the *Utility Transformation Frontier* is traced out. Define the utility transformation frontier as $U^B = U^B(U^A)$.
- Too maintain efficiency in consumption as the utility levels of the two individuals vary it is necessary to reallocate goods between the two, this traces out a curve known as the *Contract Curve*.

1.2.2 Efficiency in Production.

Suppose now that the two goods $\{X, Y\}$ are each produced using two input goods capital and labor $\{K, L\}$ respectively. the inputs are allocated to the production of the outputs according to $\{K^X, L^X\}$ and $\{K^Y, L^Y\}$ where the initial endowments of the factors are given by

$$\begin{aligned} K &= K_0^X + K_0^Y \\ L &= L_0^X + L_0^Y \end{aligned}$$

The outputs of the two goods that may be derived from the inputs are given by the production technologies

$$\begin{aligned} X &= X(K^X, L^X) \\ Y &= Y(K^Y, L^Y) \end{aligned}$$

these functions are assumed to be increasing and concave.

For the production of the two goods to be Pareto Efficient we require that we cannot reallocate the inputs between the production of the two outputs such that more of one is produced without giving up some of the other. Alternatively expressed

$$\begin{gathered} \underset{K^X, L^X}{Max}\, X(K^X, L^X) \\ s.t\; Y(K^Y, L^Y) \geq \overline{Y} \\ K = K^X + K^Y \\ L = L^X + L^Y. \end{gathered}$$

where $\overline{Y}$ is the level of production of that good which must not be reduced. By substitution and using the method of Lagrange we get

$$\underset{K^X, L^X}{Max}\, X(K^X, L^X) + \mu\left[\overline{Y} - Y(K - K^X, L - L^X)\right]$$

where μ is the Lagrange multiplier associated with the production constraint $Y(K^Y, L^Y) \geq \overline{Y}$. Maximization yields first order conditions

$$\begin{aligned} \frac{\partial X}{\partial K^X} - \mu \frac{\partial Y}{\partial K^Y} \frac{\partial K^Y}{\partial K^X} &= 0 \\ \frac{\partial X}{\partial L^X} - \mu \frac{\partial Y}{\partial L^Y} \frac{\partial L^Y}{\partial L^X} &= 0 \end{aligned}$$

using $\frac{\partial K^Y}{\partial K^X} = \frac{\partial L^Y}{\partial L^X} = -1$ and rearranging these expressions we get the condition for Pareto Efficiency in production

$$\frac{\frac{\partial X}{\partial K^X}}{\frac{\partial X}{\partial L^X}} = \frac{\frac{\partial Y}{\partial K^Y}}{\frac{\partial Y}{\partial L^Y}}$$

The left hand side (LHS) of this expression is the ratio of marginal products for the two inputs in the production of good X, the RHS of this expression is the ratio of marginal products for the two inputs in the production of good Y, alternatively expressed these are the *marginal rates of technical substitution.* The condition may be reexpressed as

$$MRTS^X_{KL} = MRTS^Y_{KL} \qquad \text{(Pareto Efficiency in Production.)}$$

Remark 4 *Recall from intermediate micro that the marginal rate of technical substitution between two inputs is equal to the slope of the isoquant, thus the condition for Pareto Efficiency in production tells us that the isoquants for the two goods must have equal slope at a pareto efficient allocation.*

The Locus of Pareto Efficient Points in Production and the Transformation Frontier. The condition for Pareto Efficiency in consumption describes an efficient allocation inputs between the production of the two outputs given some arbitrary level of production $\overline{Y}$. We can now think about varying this arbitrary production level and examining the implications,.two things will occur

- For Pareto Efficient allocations as the production of Y rises (falls) the production of X must necessarily fall (rise), hence a curve known as the *Transformation Frontier* is traced out. The equation of the transformation frontier is called the *Transformation Function* and may be written

$$F(X, Y) = 0$$

Claim 5 *The slope of the transformation function is defined (positive) as the* Marginal Rate of Transformation *and can be show too be equal to* $\frac{\partial Y(K^Y,L^Y)}{\partial K^Y} / \frac{\partial X(K^X,L^X)}{\partial K^X}$ *or* $\frac{\partial Y(K^Y,L^Y)}{\partial L^Y} / \frac{\partial X(K^X,L^X)}{\partial L^X}$.

- Too maintain efficiency in production as the production of the two goods vary it is necessary to reallocate inputs between the two, this traces out a curve known as the *Locus of Pareto Efficient Points in Production.*

Properties of the Transformation Function.

Consider the production functions

$$\begin{aligned} X &= X(K^X, L^X) \\ Y &= Y(K^Y, L^Y) \end{aligned}$$

totally differentiating these functions gives

$$\begin{aligned} dX &= \frac{\partial X}{\partial K^X} dK^X + \frac{\partial X}{\partial L^X} dL^X \\ dY &= \frac{\partial Y}{\partial K^Y} dK^Y + \frac{\partial Y}{\partial L^Y} dL^Y \end{aligned}$$

Now $dK^X = -dK^Y$ and $dL^X = -dL^Y$ so we get

$$\begin{aligned} dX &= -\frac{\partial X}{\partial K^X} dK^Y - \frac{\partial X}{\partial L^X} dL^Y \\ dY &= \frac{\partial Y}{\partial K^Y} dK^Y + \frac{\partial Y}{\partial L^Y} dL^Y \end{aligned}$$

now from the first equation and dropping the superscripts on the factors

$$dL = -\frac{dX + \frac{\partial X}{\partial K} dK}{\frac{\partial X}{\partial L}}$$

substituting this into the second equation gives

$$\begin{aligned} dY &= \frac{\partial Y}{\partial K} dK - \frac{\partial Y}{\partial L} \left[\frac{dX + \frac{\partial X}{\partial K} dK}{\frac{\partial X}{\partial L}} \right] \\ &= \left[\frac{\partial Y}{\partial K} - \frac{\frac{\partial Y}{\partial L} \frac{\partial X}{\partial K}}{\frac{\partial X}{\partial L}} \right] dK - \frac{\frac{\partial Y}{\partial L}}{\frac{\partial X}{\partial L}} dX \\ &= \left[\frac{\frac{\partial Y}{\partial K} \frac{\partial X}{\partial L} - \frac{\partial Y}{\partial L} \frac{\partial X}{\partial K}}{\frac{\partial X}{\partial L}} \right] dK - \frac{\frac{\partial Y}{\partial L}}{\frac{\partial X}{\partial L}} dX \end{aligned}$$

now the transformation function is by definition Pareto efficient so the MRTS condition implies $\frac{\frac{\partial X}{\partial K}}{\frac{\partial X}{\partial L}} = \frac{\frac{\partial Y}{\partial K}}{\frac{\partial Y}{\partial L}}$ but it is easy to see that this implies the term in square brackets in the above equation is zero, hence

$$-\frac{dY}{dX} = \frac{\frac{\partial Y}{\partial L}}{\frac{\partial X}{\partial L}}$$

and from the MRTS condition we also have

$$-\frac{dY}{dX} = \frac{\frac{\partial Y}{\partial L}}{\frac{\partial X}{\partial L}} = \frac{\frac{\partial Y}{\partial K}}{\frac{\partial X}{\partial K}}$$

this is the slope of the transformation function.

1.2.3 Efficiency in Production and Consumption.

Our first efficiency condition tell us how to efficiently allocate goods *once they are produced*, the second tells us how to efficiently produce *given* combinations of goods. We now need a condition that characterizes when the combination of goods produced is efficient vis-a-vis the combination of goods consumers wish to consume. For the combination of X and Y produced to be Pareto Efficient it is required that we cannot raise the individuals utilities by changing the output mix, expressed another way this involves the problem

$$\begin{gathered}\underset{X^A,Y^A,X^A,Y^A,X,Y}{Max} U^A(X^A, Y^A)\\ s.t\ U^B(X^B, Y^B) \geq \overline{U}^B\\ X = X^A + X^B\\ Y = Y^A + Y^B\\ F(X, Y) = 0\end{gathered}$$

forming the Lagrangian gives us

$$\begin{gathered}\underset{X^A,Y^A,X^A,Y^A,X,Y}{Max} U^A(X^A, Y^A) + \lambda\left[\overline{U}^B - U^B(X^B, Y^B)\right]\\ +\mu_1\left[X - X^A - X^B\right] + \mu_2\left[Y - Y^A - Y^B\right]\\ +\mu_3 F(X, Y)\end{gathered}$$

the first order conditions are

$$\begin{aligned}\frac{\partial U^A}{\partial X^A} - \mu_1 &= 0,\ \frac{\partial U^A}{\partial Y^A} - \mu_2 = 0,\\ -\lambda\frac{\partial U^B}{\partial X^B} - \mu_1 &= 0,\ -\lambda\frac{\partial U^B}{\partial Y^B} - \mu_2 = 0,\\ \mu_1 + \mu_3\frac{\partial F(X,Y)}{\partial X} &= 0,\ \mu_2 + \mu_3\frac{\partial F(X,Y)}{\partial Y} = 0\end{aligned}$$

simple algebra now reveals

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}} = \frac{\frac{\partial F(X,Y)}{\partial X}}{\frac{\partial F(X,Y)}{\partial Y}}$$

which tells us that the marginal rates of substitution must equal the slope of the transformation frontier or

$$MRS^A_{XY} = MRS^B_{XY} = MRT_{XY}$$

(Pareto Efficiency in Production and Consumption.)

1.3 Efficiency and Social Welfare.

If we assume that social welfare is derived from the utilities of the individuals in the economy, then we may write a social welfare function as

$$W = W(U^A, U^B)$$

social indifference curves are then simply defined by

$$W(U^A, U^B) = \text{Constant.}$$

totally differentiating we find that

$$\frac{\partial W}{\partial U^A} dU^A + \frac{\partial W}{\partial U^B} dU^B = 0$$

Hence the societal indifference curves have slope

$$\frac{dU^B}{dU^A} = -\frac{\frac{\partial W}{\partial U^A}}{\frac{\partial W}{\partial U^B}} < 0$$

To maximize social welfare we need to solve the problem

$$\begin{gathered} \underset{U^A}{Max}\, W(U^A, U^B) \\ s.t.\; U^B = U^B(U^A) \end{gathered}$$

or

$$\underset{U^A}{Max}\, W(U^A, U^B(U^A))$$

This tells us that a social welfare optimum must occur where

$$\frac{\partial W}{\partial U^A} + \frac{\partial W}{\partial U^B} \frac{\partial U^B(U^A)}{\partial U^A} = 0$$

rearranging this yields

$$-\frac{\frac{\partial W}{\partial U^A}}{\frac{\partial W}{\partial U^B}} = \frac{\partial U^B(U^A)}{\partial U^A}.$$

The social optimum thus occurs where the slope of the societal indifference curve equals the slope of the utility transformation frontier.

Remark 6 *It follows immediately that a social welfare optimum must be Pareto Efficient.*

1.4 The Fundamental Theorem of Welfare Economics.

Theorem 7 *A perfectly competitive market economy achieves a Pareto efficient allocation.*

This powerful theorem tells us that a competitive economy does not waste any resources, the minimum requirement we might want satisfied by any system of resource allocation. It does *not* tell us that a competitive equilibrium will be a social welfare optimum. To demonstrate that the theorem is true and understand why we need to show that a market system satisfies the three Pareto conditions

- Efficiency in consumption $MRS^A_{XY} = MRS^B_{XY}$.
- Efficiency in production $MRTS^X_{KL} = MRTS^Y_{KL}$.
- Efficiency in production and consumption $MRS^A_{XY} = MRT_{XY}$.

1.4.1 Perfect Competition and Efficiency in Consumption.

Suppose that the markets for the two goods $\{X, Y\}$ are perfectly competitive, then each consumer is a price taker for each good. We assume that there are given incomes $\{I^A, I^B\}$or equivalently initial endowments such that $I^A = P^X X^A_0 + P^Y Y^A_0$, $I^B = P^X X^B_0 + P^Y Y^B_0$. The consumer's utility maximization problems are

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A)$$
$$s.t\ P^X X^A + P^Y Y^A = I^A = P^X X^A_0 + P^Y Y^A_0$$

and

$$\underset{X^B,Y^B}{Max}\, U^B(X^B, Y^B)$$
$$s.t\ P^X X^B + P^Y Y^B = I^B = P^X X^B_0 + P^Y Y^B_0$$

utility maximization requires

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A) + \theta^A \left[I^A - P^X X^A - P^Y Y^A\right]$$

and

$$\underset{X^B,Y^B}{Max}\, U^B(X^B, Y^B) + \theta^B \left[I^B - P^X X^B - P^Y Y^B\right]$$

with first order conditions for the two problems

$$\begin{aligned}
\frac{\partial U^A}{\partial X^A} - \theta P^X &= 0, \\
\frac{\partial U^A}{\partial Y^A} - \theta P^Y &= 0, \\
\frac{\partial U^B}{\partial X^B} - \theta P^X &= 0, \\
\frac{\partial U^B}{\partial Y^B} - \theta P^Y &= 0
\end{aligned}$$

which reduce to

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}} = \frac{P^X}{P^Y} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

or

$$MRS^A_{XY} = \frac{P^X}{P^Y} = MRS^B_{XY}.$$

This tells us that Pareto efficiency is achieved on a competitive market via the coordinating role played by market prices.

Offer Curves.

From the budget constraints we have

$$\begin{aligned} P^X(X^A - X_0^A) + P^Y(Y^A - Y_0^A) &= 0, \\ P^X\left(X^B - X_0^B\right) + P^Y\left(Y^B - Y_0^B\right) &= 0. \end{aligned}$$

Since prices are non-negative this implies that each consumer must be a supplier of one good and demander of the other i.e. if $X^A - X_0^A > 0$ then $Y^A - Y_0^A < 0$, and if trade is to take place $X^B - X_0^B < 0$ and $Y^B - Y_0^B > 0$.

Now suppose we perform a *normalization* and express both good in terms of units of good X so that $P^X = 1$, we now have

$$\begin{aligned} X^A &= X_0^A - P^Y(Y^A - Y_0^A), \\ X^B &= X_0^B - P^Y(Y^B - Y_0^B) \end{aligned}$$

substituting these two expression into the first order conditions gives

$$P^Y \frac{\partial U^A(X_0^A - P^Y(Y^A - Y_0^A), Y^A)}{\partial X^A} = \frac{\partial U^A(X_0^A - P^Y(Y^A - Y_0^A), Y^A)}{\partial Y^A}$$

and

$$P^Y \frac{\partial U^B(X_0^B - P^Y(Y^B - Y_0^B), Y^B)}{\partial X^B} = \frac{\partial U^B(X_0^B - P^Y(Y^B - Y_0^B), Y^B)}{\partial Y^B}$$

which implicitly defines relationships between P^Y and Y^A for given $\left\{Y_0^A, X_0^A\right\}$ and P^Y and Y^B for given $\left\{Y_0^B, X_0^B\right\}$which may be written

$$\begin{aligned} Y^A &= Y^A(P^Y \mid Y_0^A, X_0^A) \\ Y^B &= Y^B(P^Y \mid Y_0^B, X_0^B) \end{aligned}$$

the budget constraints now immediately imply that we may write

$$\begin{aligned} X^A &= X^A(P^Y \mid Y_0^A, X_0^A) \\ X^B &= X^B(P^Y \mid Y_0^B, X_0^B) \end{aligned}$$

Thus individual A offers (to buy or sell)

$$Y^A(P^Y \mid Y_0^A, X_0^A) - Y_0^A$$

and in return asks for

$$X^A(P^Y \mid Y_0^A, X_0^A) - X_0^A$$

thus the offer curve of A is defined by

$$X^A(P^Y \mid Y_0^A, X_0^A) - X_0^A = -P^Y \left[Y^A(P^Y \mid Y_0^A, X_0^A) - Y_0^A\right] \qquad (A\text{'s Offer Curve.})$$

Similarly individual B offers (to buy or sell)

$$Y^B(P^Y \mid Y_0^B, X_0^B) - Y_0^B$$

and in return asks for

$$X^B(P^Y \mid Y_0^B, X_0^B) - X_0^B$$

thus B's offer curve is defined by

$$X^B(P^Y \mid Y_0^B, X_0^B) - X_0^B = -P^Y \left[Y^B(P^Y \mid Y_0^B, X_0^B) - Y_0^B\right] \qquad (B\text{'s Offer Curve.})$$

Market Equilibrium.

Ensures supply equals demand on both markets or

$$\begin{aligned} Y^A(P^Y \quad &\mid \quad Y_0^A, X_0^A) - Y_0^A + Y^B(P^Y \mid Y_0^B, X_0^B) - Y_0^B = 0, \\ X^A(P^Y \quad &\mid \quad Y_0^A, X_0^A) - X_0^A + X^B(P^Y \mid Y_0^B, X_0^B) - X_0^B = 0 \end{aligned}$$

This is simply the condition that the market equilibrium occurs where the offer curves cross. But since each offer curve is only a rewriting of the consumers first order conditions that involves $MRS_{XY} = \frac{P^X}{P^Y}$, then the point at which the offer curves cross must involve $MRS_{XY}^A = \frac{P^X}{P^Y} = MRS_{XY}^B$ and must be Pareto Efficient.

1.4.2 Perfect Competition and Efficiency in Production.

We now examine the behavior of two firms each of which produces one of the two output goods, X and Y, using the two input goods, K and L. Each input is assumed to trade in a competitive input market at the prices P^K and P^L. The problem faced by each firm is to minimize the cost of producing a given level of output.

$$\underset{K^X,L^X}{Min}\ P^K K^X + P^L L^X$$
$$s.t.\ X(K^X, L^X) = \overline{X}$$

$$\underset{K^Y,L^Y}{Min}\ P^K K^Y + P^L L^Y$$
$$s.t.\ Y(K^Y, L^Y) = \overline{Y}$$

forming the two Lagrangians gives us

$$\underset{K^X,L^X}{Min}\ P^K K^X + P^L L^X + \gamma^X \left[\overline{X} - X(K^X, L^X)\right]$$

and

$$\underset{K^Y,L^Y}{Min}\ P^K K^Y + P^L L^Y + \gamma^Y \left[\overline{Y} - Y(K^Y, L^Y)\right]$$

the four first order conditions are

$$\begin{aligned}
P^K - \gamma^X \frac{\partial X(K^X, L^X)}{\partial K^X} &= 0, \\
P^L - \gamma^X \frac{\partial X(K^X, L^X)}{\partial L^X} &= 0, \\
P^K - \gamma^Y \frac{\partial Y(K^Y, L^Y)}{\partial K^Y} &= 0, \\
P^L - \gamma^Y \frac{\partial Y(K^Y, L^Y)}{\partial L^Y} &= 0
\end{aligned}$$

dividing the first condition by the second and the third by the fourth yields

$$\frac{\frac{\partial X(K^X,L^X)}{\partial K^X}}{\frac{\partial X(K^X,L^X)}{\partial L^X}} = \frac{P^K}{P^L} = \frac{\frac{\partial Y(K^Y,L^Y)}{\partial K^Y}}{\frac{\partial Y(K^Y,L^Y)}{\partial L^Y}}$$

or

$$MRTS^X_{KL} = \frac{P^K}{P^L} = MRTS^Y_{KL}$$

Hence competitive cost minimizing firms achieve a Pareto Efficient allocation in production.

1.4.3 Perfect Competition and Efficiency in Production and Consumption.

Consider now that each firm may produce both goods if it wishes and sell each at the competitive market prices $P^X = 1$ and P^Y. Since each firm is problem is identical we may represent the problem with the analysis of a single firm

$$\begin{gathered} Max\ X + P^Y Y - P^K K - P^L L \\ s.t.\ X = X(K^X, L^X) \\ Y = Y(K^Y, L^Y) \\ K = K^X + K^Y \\ L = L^X + L^Y \end{gathered}$$

by substitution this reduces to

$$\underset{K^X, K^Y, L^X, L^Y}{Max} X(K^X, L^X) + P^Y Y(K^Y, L^Y) - P^K \left(K^X + K^Y\right) - P^L \left(L^X + L^Y\right)$$

the first order conditions to this problem are

$$\begin{aligned} \frac{\partial X(K^X, L^X)}{\partial K^X} - P^K &= 0, \\ \frac{\partial X(K^X, L^X)}{\partial L^X} - P^L &= 0, \\ P^Y \frac{\partial Y(K^Y, L^Y)}{\partial K^Y} - P^K &= 0, \\ P^Y \frac{\partial Y(K^Y, L^Y)}{\partial L^Y} - P^L &= 0 \end{aligned}$$

rearranging and dividing the third condition by the first and the fourth by the second gives

$$\begin{aligned} \frac{P^Y \frac{\partial Y(K^Y, L^Y)}{\partial K^Y}}{\frac{\partial X(K^X, L^X)}{\partial K^X}} &= 1 \\ \frac{P^Y \frac{\partial Y(K^Y, L^Y)}{\partial L^Y}}{\frac{\partial X(K^X, L^X)}{\partial L^X}} &= 1 \end{aligned}$$

or

$$\frac{\frac{\partial Y(K^Y, L^Y)}{\partial K^Y}}{\frac{\partial X(K^X, L^X)}{\partial K^X}} = \frac{\frac{\partial Y(K^Y, L^Y)}{\partial L^Y}}{\frac{\partial X(K^X, L^X)}{\partial L^X}} = \frac{1}{P^Y}$$

alternatively expressed

$$MRT_{XY} = \frac{1}{P^Y}$$

but we already know

$$MRS^A_{XY} = MRS^B_{XY} = \frac{1}{P^Y}$$

so

$$MRS^A_{XY} = MRS^B_{XY} = MRT_{XY}$$

the firms produce the pareto efficient mix of outputs.

1.5 Failures of the Fundamental Theorem.

In this section we examine the circumstances where the Fundamental Theorem will not hold. These failures of the Theorem may be associated with violations of the underlying requirements for a perfectly competitive equilibrium to exist.

1.5.1 Imperfect Competition.

Perhaps a the most obvious reason for the Fundamental theorem to fail is if there is not perfect competition. The logic of the proof of the theorem requires each agent trading on each market to take the common market prices as given. Thus each equates their private marginal valuation to the price paid for the good, since all price ratios are common so too must be all the ratios of marginal valuations, the requirement for Pareto Efficiency.

To demonstrate the effects of market power consider the two consumer two good model examined earlier. As before the two goods are $\{X, Y\}$ consumer A is a price taker in both markets, but consumer B consumes enough of good X to have market power, hence we write

$$P^X = P^X(X^B)$$

. We assume that there are given incomes $\left\{I^A, I^B\right\}$. The consumer's utility maximization problems are

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A)$$
$$s.t\ P^X(X^B)X^A + P^Y Y^A = I^A$$

and

$$\underset{X^B,Y^B}{Max}\, U^B(X^B, Y^B)$$
$$s.t\ P^X(X^B)X^B + P^Y Y^B = I^B$$

utility maximization requires

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A) + \theta^A \left[I^A - P^X(X^B)X^A - P^Y Y^A\right]$$

and

$$\underset{X^B,Y^B}{Max}\, U^B(X^B, Y^B) + \theta^B \left[I^B - P^X(X^B)X^B - P^Y Y^B\right]$$

with first order conditions for the two problems

$$\begin{aligned}
\frac{\partial U^A}{\partial X^A} - \theta^A P^X(X^B) &= 0, \\
\frac{\partial U^A}{\partial Y^A} - \theta^A P^Y &= 0, \\
\frac{\partial U^B}{\partial X^B} - \theta^B P^X(X^B) - \theta^B X^B \frac{\partial P^B}{\partial X^B} &= 0, \\
\frac{\partial U^B}{\partial Y^B} - \theta^B P^Y &= 0
\end{aligned}$$

which reduce to

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}} = \frac{P^X}{P^Y} = MRS^A_{XY}$$

but

$$\frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}} = \frac{P^X + X^B \frac{\partial P^B}{\partial X^B}}{P^Y} = MRS^B_{XY}.$$

It follows immediately that

$$MRS^A_{XY} \neq MRS^B_{XY}.$$

and the allocation cannot be Pareto efficient.

This tells us that Pareto efficiency is achieved on a competitive market via the coordinating role played by market prices.

1.5.2 Public Goods.

A public good is a good where each agent that enjoys the good may enjoy the services that flow from the *total* allocation of the good, not just their own allocation. If we again exploit our two consumer two good model and modify it such that good $X = X^A + X^B$ is now a public good we get The utilities that the two individuals derive from their endowments are described by the utility functions

$$\begin{aligned}
U^A &= U^A(X, Y^A) \\
U^B &= U^B(X, Y^B)
\end{aligned}$$

Pareto Efficiency with Public Goods.

In the presence of a public good Pareto efficiency requires that one individual's utility cannot be raised without lowering the utility of another, expressed another way this involves the problem

$$\underset{X,Y^A}{Max} U^A(X, Y^A)$$

$$s.t\; U^B(X, Y^B) \geq \overline{U}^B$$

$$F(X, Y^A + Y^B) = 0.$$

Where $\overline{U}^B$ is the level of utility that B must realize. Forming the Lagrangian we obtain

$$\underset{X,Y^A,Y^B}{Max}\, U^A(X,Y^A) + \lambda\left[\overline{U}^B - U^B(X,Y^B)\right] + \mu F(X, Y^A + Y^B)$$

where λ is the Lagrange Multiplier associated with the utility constraint and μ is the multiplier associated with the transformation function . Now maximizing yields

$$\begin{aligned}
\frac{\partial U^A}{\partial X} - \lambda\frac{\partial U^B}{\partial X} + \mu\frac{\partial F(X,Y^A+Y^B)}{\partial X} &= 0 \\
\frac{\partial U^A}{\partial Y^A} + \mu\frac{\partial F(X,Y^A+Y^B)}{\partial Y^A} &= 0 \\
-\lambda\frac{\partial U^B}{\partial Y^B} + \mu\frac{\partial F(X,Y^A+Y^B)}{\partial Y^B} &= 0
\end{aligned}$$

utilizing $\frac{\partial F(X,Y^A+Y^B)}{\partial Y^A} = \frac{\partial F(X,Y^A+Y^B)}{\partial Y^B} = \frac{\partial F(X,Y^A+Y^B)}{\partial Y}$ we have from the second and third equations

$$\frac{\partial U^A}{\partial Y^A} = -\lambda\frac{\partial U^B}{\partial Y^B} = -\mu\frac{\partial F(X,Y^A+Y^B)}{\partial Y}$$

using these terms in the first equation we may write

$$\frac{\frac{\partial U^A}{\partial X}}{\frac{\partial U^A}{\partial Y^A}} + \frac{\lambda\frac{\partial U^B}{\partial X}}{\lambda\frac{\partial U^B}{\partial Y^B}} - \frac{\mu\frac{\partial F(X,Y^A+Y^B)}{\partial X}}{\mu\frac{\partial F(X,Y^A+Y^B)}{\partial Y}} = 0$$

$$\frac{\frac{\partial U^A}{\partial X}}{\frac{\partial U^A}{\partial Y^A}} + \frac{\frac{\partial U^B}{\partial X}}{\frac{\partial U^B}{\partial Y^B}} = \sum_{A,B} MRS_{XY} = \frac{\frac{\partial F(X,Y^A+Y^B)}{\partial X}}{\frac{\partial F(X,Y^A+Y^B)}{\partial Y}} = MRT_{XY}$$

(Pareto efficiency when X is a public good.)

Hence in the presence of an public good efficiency requires that the marginal rate of transformation equal the *sum* of the marginal rates of substitution.

Failure of the Fundamental Theorem with Public Goods.

To examine if the fundamental theorem continues to hold we must now ask; does the market allocation provide $\sum_{A,B} MRS_{XY} = MRT_{XY}$ as required for efficiency?

Consumers. The consumers problem is exactly as described before except that each may now enjoy all of the public good purchased not just their own acquisitions

$$\underset{X^A,Y^A}{Max}\, U^A(X^A + X^B, Y^A)$$
$$s.t\ X^A + P^Y Y^A = I^A$$

and

$$\underset{X^B,Y^B}{Max}\, U^B(X^A + X^B, Y^B)$$
$$s.t\ X^B + P^Y Y^B = I^B$$

utility maximization requires

$$\underset{X^A,Y^A}{Max}\, U^A(X^A + X^B, Y^A) + \theta^A \left[I^A - X^A - P^Y Y^A\right]$$

and

$$\underset{X^B,Y^B}{Max}\, U^B(X^A + X^B, Y^B) + \theta^B \left[I^B - X^B - P^Y Y^B\right]$$

with first order conditions for the two problems

$$\begin{aligned}
\frac{\partial U^A}{\partial X^A} - \theta &= 0, \\
\frac{\partial U^A}{\partial Y^A} - \theta P^Y &= 0, \\
\frac{\partial U^B}{\partial X^B} - \theta &= 0, \\
\frac{\partial U^B}{\partial Y^B} - \theta P^Y &= 0
\end{aligned}$$

which reduce to

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}} = \frac{1}{P^Y} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

or

$$MRS^A_{XY} = \frac{1}{P^Y} = MRS^B_{XY}.$$

Producers As before let each firm produce both goods if it wishes and sell each at the competitive market prices $P^X = 1$ and P^Y. Since each firm's problem is identical we may represent the problem with the analysis of a single firm

$$Max\ X + P^Y Y - P^K K - P^L L$$
$$s.t.\ X = X(K^X, L^X)$$
$$Y = Y(K^Y, L^Y)$$
$$K = K^X + K^Y$$
$$L = L^X + L^Y$$

by substitution this reduces to

$$\underset{K^X,K^Y,L^X,L^Y}{Max}\ X(K^X, L^X) + P^Y Y(K^Y, L^Y) - P^K \left(K^X + K^Y\right) - P^L \left(L^X + L^Y\right)$$

the first order conditions to this problem are

$$\begin{aligned}
\frac{\partial X(K^X, L^X)}{\partial K^X} - P^K &= 0, \\
\frac{\partial X(K^X, L^X)}{\partial L^X} - P^L &= 0, \\
P^Y \frac{\partial Y(K^Y, L^Y)}{\partial K^Y} - P^K &= 0, \\
P^Y \frac{\partial Y(K^Y, L^Y)}{\partial L^Y} - P^L &= 0
\end{aligned}$$

rearranging and dividing the third condition by the first and the fourth by the second gives

$$\begin{aligned}
\frac{P^Y \frac{\partial Y(K^Y, L^Y)}{\partial K^Y}}{\frac{\partial X(K^X, L^X)}{\partial K^X}} &= 1 \\
\frac{P^Y \frac{\partial Y(K^Y, L^Y)}{\partial L^Y}}{\frac{\partial X(K^X, L^X)}{\partial L^X}} &= 1
\end{aligned}$$

or

$$\frac{\frac{\partial Y(K^Y, L^Y)}{\partial K^Y}}{\frac{\partial X(K^X, L^X)}{\partial K^X}} = \frac{\frac{\partial Y(K^Y, L^Y)}{\partial L^Y}}{\frac{\partial X(K^X, L^X)}{\partial L^X}} = \frac{1}{P^Y}$$

alternatively expressed

$$MRT_{XY} = \frac{1}{P^Y}$$

but we already know

$$MRS^A_{XY} = MRS^B_{XY} = \frac{1}{P^Y}$$

so

$$MRS^A_{XY} = MRS^B_{XY} = MRT_{XY} \neq \sum_{A,B} MRS_{XY}$$

the firms mix of outputs do *not* satisfy the efficiency condition for the level of provision of public goods..

1.5.3 Externalities.

Externalities are situations where one agents use of a good effects the utility or production of another in a way not captured on a market.

Pareto Efficiency with Externalities.

Suppose that your neighbors consumption of a good makes you jealous according to the function $J(X^B)$ this is an externality and is incorporated into the calculation of Pareto Efficiency as follows

The utility functions are now

$$\begin{aligned} U^A &= U^A(X^A, Y^A, J(X^B)) \\ U^B &= U^B(X^B, Y^B) \end{aligned}$$

the appearance of X^B in A's utility function where it has a negative effect represents jealousy .

Deriving the condition for Pareto Efficiency in the usual way involves the problem

$$\begin{gathered} \underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A, J(X^B)) \\ s.t\; U^B(X^B, Y^B) \geq \overline{U}^B \\ X = X^A + X^B \\ Y = Y^A + Y^B. \end{gathered}$$

Where $\overline{U}^B$ is the level of utility that B must realize. By substitution this problem reduces to

$$\begin{gathered} \underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A, J(X - X^A)) \\ s.t\; U^B(X - X^A, Y - Y^A) \geq \overline{U}^B \end{gathered}$$

Forming the Lagrangian we obtain

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A, J(X - X^A)) + \lambda \left[\overline{U}^B - U^B(X - X^A, Y - Y^A)\right]$$

where λ is the Lagrange Multiplier associated with the utility constraint. Now maximizing yields

$$\begin{aligned} \frac{\partial U^A}{\partial X^A} + \frac{\partial U^A}{\partial J}\frac{\partial J}{\partial X^B}\frac{\partial X^B}{\partial X^A} - \lambda \frac{\partial U^B}{\partial X^B}\frac{\partial X^B}{\partial X^A} &= 0 \\ \frac{\partial U^A}{\partial Y^A} - \lambda \frac{\partial U^B}{\partial Y^B}\frac{\partial Y^B}{\partial Y^A} &= 0 \end{aligned}$$

utilizing $\frac{\partial X^B}{\partial X^A} = \frac{\partial Y^B}{\partial Y^A} = -1$ and rearranging the expressions gives the condition for Pareto Efficiency in consumption

$$\frac{\frac{\partial U^A}{\partial X^A} - \frac{\partial U^A}{\partial J}\frac{\partial J}{\partial X^B}}{\frac{\partial U^A}{\partial Y^A}} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

The left hand side (LHS) of this expression is the ratio of marginal utilities for the two good for individual A adjusted for the effects of jealousy, the RHS is the ratio of marginal utilities for the two good for individual B.

Failure of The Fundamental Theorem with Externalities in Consumption.

From our earlier analysis we know that the market allocation will be achieved where

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}} = \frac{1}{P^Y} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

but Pareto Efficiency requires

$$\frac{\frac{\partial U^A}{\partial X^A} - \frac{\partial U^A}{\partial J}\frac{\partial J}{\partial X^B}}{\frac{\partial U^A}{\partial Y^A}} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

hence the market allocation is not Pareto Efficient.

1.5.4 Incomplete Markets.

The problem of incomplete markets is easily understood by recalling that prices coordinate agents on the correct behavior, if no market exists for one good there are no prices to coordinate behavior, indeed each agent is stuck with their initial endowment of that good. Formally the problem of incomplete markets is similar to the problem of externalities. In our previous example jealousy was a good for which there was no market, it was not priced and we saw that the market allocation was inefficient.

1.5.5 Information Based Market Failures.

Asymmetric information is behind many market failures. Suppose that one good that is being traded is health insurance, people know if they are currently sick or not, the insurance companies do not. Sick people buy insurance, healthy people may not, but the insurance companies will figure this out and raise premiums. The price will be too high and the allocation therefore inefficient. Really there need to be two markets, insurance for the currently sick and insurance for the currently healthy but because of asymmetric information only one joint market may exist and this is the heart of the problem.

1.6 Exercises.

Exercise 8 *Suppose that two consumers have the following preferences*

$$\begin{aligned} U^A &= 2(X^A Y^A)^{\frac{1}{2}} \\ U^B &= 2(X^B)^{\frac{1}{2}} + 2(Y^B)^{\frac{1}{2}} \end{aligned}$$

compute the condition for Pareto Efficiency in consumption. Suppose now that $X = X^A + X^B = 9$, $Y = Y^A + Y^B = 8$, $Y^B = 4$ *calculate the efficient levels of the variables* $\{X^A, X^B, Y^A\}$ *and the value of the MRS at the efficient allocation.*

Exercise 9 *Suppose that two producers have the following production technologies*

$$\begin{aligned} X &= 2K^X - \frac{1}{2}(K^X)^2 + 2L^X - \frac{1}{2}(L^X)^2 \\ Y &= 4K^Y - \frac{1}{2}(K^Y)^2 + 4L^Y - \frac{1}{2}(L^Y)^2 \end{aligned}$$

compute the conditions for Pareto Efficiency in production. Suppose now that $K^X + K^Y = 12$, $L^X + L^Y = 20$, $L^Y = 8$ *calculate the efficient levels of the variables* $\{L^X, K^X, K^Y\}$ *and the value of the MRTS at the efficient allocation.*

Exercise 10 *Let the transformation function be* $F(X, Y) = 2\sqrt{X} + Y - 14 = 0$ *is the solution that was found in exercise 8 an efficient allocation in both consumption and production?*

Chapter 2
PUBLIC GOODS AND PUBLIC CHOICE.

In this section we shall examine

- Pure Public Goods..
- The Efficient Provision of a Public Good.
  - Efficiency Described using Supply and Demand Curves.
  - The Free Rider Problem.
- The Under Provision of Public Goods.
- Impure Public Goods.
- Public Choice.
- Local Public Goods and Club Goods.

2.1 Pure Public Goods.

Definition 11 *A public good is a good that is non-rival in consumption. It possesses the property that its consumption by one individual at a given time does not prevent its simultaneous consumption by another.*

An example of a pure public good might be sunshine. One individuals enjoyment of a sunny does doesn't prevent its enjoyment by another. This differs from a private good such as a pencil which is rival in consumption if one individual is writing with the pencil all others are necessarily precluded from doing so.

2.2 The Efficient Provision of a Public Good.

As we demonstrated in section ** the efficient level of provision of a public good requires that the marginal rate of transformation between the public good and each private alternative be equal to the sum of the marginal rates of substitution summed across consumers. In our two good, two consumer world this requires

$$\sum_{A,B} MRS_{XY} = MRT_{XY}.$$

Since each unit of a public good is enjoyed by all individuals the marginal value of the last unit is equal to the sum of all individuals valuations. Fro the public good to be supplied efficiently it must be supplied up to the point where the sacrifice all individuals are jointly willing to make to gain it is just equal to the sacrifice that has to be made to produce it.

2.2.1 Efficiency Described using Supply and Demand Curves.

An alternative way of understanding the efficient level of provision of a public good is to construct suitable supply and demand curves.

- For a private good the market demand for that good is the sum of the units that individuals are willing to purchase at each potential price. *Demand is the horizontal summation of individual demand curves.*
- For a pure public good each unit may be consumed by all individuals, so the value of the good is the sum across all consumers of all the valuations they place on each unit. *Demand is the vertical summation of individual demand curves.*
- Efficiency in each case then requires the intersection of the supply curve for the good with the *appropriately constructed* demand curve.

2.2.2 The Free Rider Problem.

As demonstrated in the section ** the market level of provision and the efficient level of provision of a public good differ. The fundamental reason for this is the "Free

Rider " problem. Because of non-rivalry in consumption each individual can at no expense consume the provision of the good supplied by others. Clearly this reduces the incentive for each individual to supply the good for themselves. But, since all individuals face this incentive to free ride on others the good will typically be under supplied.

Example 12 *To illustrate this idea consider the potential construction of a highway which will be enjoyed by our two consumers A and B. The highway construction is to be funded by contributions from the two consumers. There are three options (i) no highway, (ii) a one lane highway, and (iii) a two lane highway. Let the highway be good X and the three provision levels be written*

$$X = \begin{cases} 0 \text{ if no highway is constructed} \\ X_1 \text{ if one lane is constructed} \\ X_2 \text{ if two lanes are constructed} \end{cases}$$

The consumers have initial incomes I^A and I^B which they may spend on contributions to highway construction or on consumption of the private good Y. The cost of constructing a two lane highway is $C(X_2)$, the cost of constructing a one lane highway is simply $C(X_1) = C(X_2)/(2)$. We shall assume that all goods are priced in units of the private good, i.e. $P^Y = 1$. The preferences of each of the consumers are as follows

$$U(X_1, I) > U\left(X_2, I - \frac{C(X_2)}{2}\right) > U\left(X_1, I - \frac{C(X_2)}{4}\right) > U(0, I) > U\left(X_1, I - \frac{C(X_2)}{2}\right).$$

We see that if the two consumers contribute equally a two lane road is better than a one lane one, which is in turn better than no road, i.e.

$$U\left(X_2, I - \frac{C(X_2)}{2}\right) > U\left(X_1, I - \frac{C(X_2)}{4}\right) > U(0, I).$$

In table form the options may be presented as

		Consumer A	
		No Contribution	*Contribute*
Consumer B	*No Contribution*	$U(0, I)$, $U(0, I)$	$U\left(X_1, I - \frac{C(X_2)}{4}\right)$, $U(X_1, I)$
	Contribute	$U(X_1, I)$, $U\left(X_1, I - \frac{C(X_2)}{4}\right)$	$U\left(X_2, I - \frac{C(X_2)}{2}\right)$, $U\left(X_2, I - \frac{C(X_2)}{2}\right)$

Notice that if one individual contributes $\frac{C(X_2)}{2}$ this is sufficient to finance a one lane road and the other individual may then refuse to contribute and obtain $U(X_1, I)$, but faced with this behavior the other individual will refuse to contribute since

$$U(0, I) > U\left(X_1, I - \frac{C(X_2)}{2}\right)$$

the equilibrium involves $X = 0$ while the efficient allocation involves $X = X_2$. The problem is that each individual literally wants to be a free rider on the road, but as a consequence of this no road is built.

2.3 The Under Provision of Public Goods.

To demonstrate that the market will typically under provide public goods we consider a simplified version of our earlier model. suppose that our two individuals A and B are each endowed with a given quantity of one good, *putty* denoted $\{X^A, X^B\}$. This putty may be used to make plain bowls etc. for personal use denoted $\{X_R^A, X_R^B\}$ or may be shaped into pieces of art that are placed on public display denoted $\{X_L^A, X_L^B\}$ naturally

$$\begin{aligned} X^A &= X_R^A + X_L^A \\ X^B &= X_R^B + X_L^B \end{aligned}$$

the art is a public good and we write the individuals utilities a

$$\begin{aligned} U^A &= U^A(X_R^A, X_L^A + X_L^B) \\ U^B &= U^B(X_R^B, X_L^A + X_L^B) \end{aligned}$$

by substitution these reduce to

$$\begin{aligned} U^A &= U^A(X_R^A, X^A - X_R^A + X^B - X_R^B) \\ U^B &= U^B(X_R^B, X^A - X_R^A + X^B - X_R^B) \end{aligned}$$

2.3.1 Market Provision of the Public Good.

Individual or market choices must satisfy

$$\underset{X_R^A}{Max} U^A(X_R^A, X^A - X_R^A + X^B - X_R^B)$$
$$\underset{X_R^B}{Max} U^B(X_R^B, X^A - X_R^A + X^B - X_R^B)$$

the first order conditions to which are

$$\begin{aligned} \frac{\partial U^A}{\partial X_R^A} - \frac{\partial U^A}{\partial X_L^A} &= 0 \\ \frac{\partial U^B}{\partial X_R^B} - \frac{\partial U^B}{\partial X_L^B} &= 0 \end{aligned}$$

simple manipulation of these expressions reveals

$$\frac{\frac{\partial U^A}{\partial X_L^A}}{\frac{\partial U^A}{\partial X_R^A}} = \frac{\frac{\partial U^B}{\partial X_L^B}}{\frac{\partial U^B}{\partial X_R^B}} = 1$$

so the $MRS^A = MRS^B = MRT (= 1)$.

2.3.2 Efficient Provision of the Public Good.

An efficient allocation must make one individual as well off as possible without harming the other or

$$\underset{X_R^A, X_R^B}{Max}\, U^A(X_R^A, X_L^A + X_L^B)$$
$$s.t.\ U^B(X_R^B, X_L^A + X_L^B) \geq \overline{U}^B$$

where if Ψ is the lagrange multiplier associated with B's utility constraint,$\overline{U}^B$, the problem becomes

$$\underset{X_R^A, X_R^B, X_L^A, X_L^B}{Max} U^A(X_R^A, X_L^A + X_L^B) - \Psi\left[\overline{U}^B - U^B(X_R^B, X_L^A + X_L^B)\right]$$
$$-\lambda^A\left[X^A - X_R^A - X_L^A\right] - \lambda^B\left[X^B - X_R^B - X_L^B\right]$$

The first order conditions are

$$\begin{aligned} \frac{\partial U^A}{\partial X_R^A} + \lambda^A &= 0 \\ \Psi\frac{\partial U^B}{\partial X_R^B} + \lambda^B &= 0 \\ \frac{\partial U^A}{\partial X_L} + \Psi\frac{\partial U^B}{\partial X_L} + \lambda^A &= 0 \\ \frac{\partial U^A}{\partial X_L} + \Psi\frac{\partial U^B}{\partial X_L} + \lambda^B &= 0 \end{aligned}$$

so from the second pair of these equations we get

$$\lambda^A = \lambda^B$$

using this and the first pair of equations we get

$$\frac{\partial U^A}{\partial X_R^A} = \Psi\frac{\partial U^B}{\partial X_R^B} = \lambda^A = \lambda^B$$

we may immediately write the third equation as

$$\frac{\frac{\partial U^A}{\partial X_L}}{\frac{\partial U^A}{\partial X_R^A}} + \frac{\frac{\partial U^B}{\partial X_L}}{\frac{\partial U^B}{\partial X_R^B}} = 1$$

so the $\sum MRS = MRT(= 1)$.

2.3.3 The Intuition of Under Provision.

Fro the previous two sections we know

$$\left.\frac{\frac{\partial U^A}{\partial X_L}}{\frac{\partial U^A}{\partial X_R^A}}\right|_e + \left.\frac{\frac{\partial U^B}{\partial X_L}}{\frac{\partial U^B}{\partial X_R^B}}\right|_e = 1 = \left.\frac{\frac{\partial U^A}{\partial X_L^A}}{\frac{\partial U^A}{\partial X_R^A}}\right|_p = \left.\frac{\frac{\partial U^B}{\partial X_L^B}}{\frac{\partial U^B}{\partial X_R^B}}\right|_p$$

where e and p indicate efficient and private provision of the good respectively. From these equalities we can immediately write

$$\left.\frac{\frac{\partial U^A}{\partial X_L}}{\frac{\partial U^A}{\partial X_R^A}}\right|_e < \left.\frac{\frac{\partial U^A}{\partial X_L^A}}{\frac{\partial U^A}{\partial X_R^A}}\right|_p .$$

This tells us that private marginal rate of substitution between the two good is too high, the marginal utility of the public good $\left.\frac{\partial U^A}{\partial X_L^A}\right|_p$ is too high relative to the marginal utility of the private good $\left.\frac{\partial U^A}{\partial X_R^A}\right|_p$. Since the marginal utilities of the goods are decreasing this tells us that for a given level of provision of the public good by individual B then individual A will have an incentive to under provide the good, and similarly B will also have an incentive to under provide the good.

2.4 Impure Public Goods.

Definition 13 *An impure public good is a public good that is has both a public and private component. As individuals consume the good it yields individual private utility and functions as a public good.*

Example 14 *Expenditure on preventing yourself incurring an infectious disease. You benefit privately from not getting the disease and you help provide a public good in terms of eradication of the disease for all.*

In our two consumer world we may now represent an impure public good as a third good which is automatically supplied to all with the provision of one of the private goods. Let X be a purely private good as before, and let Y be privately consumed and lead to the provision of a public good Z according to

$$Z = \alpha Y$$

where α is a constant. The utilities of the two agents are now given by

$$\begin{aligned} U^A &= U^A(X^A, Y^A, Z) \\ U^B &= U^B(X^B, Y^B, Z) \end{aligned}$$

we need first to define Pareto efficiency for the economy with an impure Public good.

2.4.1 Efficient Provision of an Impure Public Good.

As always we find the efficiency conditions by maximizing the utility of one agent subject to the other not being hurt and subject to the resource constraints or

$$\begin{gathered}\underset{X^A,Y^A,X^B,Y^B,Z}{Max} U^A(X^A,Y^A,Z) \\ s.t\; U^B(X^B,Y^B,Z) \geq \overline{U}^B \\ Y^A+Y^B=Y \\ X^A+X^B=X \\ Z=\alpha(Y) \\ F(X,Y)=0.\end{gathered}$$

By substitution we can reduce this problem to

$$\begin{gathered}\underset{X^A,Y^A,X,Y}{Max} U^A(X^A,Y^A,\alpha Y) \\ s.t\; U^B(X-X^A,Y-Y^A,\alpha Y) \geq \overline{U}^B \\ F(X,Y)=0.\end{gathered}$$

Where $\overline{U}^B$ is the level of utility that B must realize. Forming the Lagrangian we obtain

$$\underset{X^A,Y^A,X,Y}{Max} U^A(X^A,Y^A,\alpha Y)+\lambda\left[\overline{U}^B-U^B(X-X^A,Y-Y^A,\alpha Y)\right]+\mu F(X,Y)$$

where λ is the Lagrange Multiplier associated with the utility constraint and μ is the multiplier associated with the transformation function . Now maximizing yields

$$\begin{gathered}\frac{\partial U^A}{\partial X^A}+\lambda\frac{\partial U^B}{\partial X^B}\frac{\partial X^B}{\partial X^A}=0 \\ \frac{\partial U^A}{\partial Y^A}+\lambda\frac{\partial U^B}{\partial Y^B}\frac{\partial Y^B}{\partial Y^A}=0 \\ -\lambda\frac{\partial U^B}{\partial X^B}\frac{\partial X^B}{\partial X}+\mu\frac{\partial F}{\partial X}=0 \\ \frac{\partial U^A}{\partial Z}\alpha-\lambda\alpha\frac{\partial U^B}{\partial Z}-\lambda\frac{\partial U^B}{\partial Y^B}\frac{\partial Y^B}{\partial Y}+\mu\frac{\partial F}{\partial Y}=0\end{gathered}$$

Now $\frac{\partial X^B}{\partial X^A} = \frac{\partial Y^B}{\partial Y^A} = -1$ and $\frac{\partial Y^B}{\partial Y} = \frac{\partial X^B}{\partial X} = 1$ so

$$\frac{\partial U^A}{\partial X^A} - \lambda \frac{\partial U^B}{\partial X^B} = 0$$
$$\frac{\partial U^A}{\partial Y^A} - \lambda \frac{\partial U^B}{\partial Y^B} = 0$$
$$-\lambda \frac{\partial U^B}{\partial X^B} + \mu \frac{\partial F}{\partial X} = 0$$
$$\frac{\partial U^A}{\partial Z}\alpha - \lambda\alpha \frac{\partial U^B}{\partial Z} - \lambda \frac{\partial U^B}{\partial Y^B} + \mu \frac{\partial F}{\partial Y} = 0$$

So the efficiency conditions become (note these are in terms of Y for X rather than X for Y)

$$MRS^A_{YX} = \frac{\frac{\partial U^A}{\partial Y^A}}{\frac{\partial U^A}{\partial X^A}} = \frac{\frac{\partial U^B}{\partial Y^B}}{\frac{\partial U^B}{\partial X^B}} = MRS^B_{YX}$$

this tells us that once quantities of the two goods X and Y exist they should be allocated between the two individuals such that marginal rates of substitution are equalized. This follows from noting that who possesses the good Y doesn't effect it's utility as a public good. But

$$MRT_{YX} = \frac{\frac{\partial F}{\partial Y}}{\frac{\partial F}{\partial X}} = \frac{\frac{\partial U^B}{\partial Y^B} - \frac{\partial U^A}{\partial Z}\frac{\alpha}{\lambda} + \alpha\frac{\partial U^B}{\partial Z}}{\frac{\partial U^B}{\partial X^B}} = MRS^B_{YX} + \frac{\alpha\frac{\partial U^B}{\partial Z} - \frac{\partial U^A}{\partial Z}\frac{\alpha}{\lambda}}{\frac{\partial U^B}{\partial X^B}}$$

Note that $\frac{\alpha\frac{\partial U^B}{\partial Z} - \frac{\partial U^A}{\partial Z}\frac{\alpha}{\lambda}}{\frac{\partial U^B}{\partial X^B}} > 0$, now if we evaluate these conditions at the efficient point $MRT_{YX} > MRS^B_{YX}$ in essence this reveals that the MRS^B_{YX} needs to be lower than it would be in the case with private goods hence there must be more Y relative to X.

We call

$$\frac{\frac{\partial U^B}{\partial Y^B} - \frac{\partial U^A}{\partial Z}\frac{\alpha}{\lambda} + \alpha\frac{\partial U^B}{\partial Z}}{\frac{\partial U^B}{\partial X^B}} = SMRS_{YX}$$

the *Social Marginal Rate of Substitution* it is the rate at which good Y should be traded for good X to achieve Pareto Efficiency.

In our example you should look after your health not up to the point where the private cost and benefits are equal, but you should also consume more Y so as to help with the eradication of the disease.

2.4.2 Market Provision of an Impure Public Good.

In the market environment each individual faces the problem (assuming $P^X = 1$)

$$\underset{X^A, Y^A}{Max}\, U^A(X^A, Y^A, Z)$$
$$s.t\ X^A + P^Y Y^A = I^A$$
$$Z = \alpha(Y^A + Y^B)$$

which may be rewritten

$$\underset{Y^A}{Max} U^A(I^A - P^Y Y^A, Y^A, \alpha(Y^A + Y^B))$$

the first order condition is thus

$$-P^Y \frac{\partial U^A}{\partial X^A} + \frac{\partial U^A}{\partial Y^A} + \alpha \frac{\partial U^A}{\partial Z} = 0$$

For individual B we get

$$-P^Y \frac{\partial U^B}{\partial X^B} + \frac{\partial U^B}{\partial Y^B} + \alpha \frac{\partial U^B}{\partial Z} = 0$$

so we may write

$$\frac{\frac{\partial U^A}{\partial Y^A} + \alpha \frac{\partial U^A}{\partial Z}}{\frac{\partial U^A}{\partial X^A}} = P^Y = \frac{\frac{\partial U^B}{\partial Y^B} + \alpha \frac{\partial U^B}{\partial Z}}{\frac{\partial U^B}{\partial X^B}}$$

Now we have shown earlier that the market provision by a firm will involve

$$MRT_{YX} = \frac{\frac{\partial F}{\partial Y}}{\frac{\partial F}{\partial X}} = P^Y$$

so in the case of private provision we get

$$MRT_{YX} = \frac{\frac{\partial F}{\partial Y}}{\frac{\partial F}{\partial X}} = \frac{\frac{\partial U^B}{\partial Y^B} + \alpha \frac{\partial U^B}{\partial Z}}{\frac{\partial U^B}{\partial X^B}} = MRS^B_{YX} + \frac{\alpha \frac{\partial U^B}{\partial Z}}{\frac{\partial U^B}{\partial X^B}} = MRS^A_{YX} + \frac{\alpha \frac{\partial U^A}{\partial Z}}{\frac{\partial UA}{\partial X^A}}$$

Notice that

$$MRT_{YX} = \frac{\frac{\partial F}{\partial Y}}{\frac{\partial F}{\partial X}} = \frac{\frac{\partial U^B}{\partial Y^B} + \alpha \frac{\partial U^B}{\partial Z}}{\frac{\partial U^B}{\partial X^B}} = MRS^B_{YX} + \frac{\alpha \frac{\partial U^B}{\partial Z}}{\frac{\partial U^B}{\partial X^B}}$$

$$MRS^B_{YX} + \frac{\alpha \frac{\partial U^B}{\partial Z} - \frac{\partial U^A}{\partial Z} \frac{\alpha}{\lambda}}{\frac{\partial U^B}{\partial X^B}} = SMRS_{YX}$$

Now $\frac{\alpha \frac{\partial U^B}{\partial Z} - \frac{\partial U^A}{\partial Z} \frac{\alpha}{\lambda}}{\frac{\partial U^B}{\partial X^B}} > \frac{\alpha \frac{\partial U^B}{\partial Z}}{\frac{\partial U^B}{\partial X^B}}$ so if we evaluate these conditions at the efficient point this reveals that the MRS^B_{YX} needs to be lower than it would be in the case with private provision of the public good hence there must be more Y relative to X. So we conclude that the market solution supplies too little Y and too much X.

2.5 Exclusion.

While the essence of the public good problem is that the goods are non-rival in consumption and thus prone to the free rider problem, this does not mean that you cannot exclude people from consuming them. Private goods are by definition goods where their consumption by one individual excludes another from consuming them, and thus this exclusion is free. For some public goods it is possible to exclude individual consumption. For example the entrance booths on the access roads to a national park can be used to exclude individuals from using the park. But notice that exclusion is costly, the ranger manning the toll booth to a park has to be paid. If you can exclude someone from enjoying a public good then you can charge them a fee for use or access, thus you can make them contribute and avoid the free rider problem.

2.5.1 Public Good Pricing.

Via a Tax on the Private Good.

Suppose that our two individuals may spend incomes I^A, I^B on a public good Y or a private good X which are supplied at the prices $P^X = 1$, P^Y, and let the government have the option of taxing the private good at the rates τ^A, τ^B.

Individual choices must satisfy

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A + Y^B)$$
$$s.t.\ (1+\tau^A)X^A + P^Y Y^A = I^A$$

or

$$\underset{X^A,Y^A}{Max}\, U^A\left[\frac{(I^A - P^Y Y^A)}{(1+\tau^A)}, Y^A + Y^B\right]$$

the first order conditions to which are

$$-\frac{\partial U^A}{\partial X^A}\frac{P^Y}{(1+\tau^A)} + \frac{\partial U^A}{\partial Y^A} = 0$$

For individual B identical methods yield

$$-\frac{\partial U^B}{\partial X^B}\frac{P^Y}{(1+\tau^B)} + \frac{\partial U^B}{\partial Y^B} = 0$$

simple manipulation of these expressions reveals

$$\frac{\frac{\partial U^A}{\partial Y^A}}{\frac{\partial U^A}{\partial X^A}} = \frac{P^Y}{(1+\tau^A)}$$
$$\frac{\frac{\partial U^B}{\partial Y^B}}{\frac{\partial U^B}{\partial X^B}} = \frac{P^Y}{(1+\tau^B)}$$

Now we know that for efficiency we require

$$\frac{\frac{\partial U^A}{\partial Y^A}}{\frac{\partial U^A}{\partial X^A}} + \frac{\frac{\partial U^B}{\partial Y^B}}{\frac{\partial U^B}{\partial X^B}} = P^Y = MRT_{YX}$$

or

$$\frac{1}{(1+\tau^A)} + \frac{1}{(1+\tau^B)} = 1$$

If applied these taxes would induce a Pareto efficient allocation.

Via User Fees.

We know that the public good will be efficiently provided if

$$\begin{aligned} \frac{\frac{\partial U^A}{\partial Y^A}}{\frac{\partial U^A}{\partial X^A}} &= \frac{P^Y}{(1+\tau^A)} \\ \frac{\frac{\partial U^B}{\partial Y^B}}{\frac{\partial U^B}{\partial X^B}} &= \frac{P^Y}{(1+\tau^B)} \end{aligned}$$

with the $\tau' s$ defined as in the previous section. So

$$\begin{aligned} P^{YA} &= \frac{P^Y}{(1+\tau^A)} \\ P^{YB} &= \frac{P^Y}{(1+\tau^B)} \end{aligned}$$

define entry fees for the park such that an efficient allocation may be achieved. Notice that this involves charging individual tax prices.

2.5.2 Costly Exclusion and User Fees.

We know that the taxes/user fees derived above would induce efficient provision/use of a public good, but the collection of these fees is costly and must be taken into account when deciding on how much of a public good is to be supplied. Suppose we think of Y^A as visits to a park such that the user fee has to be collected at every visit, let C be the per fee cost of fee collection.

Efficiency with Costly Exclusion.

Suppose that a policy maker has the necessary tax instruments to manipulate marginal rate of substitution as in the section above, but there is a resource cost of cY

to the exclusion necessary to charge the user fees, here Pareto efficiency involves the problem

$$\underset{X,Y^A}{Max} U^A(X^A, Y(1-c))$$
$$s.t\ U^B(X^B, Y(1-c)) \geq \overline{U}^B$$
$$F(X^A + X^B, Y) = 0.$$

Where $\overline{U}^B$ is the level of utility that B must realize. Forming the Lagrangian we obtain

$$\underset{X^A,X^B,Y}{Max} U^A(X^A, Y(1-c)) + \lambda\left[\overline{U}^B - U^B(X^B, Y(1-c))\right] + \mu F(X^A + X^B, Y)$$

where λ is the Lagrange Multiplier associated with the utility constraint and μ is the multiplier associated with the transformation function . Now maximizing yields

$$(1-c)\frac{\partial U^A}{\partial Y} - \lambda(1-c)\frac{\partial U^B}{\partial Y} + \mu\frac{\partial F}{\partial Y} = 0$$
$$\frac{\partial U^A}{\partial X^A} + \mu\frac{\partial F}{\partial X^A} = 0$$
$$-\lambda\frac{\partial U^B}{\partial X^B} + \mu\frac{\partial F}{\partial X^B} = 0$$

utilizing $\frac{\partial F}{\partial X^A} = \frac{\partial F}{\partial X^B} = \frac{\partial F}{\partial X}$ we have from the second and third equations

$$\frac{\partial U^A}{\partial X^A} = -\lambda\frac{\partial U^B}{\partial X^B} = -\mu\frac{\partial F}{\partial X}$$

using these terms in the first equation we may write

$$(1-c)\frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} + \frac{\lambda(1-c)\frac{\partial U^B}{\partial Y}}{\lambda\frac{\partial U^B}{\partial X}} - \frac{\mu\frac{\partial F}{\partial Y}}{\mu\frac{\partial F}{\partial X}} = 0$$

$$(1-c)\left(\frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} + \frac{\frac{\partial U^B}{\partial Y}}{\frac{\partial U^B}{\partial X^B}}\right) = (1-c)\sum_{A,B} MRS_{YX} = \frac{\frac{\partial F}{\partial Y}}{\frac{\partial F}{\partial X}} = MRT_{YX}$$

(Pareto efficiency when Y is a public good with costly exclusion.)

Hence with costly exclusion efficient provision of public good efficiency requires that the marginal rate of transformation equal the weighted *sum* of the marginal rates of substitution.

2.6 Weak Links, Best Shots, and Other Funny Goods.

2.6.1 Weak-Link Public Goods.

Definition 15 *A Weak Link Public Good. Is a good where the level of provision is that supplied by the smallest contributor. They are thus determined by the minimum contribution.*

Example 16 *Dikes are a classic example. If one individual builds a dike 10ft above flood level, while his neighbor builds one only 5ft above flood level, then both will be flooded if the river level rises by more than 5ft. In essence the last 5ft of the first builders dike is useless.*

Efficiency for Weak-Link Public Goods.

In a weak link public good the utility of the agents is given by

$$U^A(X^A, \min\{Y^A, Y^B\})$$
$$U^B(X^B, \min\{Y^A, Y^B\})$$

Notice immediately that the efficient provision of the good necessarily involves $Y^A = Y^B$ since anything else would involve waste. Hence we get $Y^A = Y^B = \frac{Y}{2}$, so for efficiency we require

$$Max\ U^A(X^A, \frac{Y}{2})$$
$$s.t.\ U^B(X^B, \frac{Y}{2}) \geq \overline{U}^B$$
$$X^A + X^B = X$$
$$F(X^A + X^B, Y) = 0.$$

or

$$\underset{X^A,X,Y}{Max}\ U^A(X^A, \frac{Y}{2})$$
$$s.t.\ U^B(X - X^A, \frac{Y}{2}) \geq \overline{U}^B$$
$$F(X, Y) = 0.$$

with FOC's

$$\frac{\partial U^A}{\partial X^A} + \lambda \frac{\partial U^B}{\partial X^B} = 0$$
$$\frac{1}{2}\frac{\partial U^A}{\partial Y} - \frac{\lambda}{2}\frac{\partial U^B}{\partial Y} + \mu \frac{\partial F}{\partial Y} = 0$$
$$-\lambda \frac{\partial U^B}{\partial X^B} + \mu \frac{\partial F}{\partial X} = 0$$

from the first and third equations we immediately get $\lambda \frac{\partial U^B}{\partial X} = \mu \frac{\partial F}{\partial X} = -\frac{\partial U^A}{\partial X^A}$ so we may use the to rewrite the second equation as

$$-\frac{1}{2}\frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} - \frac{1}{2}\frac{\frac{\partial U^B}{\partial Y}}{\frac{\partial U^B}{\partial X^B}} + \frac{\frac{\partial F}{\partial Y}}{\frac{\partial F}{\partial X}} = 0$$

so

$$\frac{1}{2}\sum MRS_{YX} = MRT_{YX}$$

is the efficiency condition. It reflects that fact that any provision of the good by one agent must be matched by an equal provision by the other for it to have any value. This implies that the efficient provision of a public good with this characteristic would be less than that of a public good identical in all aspects except the weak-link property.

Market Provision of Weak-Link Public Goods.

Notice that no individual has an incentive to provide a weak-link public good in excess of the provision by others. It follows that there are two possibilities for market provision

- If no agent is supplying the good them nobody has any incentive to start, hence zero provision is an equilibrium.
- Each agent supplies the good either up to the point where the private marginal cost equals the private marginal benefit, or, up to the point of provision by the other agent. It follows immediately that there is another equilibrium where the level of provision is that chosen by the agent that values the good the least.

From the two arguments outlined above we see that the level of ,market provision is in sense further from the efficient level than in the standard pure public good model. There seem to be a strong argument for public provision of goods such as flood prevention measures.

2.6.2 Best-Shot Public Goods.

Definition 17 *A Best Shot public good is one where the level of provision is that provided by the highest contribution.*

Example 18 *A society is protected by an anti-missile shield, but only the most accurate anti-missile missile actually provides the protection.*

Example 19 *Academic research that leads to the solution to a medical problem.*

Efficiency for a Best-Shot Public Good.

In a weak link public good the utility of the agents is given by

$$U^A(X^A, \max\{Y^A, Y^B\})$$
$$U^B(X^B, \max\{Y^A, Y^B\})$$

Notice immediately that the efficient provision of the good necessarily involves either $Y^A = 0$ or $Y^B = 0$ since anything else would involve waste. Hence suppose $Y^A = 0$, so for efficiency we require

$$
\begin{gathered}
Max\ U^A(X^A, Y^B) \\
s.t.\ U^B(X^B, Y^B) \geq \overline{U}^B \\
X^A + X^B = X \\
F(X^A + X^B, Y^B) = 0.
\end{gathered}
$$

or

$$
\begin{gathered}
\underset{X^A, X, Y}{Max}\ U^A(X^A, Y^B) \\
s.t.\ U^B(X - X^A, Y^B) \geq \overline{U}^B \\
F(X, Y^B) = 0.
\end{gathered}
$$

with FOC's

$$
\begin{gathered}
\frac{\partial U^A}{\partial X^A} + \lambda \frac{\partial U^B}{\partial X^B} = 0 \\
\frac{\partial U^A}{\partial Y^B} - \lambda \frac{\partial U^B}{\partial Y^B} + \mu \frac{\partial F}{\partial Y^B} = 0 \\
-\lambda \frac{\partial U^B}{\partial X^B} + \mu \frac{\partial F}{\partial X} = 0
\end{gathered}
$$

from the first and third equations we immediately get $\lambda \frac{\partial U^B}{\partial X} = \mu \frac{\partial F}{\partial X} = -\frac{\partial U^A}{\partial X^A}$ so we may use the to rewrite the second equation as

$$
-\frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} - \frac{\frac{\partial U^B}{\partial Y}}{\frac{\partial U^B}{\partial X^B}} + \frac{\frac{\partial F}{\partial Y}}{\frac{\partial F}{\partial X}} = 0
$$

so

$$
\sum MRS_{YX} = MRT_{YX}
$$

is the efficiency condition *as usual.*

Market Provision of Best-Shot Public Goods.

Notice that best shot public goods are extremely prone to the free rider problem. Each agent has an incentive to try too be the contributor that supplies zero units of the good. it is not clear how this good is supplied by the market we may have the problem of the non-existence of an equilibrium. If no agent supplies the good then each has an incentive to supply it up to the point where private marginal costs and benefits are equalized. but if all agents supply the good each may then have an incentive to reduce their supply to zero. If, however, one agent gets "caught" supplying the good while the others do not, then this may be an equilibrium Who supplies the good and at what level is somewhat arbitrary. Again e conclude that public provision may be necessary.

2.7 Local Public Goods.

2.7.1 Local Public Goods.

Definition 20 *Local public goods are public goods whose enjoyment is geographically or otherwise limited.*

Example 21 *A museum in a town or a city park will typically only be enjoyed by those that live within a certain distance from the good.*

The key element in analyzing the provision of local public goods is whether or not the jurisdiction of the public good lies entirely within a single political jurisdiction. If the individuals who enjoy a park all live within the boundaries of the political jurisdiction of the municipality that supplies it, then the public good (whether pure or impure) may be analyzed exactly as in our previous cases. However, if the good is enjoyed by individuals outside the area of political jurisdiction then we have spillovers, and there will be problems with achieving an efficient level of provision of the good. Two counties bordering a polluted lake would each benefit from any clean up measures employed and paid for by the other. Each has an incentive to free ride. The problem is one of governmental free riding.

Suppose that we have two political jurisdictions A, B further suppose the objectives of the policy, makers in the two jurisdictions involve levying total taxes t^A, t^B which allow them to supply a single public good $P = P^A(t^A + t^B)$, the municipalities utilities are given by

$$J^A(P(t^A + t^B), t^A)$$
$$J^B(P(t^A + t^B), t^B)$$

with $\frac{\partial J^A}{\partial P}, \frac{\partial J^B}{\partial P} > 0, \frac{\partial J^A}{\partial t}, \frac{\partial J^B}{\partial t} < 0$ we assume sufficient concavity to ensure the solution for each municipalities optimization problem is interior. The taxes t^A, t^B represent private goods forgone to the taxpayers in each municipality.

Efficiency.

As is usual we find the efficiency condition by

$$\underset{t^A, t^B}{Max}\ J^A(P(t^A + t^B), t^A)$$
$$s.t.\ J^B(P(t^A + t^B), t^B) \geq \overline{J}^B$$

the first order conditions to which are

$$\begin{aligned} \frac{\partial J^A}{\partial P}\frac{\partial P}{\partial t^A} + \frac{\partial J^A}{\partial t^A} - \lambda\frac{\partial J^B}{\partial P}\frac{\partial P}{\partial t^A} &= 0 \\ \frac{\partial J^A}{\partial P}\frac{\partial P}{\partial t^B} - \lambda\frac{\partial J^B}{\partial P}\frac{\partial P}{\partial t^B} - \lambda\frac{\partial J^B}{\partial t^B} &= 0 \end{aligned}$$

note that $\frac{\partial P}{\partial t^B} = \frac{\partial P}{\partial t^A}$ so we immediately have $\frac{\partial J^A}{\partial t^A} = -\lambda\frac{\partial J^B}{\partial t^B}$, so the first condition may be written as

$$\frac{\frac{\partial J^A}{\partial P}\frac{\partial P}{\partial t^A}}{\frac{\partial J^A}{\partial t^A}} + 1 + \frac{\frac{\partial J^B}{\partial P}\frac{\partial P}{\partial t^A}}{\frac{\partial J^B}{\partial t^B}} = 0$$

or

$$\frac{1}{\frac{\partial P}{\partial t^A}} = -\left(\frac{\frac{\partial J^A}{\partial P}}{\frac{\partial J^A}{\partial t^A}} + \frac{\frac{\partial J^B}{\partial P}}{\frac{\partial J^B}{\partial t^B}}\right)$$

the LHS of this expression is the marginal rate of transformation between the public good and the private good. The RHS is the sum of the marginal rates of substitution. Recall that $\frac{\partial J^A}{\partial t^A} < 0$ and represents the utility value of private goods foregone. So we get the familiar condition for efficiency.

Provision by Independent Municipalities.

Each independent municipality maximizes

$$\begin{aligned} &\underset{t^A}{Max}\ J^A(P(t^A + t^B), t^A) \\ &\underset{t^B}{Max}\ J^B(P(t^A + t^B), t^B) \end{aligned}$$

with the pair of first order conditions

$$\begin{aligned} \frac{\partial J^A}{\partial P}\frac{\partial P}{\partial t^A} + \frac{\partial J^A}{\partial t^A} &= 0 \\ \frac{\partial J^B}{\partial P}\frac{\partial P}{\partial t^B} + \frac{\partial J^B}{\partial t^B} &= 0 \end{aligned}$$

we see immediately that this implies that provision satisfies

$$\frac{1}{\frac{\partial P}{\partial t^A}} = -\frac{\frac{\partial J^A}{\partial P}}{\frac{\partial J^A}{\partial t^A}} = -\frac{\frac{\partial J^B}{\partial P}}{\frac{\partial J^B}{\partial t^B}}$$

This is the standard public good problem and we know from our earlier analysis that this good will be underprovided by the market.

2.8 Club Goods.

Definition 22 *A club is a group of individuals who jointly enjoy benefits from their membership. A club may be based on, (1) the sharing of costs, (2) the members characteristics, or, (3) the sharing of a good that has excludable benefits, and of course combinations of (1)-(3).*

Example 23 *A club based on the sharing of costs might be a private swimming club, no single member may be able to afford the cost of minimum provision and maintenance of a pool, but jointly they can do so.*

Example 24 *A club based on members characteristics might be a chess club, here clearly the characteristic is a capacity to play chess.*

Example 25 *A club based on excludable benefits might be a private golf club.*

In the spirit of our earlier analysis we shall examine clubs that supply an excludable public good. Is useful to first out line the key features of such clubs

- Participation in the club is *voluntary* so those that choose to be members satisfy a *participation constraint.* Notice that with a standard pure public good participation is unavoidable.
- As the membership of a club increases so too do the costs and benefits of participating.
- Clubs offer goods that non-members would typically like to enjoy, thus an *exclusion mechanism* is required to prevent this.
- The level of provision of a club good is related to the size and characteristics of the membership. This is not true of a pure public good.

2.8.1 Homogeneous Clubs.

We assume there are two goods X and Y. X is a private good while Y is a club good. We further assume that each club member possesses the same tastes and endowment and uses the club an equal amount, a members utility function may be written

$$U^A = U^A(X^A, Y, N)$$

where N is the number of club members. We assume that $\frac{\partial U^A}{\partial N} > 0$ for $N \to 0$ and $\frac{\partial U^A}{\partial N} < 0$ for $N \to \infty$ and that $\frac{\partial^2 U^A}{\partial N^2} < 0$. We assume that each club member attempts to maximize their utility subject to a resource constraint

$$F^A(X^A, Y, N) = 0$$

which has the properties

- $\frac{\partial F^A}{\partial N} < 0$ as the number of club members increases the cost per member declines

- $\frac{\partial F^A}{\partial X^A} > 0$ the private good is costly in terms of resources

- $\frac{\partial F^A}{\partial Y} > 0$ the club good is costly in terms of resources.

The Lagrangian that represents the consumers optimization problem now becomes

$$\underset{X^A,Y,N}{Max}\ \Im = U^A(X^A, Y, N) + \lambda F^A(X^A, Y, N)$$

the first order conditions to which are

$$\begin{aligned} \frac{\partial \Im}{\partial X^A} &= \frac{\partial U^A}{\partial X^A} + \lambda \frac{\partial F^A}{\partial X^A} = 0 \\ \frac{\partial \Im}{\partial Y} &= \frac{\partial U^A}{\partial Y} + \lambda \frac{\partial F^A}{\partial Y} = 0 \\ \frac{\partial \Im}{\partial N} &= \frac{\partial U^A}{\partial N} + \lambda \frac{\partial F^A}{\partial N} = 0 \end{aligned}$$

by our usual methods these equation may be rearranged to yield

$$\begin{aligned} MRS^A_{YX^A} &= \frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} = \frac{\frac{\partial F^A}{\partial Y}}{\frac{\partial F^A}{\partial X^A}} = MRT^A_{YX^A} \quad \text{(provision condition)} \\ MRS^A_{NX^A} &= \frac{\frac{\partial U^A}{\partial N}}{\frac{\partial U^A}{\partial X^A}} = \frac{\frac{\partial F^A}{\partial N}}{\frac{\partial F^A}{\partial X^A}} = MRT^A_{NX^A} \quad \text{(membership condition)} \end{aligned}$$

- The provision condition tells us that for each member of the club the marginal rate of substitution between the club good and the private good must be equated to the marginal rate of transformation between the two goods.

- The membership condition tells us that for within club optimality the marginal rate of substitution between group size and the private good must be equated to the marginal rate of transformation between group size and the private good. Hence marginal benefits and marginal costs from having one more member in the club are equated. Typically both of these will be negative, one more member means increased congestion, but the club cost will go down.

- These conditions look a lot like efficiency conditions, but they are *not* because there is no account taken of the utility of non-club members.

2.8.2 An Interpretation of Club Theory.

One interpretation of club theory is that a club is a jurisdiction supplying local impure public goods. For example education or parks. We might then ask; What is the optimal number and type of clubs in an economy? Which translates in to the question of how local impure public good should be supplied. It can be shown that if we divide the individuals in a population in to clubs and no individual or group of individuals wish to transfer between clubs or form a new club, then the set of clubs is Pareto efficient.

2.8.3 The Tiebout Hypothesis.

Tiebout proposed that "voting with your feet" might lead to a pareto efficient level of provision of local public goods. Suppose that different communities offered different mixes of local public goods, for example some were suited to young families having lots of public parks and play areas, others were more suited to needs of the retired having ample health care and public golf courses. If people relocate to the communities that suit them best then the resulting allocation will (under some further assumptions) be Pareto efficient.

2.9 Public Choice and Public Good Provision Mechanisms.

We now know that there are severe problems with the private provision of a public good of any sort. We next turn our attention to ways in which the good may be supplied publicly by a government. The government must find a mechanism to solve the free rider problem, and the inherent problem of individuals misrepresenting how much they value the public good.

2.9.1 Lindahl Equilibrium.

Suppose that a government wanted to find out the efficient level of provision of a public good and how the cost of provision should be distributed across the population. It might adopt the voting procedure suggested by Lindahl. The mechanism works like this

- Tell each individual that they will pay an individual "tax price" as a percentage of the cost per unit of supplying a public good.
- Have each individual vote on the number of units they wish to have supplied.
- Those individuals who vote for more than the average have their tax prices raised those who vote for less have their tax prices lowered.
- Repeat this process as many times as necessary until all individuals vote for the same level of provision of the good. This is the Lindahl Equilibrium.

Properties of The Lindahl Equilibrium.

To derive the Lindahl Equilibrium and examine it's properties is quite easy, consider the problems faced by the our two individuals A and B each wishes to cast their vote for the level of provision of the public good that would maximize their own utility given the share of the cost they must pay. Suppose that our two individuals may spend incomes I^A, I^B on a public good Y or a private good X. The private and public goods are good is supplied at the price $P^X = P^Y = 1$, the tax prices of the public good to each of the two individuals are given by τ^A, τ^B, and note that $\tau^A + \tau^B = 1$, i.e. the good must be paid for.

Individual choices must satisfy

$$\underset{X^A,Y}{Max} U^A(X^A, Y)$$
$$s.t.\ X^A + \tau^A Y = I^A$$

or

$$\underset{Y}{Max} U^A\left(I^A - \tau^A Y, Y\right)$$

the first order conditions to which are

$$-\tau^A \frac{\partial U^A}{\partial X^A} + \frac{\partial U^A}{\partial Y} = 0$$

For individual B identical methods yield

$$-\tau^B \frac{\partial U^B}{\partial X^B} + \frac{\partial U^B}{\partial Y} = -(1-\tau^A)\frac{\partial U^B}{\partial X^B} + \frac{\partial U^B}{\partial Y} = 0$$

simple manipulation of these expressions reveals

$$\begin{aligned} \frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} &= \tau^A \\ \frac{\frac{\partial U^B}{\partial Y}}{\frac{\partial U^B}{\partial X^B}} &= (1-\tau^A) \end{aligned}$$

Substituting for τ^A from the first expression into the second gives

$$\frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} + \frac{\frac{\partial U^B}{\partial Y}}{\frac{\partial U^B}{\partial X^B}} = \sum_{A,B} MRS_{YX} = 1$$

Now we know $\frac{P^Y}{P^X} = 1 = MRT_{YX}$ so we find that

$$\sum_{A,B} MRS_{YX} = MRT_{YX}$$

so *the Lindahl mechanism induces a Pareto efficient allocation.*

Problems with the Lindahl Mechanism.

1. The mechanism is obviously complicated. Each individual must pay a personal "tax price" and this tax price must be established through successive rounds of voting until unanimity is achieved. To actually implement this mechanism in any large economy would obviously be very, perhaps prohibitively, costly.

2. The mechanism can be strategically manipulated by the individuals. If an individual recognizes that their tax price will be lowered if they vote for a lower level of provision of the good they will typically do so. They can still hope to get a high level of provision because of the behavior of the other agent. In essence a more sophisticated version of the Free Rider problem remains.

2.9.2 Majority Voting.

In the previous section we say how unanimous voting in the form of the Lindahl mechanism can lead to an efficient level of provision of public goods by the government. But in reality the mechanism stands little chance of success in all but a few circumstances. Suppose now that instead of unanimity we required only a simple majority to determine the level of provision.

To facilitate our analysis assume that there are three individuals in our economy, A, B and C, and two goods one public Y and one private X. We further assume that each individual faces an equal share in the cost of the provision of the good. Each individual has a most preferred level of provision which they propose be adopted by society. Denote their proposals as $Y^A > Y^B > Y^C$. We assume pairwise voting so that each option has to beat the other two in separate votes.

Single Peaked Preferences.

We first examine decision making when preferences are described as "single peaked", loosely speaking this means that no individual likes *both* of the extreme levels of provision more than the intermediate level. Preferences that satisfy this condition might be given by

		Individual		
		A	B	C
Preference Ranking	1^{st}	Y^A	Y^B	Y^C
	2^{nd}	Y^B	Y^C	Y^B
	3^{rd}	Y^C	Y^A	Y^A

We see immediately from this ranking that in pairwise comparisons

- Y^A vs. Y^B then Y^B wins 2:1.
- Y^B vs. Y^C then Y^B wins 2:1.

- Y^C vs. Y^A then Y^C wins 2:1.

It then follows for any sequence of votes the choice will be Y^B. With this mechanism at least we get a clear unambiguous decision.

Multi-Peaked Preferences.

With multi peaked preferences at least one individual likes both extremes more than the intermediate case. This might be the "I'd rather you didn't do it but if your going to do it, do it properly" type of reasoning. In such circumstances there are problems as illustrated below

		Individual		
		A	B	C
	1^{st}	Y^A	Y^B	Y^C
Preference Ranking	2^{nd}	Y^B	Y^C	Y^A
	3^{rd}	Y^C	Y^A	Y^B

We see immediately from this ranking that in pairwise comparisons

- Y^A vs. Y^B then Y^A wins 2:1.
- Y^A vs. Y^C then Y^C wins 2:1.
- Y^C vs. Y^B then Y^B wins 2:1.

This leads immediately to two problems

1. Voting Cycles - If voting continues until a clear winner emerges the process will just cycle round *ad* nauseam.

2. Agenda Manipulation - If the voting sequence is Y^A vs. Y^B then Y^A vs. Y^C we end up with the decision Y^C. If the voting sequence is Y^C vs. Y^B then Y^A vs. Y^B we end up with the decision Y^A. If the voting sequence is Y^A vs. Y^C then Y^C vs. Y^B we end up with the decision Y^B. So the sequence in which the votes takes place determinate the outcome!

Efficiency.

The selection of the proposed levels of public good provision $Y^A > Y^B > Y^C$ each satisfy the requirement that they maximize the level of utility of one voter given the cost sharing rule. There is nothing in the majority voting mechanism that ensures there will be efficiency. the easiest way to see this is via an example. Suppose that individuals now have the original preferences

		Individual		
		A	B	C
	1^{st}	Y^A	Y^B	Y^C
Preference Ranking	2^{nd}	Y^B	Y^A	Y^A
	3^{rd}	Y^C	Y^C	Y^B

Suppose individual C gets infinite utility from the choice Y^C, while the other two individuals receive only an arbitrarily small benefit from the majority choice Y^A. Individual C would be willing to sacrifice all the private good he/she owns for the decision Y^C and still be better off. The other two individuals would gain from trading the choice Y^C for more of the private good, hence here the allocation under majority voting cannot be efficient.

Log Rolling.

Our criticism of the efficiency properties of majority voting points to one of it's major weakness' in that it doesn't allow voters to express the strength of their preferences. If I prefer option Y^A to Y^B by a litter or a lot it's all the same under majority voting. Log rolling allows voters to trade votes on different issues. This allows individuals to express the strength of preferences giving up votes on issues they care little about in return for votes on issues about which they care a lot. Vote trading essentially sets up a market that can in the correct circumstances lead to pareto efficient decisions. However, it should be noted that *vote trading is not necessarily pareto improving* if A and B trade votes such that C is badly hurt the outcome is clearly not pareto improving.

Downs Theorem

Down theorem simply states that vote maximizing behavior on the part of political candidates will result in the preferred program of the median vote being adopted. Which may be bad news in the light of our earlier criticisms of median voting.

2.9.3 The Clarke-Groves Mechanism.

The Clarke-Groves mechanism provides a way a government may solicit the true preferences of individuals over the level of the provision of a public good. The mechanism stems from a simple observation, if the amount individuals have to pay towards a public good is linked to their reported preferences for this good, then their incentives to report these preferences honestly are distorted. To solve this problem the amount individuals have to pay must be separated from their reported preferences. The proposed mechanism works as follows

1. Each individual is asked to report to a planner the value to themselves of a given public good.
2. The planner adds up the valuations reported, if they exceed the cost the good is supplied, if not it isn't.
3. To pay for the good the planner then charges each individual the cost minus the sum of all the other agents valuations.

Let

C be the cost of supplying the public good.

r^A, r^B be the valuations of the public good the two individuals *report* to the planner.

v^A, v^B be the individuals *true* valuations of the public good (which may or may not equal the $r's$).

Efficiency

Requires that the good be supplied if

$$v^A + v^B \geq C$$

Decision Rule.

The good will actually be supplied if

$$r^A + r^B \geq C$$

The Mechanism.

Each individual is assessed a tax to pay for the good of

$$\begin{aligned} \tau^A &= C - r^B \\ \tau^B &= C - r^A \end{aligned}$$

The key point is that the tax paid by A depends on the reported valuation of B and vice versa. This means that what an individual pays for the public good and how much he enjoys it are uncoupled.

Individual Incentives to Report their Valuations.

We now address the key issue; given the tax mechanism above will individuals truthfully report their valuations of the public good. Let us assume that individuals utility functions take the following simple form

$$U^A = \begin{array}{cc} I^A & \text{if the project is not undertaken} \\ I^A - \tau^A + v^A = I^A - C + r^B + v^A & \text{if the project is undertaken} \end{array}$$

$$U^B = \begin{array}{cc} I^B & \text{if the project is not undertaken} \\ I^B - \tau^B + v^B = I^B - C + r^A + v^B & \text{if the project is undertaken} \end{array}$$

so we see that A will want the project undertaken if

$$I^A - C + r^B + v^A \geq I^A$$
$$v^A \geq C - r^B$$

similarly B will want the project undertaken if

$$v^B \geq C - r^A$$

Now both A and B know that the planner will undertake the project if

$$r^A + r^B \geq C$$

and they also know the reported values of each other. Consider the incentives faced by A they take $C - r^B$ as given and so want the good provided if $v^A \geq C - r^B$ hence they have no incentive to misrepresent their valuation. Similarly for B they take $C - r^A$ as given and so want the good provided if $v^B \geq C - r^A$ hence they also have no incentive to misrepresent their valuation. Hence they both tell the truth, and the solution is

$$\begin{aligned} r^A &= v^A \\ r^B &= v^B \\ \tau^A &= C - v^B \\ \tau^B &= C - v^A \end{aligned}$$

which is also efficient.

A Problem.

The revenues the government raises is

$$\tau^A + \tau^B = 2C - v^B - v^A$$

there is no guarantee that

$$\tau^A + \tau^B = 2C - v^B - v^A \geq C$$

or

$$C - v^B - v^A \geq 0$$

indeed since

$$r^A + r^B \geq C$$

is the decision rule then

$$C - v^B - v^A \leq 0$$

and the government does not raise enough revenue to fund the project.

A Solution.

The problem with the mechanism as it stands is that it does not generate sufficient tax revenue. To solve this problem an extra term is added to the tax formula, this takes the form of

$$\begin{aligned} \tau^A &= C - r^B + \max\left\{0, r^B - \frac{1}{2}C\right\} \\ \tau^B &= C - r^A + \max\left\{0, r^A - \frac{1}{2}C\right\} \end{aligned}$$

For the public good to be funded we require $\tau^A + \tau^B \geq C$, now if $r^B - \frac{1}{2}C > 0$ individual A pays a further tax of this amount,while if $r^A - \frac{1}{2}C > 0$ individual B, also pays a further tax of this amount. Running through the potential cases we get

Good Should be Provided			Revenue
$C - v^B - v^A < 0$	$r^B - \frac{1}{2}C > 0$	$r^A - \frac{1}{2}C > 0$	C
$C - v^B - v^A < 0$	$r^B - \frac{1}{2}C \leq 0$	$r^A - \frac{1}{2}C > 0$	$C + \frac{1}{2}C - r^B$
$C - v^B - v^A < 0$	$r^B - \frac{1}{2}C > 0$	$r^A - \frac{1}{2}C \leq 0$	$C + \frac{1}{2}C - r^A$

from which we see that the good is funded in each case. Notice also that A's payments (similarly B's) are still unaffected by their own reported valuation, and so there remains no incentive to lie.

More Problems.

The Clarke-Groves mechanism has the advantages that; (1) It *always* induces the truthful revelation of preferences, and, (2) there are *always* sufficient funds to provide the public good. However it also suffers from two significant drawbacks

1. The tax revenues raised are excessive, and cannot be paid back to the individuals without damaging the incentives to tell the truth. As such the scheme, while it provides truthful revelation of preferences and the "correct" level of provision of the public good, is not pareto efficient because the tax itself is wasteful.

2. The mechanism only works for some types of preferences, those with "transferable utility", in essence utility that can be expressed in terms of money.

2.9.4 The Groves-Ledyard Mechanism.

In an attempt to improve on the Clarke-Groves mechanism, Groves and Ledyard propose an alternative tax scheme. This scheme has the properties that; (1) the government's budget balances, so the tax scheme is not inherently wasteful, and, (2) a pareto efficient allocation is achieved. However, to obtain this result they must sacrifice the property that everyone always tells the truth. As we shall see, with the Groves-Ledyard mechanism, each individual only has an incentive to tell the truth if all the other individuals are already telling the truth.

The Mechanism.

The Groves-Ledyard mechanism consists of two components , a public good supply function, and a tax function.

The Supply Function. Each individual in the economy has to send a message to the provider of the public good (presumably the government), this message is how much of the public good they would like to have supplied *over and above* the total requested by everyone else. The provision level of the good is the sum of these messages In our two individual world the individuals send the messages m^A, and m^B, and the level of provision is then

$$Y = m^A + m^B$$

The Tax Function (Do three individual case in book). The original tax function proposed by G&L was of the following form

$$\tau^i = \alpha^i Y + \frac{\gamma}{2}\left[\frac{n-1}{n}(m^i - \mu^{-i})^2 - \left(\sigma^{-i}\right)^2\right]$$

where there are $i = 1, .., n$ individuals, $\sum \alpha^i = 1$ and γ are positive constants, μ^{-i} is the average signal sent by all individuals other than person i, and σ^{-i} is the variance of all the messages sent by all individuals other than person i. In our simple two individual world this reduces to (we have had to introduce the constant Z because otherwise the variance term vanishes in the two individual case)

$$\tau^A = \alpha^A Y + \frac{\gamma}{2}\left[\frac{1}{2}(m^A - m^B)^2 - Z\right]$$

Equilibrium and The Samuelson Condition. Our individuals may choose between the consumption of a private good X and the public good Y , and have initial income levels I^A, I^B. We further assume prices $p^X = p^Y = 1$ so that the MRT between the two goods is 1. Now the utility maximization problems are

$$\begin{aligned} Max\ U^A &= U(I^A - \tau^A, Y) \\ s.t.\ \tau^A &= \alpha^A Y + \frac{\gamma}{2}\left[\frac{1}{2}(m^A - m^B)^2 - Z\right] \\ Y &= m^A + m^B \end{aligned}$$

by substitution this reduces to

$$\underset{m^A}{Max}\ U^A = U(I^A - \alpha^A\left(m^A + m^B\right) + \frac{\gamma}{2}\left[\frac{1}{2}(m^A - m^B)^2 - Z\right], m^A + m^B)$$

the FOC to which is

$$\left(\frac{\gamma}{2}(m^A - m^B) - \alpha^A\right)\frac{\partial U^A}{\partial X^A} + \frac{\partial U^A}{\partial Y} = 0 \qquad (*)$$

By identical methods we get for individual B

$$\left(\frac{\gamma}{2}(m^B - m^A) - \alpha^B\right)\frac{\partial U^B}{\partial X^B} + \frac{\partial U^B}{\partial Y} = 0$$

recall $\sum \alpha^i = 1$ so $\alpha^B = 1 - \alpha^A$ and we get for B

$$\left(\frac{\gamma}{2}(m^B - m^A) - \left(1 - \alpha^A\right)\right)\frac{\partial U^B}{\partial X^B} + \frac{\partial U^B}{\partial Y} = 0 \qquad (**)$$

Now dividing (*) by $\frac{\partial U^A}{\partial X^A}$ and (**) by $\frac{\partial U^A}{\partial X^A}$ gives

$$\begin{aligned}\left(\frac{\gamma}{2}(m^A - m^B) - \alpha^A\right) + \frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} &= 0\\ \left(\frac{\gamma}{2}(m^B - m^A) - \left(1 - \alpha^A\right)\right) + \frac{\frac{\partial U^B}{\partial Y}}{\frac{\partial U^B}{\partial X^B}} &= 0\end{aligned}$$

adding the two expressions together gives

$$\sum_{A,B} MRS_{XY} = \frac{\frac{\partial U^A}{\partial Y}}{\frac{\partial U^A}{\partial X^A}} + \frac{\frac{\partial U^B}{\partial Y}}{\frac{\partial U^B}{\partial X^B}} = 1 = MRT_{XY}$$

Hence the allocation satisfies the Samuelson Condition.

Budget Balance. We now wish to show that the sum of the tax contributions just funds the provision of the public good, this requires we demonstrate that

$$Y = m^A + m^B = \tau^A + \tau^B$$

Starting with the tax functions we note that

$$\begin{aligned}\tau^A &= \alpha^A Y + \frac{\gamma}{2}\left[\frac{1}{2}(m^A - m^B)^2 - Z\right]\\ \tau^B &= \left(1 - \alpha^A\right)Y + \frac{\gamma}{2}\left[\frac{1}{2}(m^B - m^A)^2 - Z\right]\end{aligned}$$

adding these together we get

$$\tau^A + \tau^B = Y + \gamma\left[\frac{1}{2}(m^A - m^B)^2 - Z\right]$$

hence the budget balances provided Z is chosen such that

$$2Z = (m^A - m^B)^2$$

It then immediately follows that since the tax is non-wasteful and the Samuelson condition is satisfied then the allocation is Pareto efficient.

2.9.5 The Bayes-Nash Mechanism.

Suppose our two individuals A and B have the linear preferences over private and public goods X and Y as

$$\begin{aligned} U^A &= X^A + v^A Y \\ U^B &= X^B + v^B Y \end{aligned}$$

where v^A, v^B are their marginal valuations of the public good. Suppose that the public good may be provided, in which case $Y = 1$, or it may not, so $Y = 0$. Let the valuations of each individual, v^A, v^B, be their own private information. Suppose that all they know of the other's valuation is $-1 \geq v^A, v^B \geq 1$, and that the distribution of the valuation is uniform.

Samuelson's condition requires the public good be provided if

$$v^A + v^B \geq 1$$

The Mechanism.

There are two components to the mechanism, the reported valuation given by the individual to the government, and then a transfer given by the government to each individual.

1. Each individual is asked to report their valuation r^A, r^B. If A reports r^A they know the project will be undertaken if

   $$r^B \geq 1 - r^A$$

   this probability of this is

   $$\frac{2 - \left(1 - r^A\right)}{2} = \frac{1 + r^A}{2}$$

   If the project is undertaken then the expected benefit to B must lie between $-r^A$ and $+1$ (or the project would not have been undertaken). The expected valuation of B is then

   $$\frac{1 - r^A}{2}$$

   hence the expected benefit to B may be written

   $$\left(\frac{1 + r^A}{2}\right)\left(\frac{1 - r^A}{2}\right) = \frac{1 - \left(r^A\right)^2}{2}$$

   which is just the probability of the project times the expected valuation. By the same method the expected benefit to A may be written

   $$\left(\frac{1 + r^B}{2}\right)\left(\frac{1 - r^B}{2}\right) = \frac{1 - \left(r^B\right)^2}{4}$$

2. Each individual is then given a transfer from the government of the form

$$t^A = \frac{1-\left(r^A\right)^2}{2} - \frac{1-\left(r^B\right)^2}{4}$$
$$t^B = \frac{1-\left(r^B\right)^2}{2} - \frac{1-\left(r^A\right)^2}{4}$$

notice that the sum of these transfers is zero so the governments budget balances.

Utility Maximization.

Each individual now chooses their report to maximize their expected utility given the transfer scheme. For individual A this involves

$$\underset{r^A}{Max}\ t^A + \left(\frac{1+r^A}{2}\right)v^A$$
$$s.t.\ t^A = \frac{1-\left(r^A\right)^2}{2} - \frac{1-\left(r^B\right)^2}{4}$$

substituting in the transfer function gives

$$\underset{r^A}{Max}\ \frac{1-\left(r^A\right)^2}{2} - \frac{1-\left(r^B\right)^2}{4} + \left(\frac{1+r^A}{2}\right)v^A$$

the FOC for r^A becomes

$$\frac{\partial U^A}{\partial r^A} = -\frac{r^A}{2} + \frac{v^A}{2} = 0$$
$$\Rightarrow r^A = v^A$$

A tells the truth.

By identical methods

$$\frac{\partial U^B}{\partial r^B} = -\frac{r^B}{2} + \frac{v^B}{2} = 0$$
$$\Rightarrow r^B = v^B$$

and B also tells the truth.

Since the scheme is non-wasteful and the two individuals tell the truth, the allocation will be Pareto efficient.

2.10 Charities and Lotteries (Not on the 440/540 course).

2.10.1 Altruism and Warm Glow.

2.10.2 Morgan's Lottery Solution.

Chapter 3
EXTERNALITIES.

3.1 The Fundamental Sources of Externality Problems.

Definition 26 *An externality is an effect that the decision of one economic agent has upon the benefits (or costs) of another upon which the effected agent has no influence.*

Usually we translate this vague expression into the statement that there is an economic effect between economic agents that is not allocated via competitive markets.

Example 27 *Global Warming - Each countries emissions of* CO^2 *into the atmosphere help raise the temperature all over the globe.*

Example 28 *Acid Rain - The industries in the Ohio valley emit sulphurous gases that cause the rain in the Adirondacks to be acidic killing trees, fish etc.*

3.1.1 Missing Markets and Transactions Costs.

Whenever there is an externality between two economic agents it would seem that there is an economic "good" that could be traded to the benefit of the trading partners which for some reason is not traded. For example I could charge you for the right to smoke in my house. If you choose to pay we would be both better off since this is a voluntary trade. Why do such markets not exist? One answer is transactions costs. Suppose there is a cost to establishing a market C, and further let the price an individual is willing to pay for a good be P^d (the demand price). Let P^s be the price the seller is willing to accept to part with the particular good. we may write the surplus obtained from the trade as $P^d - P^s \geq 0$, but if $P^d - P^s < C$ we see that the transactions cost eats up all of the surplus and the trade will not be made.

3.1.2 The Absence of Property Rights.

In some circumstances it is impossible or too expensive to establish property rights. If there are no property rights then nobody owns the good hence trading it on a market becomes impossible.

Example 29 *In using the air some users pollute the atmosphere for others but excluding individuals from using the air is clearly ridiculous.*

Example 30 *Excluding individuals from enjoying the benefits of national defence is impossible.*

Notice, however, that the absence of a market does not imply that the allocation is pareto inefficient *per se,* if it is too expensive to establish private property rights then "common ownership" may be efficient.

Example 31 *Consider fishing in the pacific, the costs of enforcing private ownership over vast areas of ocean would probably cost more than the benefits obtained, the absence of private property rights is thus efficient!!*

3.1.3 Nonconvexities.

Non-convexities often lead to situations where socially desirable actions are not undertaken because private individuals or firms do not find it profitable to follow them.

Example 32 *The classic example is that of a firm that faces fixed set up costs so large that it cannot make a profit despite the fact that production of the good is socially desirable.*

Example 33 *A paper mill may pollute a river causing damage to the fishing industry. The fishermen may pay the paper mill not to pollute and thus not produce. Yet paper is socially desirable.*

3.2 Types of Externalities.

Externalities may arise between economic agents of any type, firms, households, governments, cities etc. and may be positive or negative. What is key is that they affect each other in ways that are not priced on a market

3.2.1 Consumption.

Suppose that your neighbor has a barbecue you can smell the cooking and don't like it, this is an externality and is incorporated into the calculation of Pareto Efficiency as follows. We assume our usual two good model and make the convenient assumption $P^X = 1$.

The utility functions are now

$$\begin{aligned} U^A &= U^A(X^A, Y^A, X^B) \\ U^B &= U^B(X^B, Y^B) \end{aligned}$$

the appearance of X^B in A's utility function where it has a negative effect represents the externality.

Deriving the condition for Pareto Efficiency in the usual way involves the problem

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A, X^B)$$
$$s.t\ U^B(X^B, Y^B) \geq \overline{U}^B$$
$$X = X^A + X^B$$
$$Y = Y^A + Y^B.$$

Where $\overline{U}^B$ is the level of utility that B must realize. By substitution this problem reduces to

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A, X - X^A)$$
$$s.t\ U^B(X - X^A, Y - Y^A) \geq \overline{U}^B$$

Forming the Lagrangian we obtain

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A, X - X^A) + \lambda \left[\overline{U}^B - U^B(X - X^A, Y - Y^A)\right]$$

where λ is the Lagrange Multiplier associated with the utility constraint. Now maximizing yields

$$\begin{aligned} \frac{\partial U^A}{\partial X^A} + \frac{\partial U^A}{\partial X^B}\frac{\partial X^B}{\partial X^A} - \lambda\frac{\partial U^B}{\partial X^B}\frac{\partial X^B}{\partial X^A} &= 0 \\ \frac{\partial U^A}{\partial Y^A} - \lambda\frac{\partial U^B}{\partial Y^B}\frac{\partial Y^B}{\partial Y^A} &= 0 \end{aligned}$$

utilizing $\frac{\partial X^B}{\partial X^A} = \frac{\partial Y^B}{\partial Y^A} = -1$ and rearranging the expressions gives the condition for Pareto Efficiency in consumption

$$\frac{\frac{\partial U^A}{\partial X^A} - \frac{\partial U^A}{\partial X^B}}{\frac{\partial U^A}{\partial Y^A}} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

The left hand side (LHS) of this expression is the ratio of marginal utilities for the two good for individual A adjusted for externality, the RHS is the ratio of marginal utilities for the two good for individual B.

Failure of The Fundamental Theorem with Externalities in Consumption.

From our earlier analysis we know that the market allocation will be achieved where

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}} = \frac{1}{P^Y} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

but Pareto Efficiency requires

$$\frac{\frac{\partial U^A}{\partial X^A}-\frac{\partial U^A}{\partial X^B}}{\frac{\partial U^A}{\partial Y^A}}=\frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

hence the market allocation is not Pareto Efficient. Indeed it is easy to see that the market allocation involves individual B involves in too much of the externality causing activity.

3.2.2 Production.

Production externalities often involve effects that one firm's output decisions have on another firm's costs. A classic example is the industrial pollution of ground water, in this case farmers may have to either filter the water or purchase clean water from an alternative source for their stock. We shall use this example for our analysis, let X be industrial production and Y agricultural production. Each producer uses two inputs water W and labor L to produce outputs via the production technologies $X(L^X,W^X), Y(L^Y,W^Y)$. All prices are assumed to be unity and determined on competitive markets. For the industry profit maximization involves

$$\begin{aligned} Max\ \pi^X &= X - C^X \\ s.t.\ X &= X(L^X,W^X) \\ C^X &= L^X + W^X \end{aligned}$$

by substitution this reduces to

$$\underset{L^X,W^X}{Max}\ \pi^X = X(L^X,W^X) - \left(L^X + W^X\right)$$

and the first order conditions are simply

$$\begin{aligned} \frac{\partial X}{\partial L^X} - 1 &= 0 \\ \frac{\partial X}{\partial W^X} - 1 &= 0 \end{aligned}$$

For agriculture profit maximization involves

$$\begin{aligned} Max\ \pi^Y &= Y - C^Y \\ s.t.\ Y &= Y(L^Y,W^Y) \\ C^Y &= L^Y + W^Y(1 + X/N) \end{aligned}$$

The term $(1 + X/N)$ captures the effect of the production of X on the cost of water to the producer of Y. By substitution this reduces to

$$\underset{L^Y,W^Y}{Max}\ \pi^Y = Y(L^Y,W^Y) - \left(L^Y + W^Y(1 + X/N)\right)$$

with FOC's

$$\frac{\partial Y}{\partial L^Y} - 1 = 0$$
$$\frac{\partial Y}{\partial W^Y} - (1 + X/N) = 0$$

Notice that for a greater X the more expensive will be water to Y, typically this will involve Y using less water and producing less the more X is produced.

Pareto Efficiency.

In this case pareto efficiency requires one firms profits be maximized subject to the others not being damaged or

$$\underset{L^X,W^X,L^Y,W^Y}{Max} \; \pi^X = X(L^X, W^X) - \left(L^X + W^X\right)$$
$$s.t. \; Y(L^Y, W^Y) - \left[L^Y + W^Y\left(1 + \frac{X(L^X, W^X)}{N}\right)\right] \geq \overline{\pi}^Y$$

the first order conditions to this problem involve

$$\begin{aligned}
\frac{\partial X}{\partial L^X} - 1 + \lambda\left[\frac{W^Y}{N}\frac{\partial X}{\partial L^X}\right] &= 0 \\
\frac{\partial X}{\partial W^X} - 1 + \lambda\left[\frac{W^Y}{N}\frac{\partial X}{\partial W^X}\right] &= 0 \\
\frac{\partial Y}{\partial L^Y} - 1 &= 0 \\
\frac{\partial Y}{\partial W^Y} - (1 + X/N) &= 0
\end{aligned}$$

Which clearly differs from he market solution, hence the market solution is typically not pareto efficient. The Producer of X neglects the effects his production has on Y's costs and typically over produces in a paretian sense.

3.2.3 Consumption and Production.

3.2.4 Reciprocal Externalities; The Common Pool Resource Problem.

3.3 Solutions to Externality Problems.

3.3.1 Pigouvian Solutions.

Pigouvian solutions rely on "pricing" the externality via a tax on the externality generating activity. The agent generating the externality is then induced to "internalize " it as the tax makes him face the full social cost of his actions/decisions.

Example. Suppose there are two firms one produces banjos and the other produces books. Readers of books are adversely effected by the sound of bad banjo playing, and thus both read less and purchase fewer books. The reading of books does not effect the enjoyment or sales of banjos except through standard market mechanisms. Let the profits of the firm that makes banjos be written

$$\pi_1 = rx - c(x)$$

where r is the competitive market price of banjos, x the number of banjos produced, $c(x)$ is an increasing convex cost function. For the firm that produces books profits are given by

$$\pi_2 = \Pi - e(x)$$

where Π is a constant and $e(x)$ is an increasing convex function that captures the negative effect of banjo sales on the book producers profits. (Note this is Varian's simple model with the addition of a constant).

In the absence of any corrective measures the banjo manufacturer chooses to

$$\underset{x}{Max} rx - c(x)$$

the first order condition to which defines the privately optimal action $\widetilde{x}$ as satisfying

$$r - c'(\widetilde{x}) = 0$$

Efficiency requires the banjo producer maximizes profit taking into account the full costs of his actions or

$$\underset{x}{Max} rx - c(x) - e(x)$$

the first order condition to which defines the optimal action x^*as satisfying

$$r - c'(x^*) - e'(x^*) = 0$$

The Pigouvian solution then requires setting a pigouvian tax p such that $p = e'(x^*)$. the banjo manufacturers profit maximization problem becomes

$$\underset{x}{Max} rx - c(x) - px$$

and we immediately get

$$r - c'(x^*) - p = 0 = r - c'(x^*) - e'(x^*)$$

and efficiency is obtained.

Problems with the Pigouvian Solution. To set $p = e'(x^*)$ it is necessary that the government know $e'(x^*)$. This is difficult for the government to know,

1. The government may not be able to observe the effect $e(x)$.
2. Even if $e(x)$ has been observed it has only been seen at the point $\widetilde{x}$. It needs to know the slope of the function at x^*.

This strongly suggests that Pigouvian taxes while they are observed in the real economy are an imprecise way of dealing with externalities.

3.3.2 Coasian Solutions.

The classic Coasian solution to an externality problem simply involves establishing property rights and then letting the agents concerned bargain (a market is a special case of bargaining where both sides make take-it-or-leave-it offers at the going market price). The Coase theorem then goes on to say that the result will be efficient whatever the allocation of property rights. To see that this work consider our banjo/books example from above. We shall assume that after property rights are established the agents bargain and reach agreement according to the Nash bargaining solution.

Remark 34 *The Nash bargaining solution involves the two bargainers maximizing the product of their joint surplus over and above that which they could receive in the absence of agreement. It can be justified via a bargaining process of alternating concessions where each agent concedes to the other until further concessions on their part are more costly than concessions by the other agent.*

1. Property rights allocated to the banjo producer.

Here the outcome (*threat points*) in the absence of agreement is

$$\widetilde{\pi}_1 = r\widetilde{x} - c(\widetilde{x})$$

and

$$\widetilde{\pi}_2 = \Pi - e(\widetilde{x})$$

the Nash bargaining solution involves

$$\begin{aligned} & \underset{x,A}{Max}(\pi_1 - \widetilde{\pi}_1)(\pi_2 - \widetilde{\pi}_2) \\ = \; & [rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})]\,[\Pi - e(x) - A - \Pi + e(\widetilde{x})] \\ = \; & [rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})]\,[e(\widetilde{x}) - e(x) - A] \end{aligned}$$

where A is a *side payment* agreed between the two bargainers. The first order conditions are

$$[r - c'(x)]\,[e(\widetilde{x}) - e(x) - A] - e'(x)\,[rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})] = 0 \quad (3.1)$$

$$[e(\widetilde{x}) - e(x) - A] - [rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})] = 0 \quad (3.2)$$

from (2) we have

$$[e(\widetilde{x}) - e(x) - A] = [rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})] \tag{3.3}$$

substituting (3) into (1) gives

$$[r - c'(x) - e'(x)]\,[e(\widetilde{x}) - e(x) - A] = 0$$

dividing both sides by $[e(\widetilde{x}) - e(x) - A]$ gives

$$r - c'(x) - e'(x) = 0$$

hence we have efficiency when the property rights are allocated to the banjo producer.

2. Property rights allocated to the book producer.

If the book producer were allocated the property rights then prior to any bargaining the banjo producer would be denied the right to produce hence the outcome in the absence of agreement is given by

$$\overline{\pi}_1 = 0$$

and

$$\overline{\pi}_2 = \Pi$$

the Nash bargaining solution now involves

$$\begin{aligned} & \underset{x,B}{Max}(\pi_1 - \overline{\pi}_1)(\pi_2 - \overline{\pi}_2) \\ = \; & [rx - c(x) - B]\,[\Pi - e(x) + B - \Pi] \\ = \; & [rx - c(x) - B]\,[B - e(x)] \end{aligned}$$

notice that the side payment B is now a payment from the banjo producer to the book producer. The first order conditions are

$$[r - c'(x)]\,[B - e(x)] - e'(x)\,[rx - c(x) - B] = 0 \tag{3.4}$$

$$[rx - c(x) - B] - [B - e(x)] = 0 \tag{3.5}$$

from (5) we have

$$[rx - c(x) - B] = [B - e(x)] \tag{3.6}$$

substituting (6) into (4) gives

$$[r - c'(x) - e'(x)]\,[B - e(x)] = 0$$

dividing both sides by $[B - e(x)]$ gives

$$r - c'(x) - e'(x) = 0$$

hence we have efficiency when the property rights are allocated to the book producer.

Problems with Coasian Solutions. As we previously noted the problems with Coasian solutions are twofold.

1. Everyone will want to obtain the initial property rights so establishing them is not a trivial problem.

2. In some cases, perhaps due to non-convexities, the market solution will be a corner solution rather than the interior pareto efficient solution.

3.3.3 Varian's Solution.

Suppose we are in situation where a government does not have the information necessary to design pigouvian taxes, nor is it able (perhaps due to political constraints) to adopt a Coasian solution. Varian suggests that if the agents involved in the externality problem have full information concerning its causes and effects then the government may exploit this to achieve efficiency.

The mechanism consists of two stages

Announcement Stage: Each firm simultaneously announces a pigouvian tax. Firm 1's announcement is p_1, firm two announces p_2.

Choice Stage: Regulator enforces the tax schedules announced by the two firms according the next set of equations, the firms then choose their output levels.

$$\Pi_1 = rx - c(x) - p_2 x - \alpha_1 (p_1 - p_2)^2$$
$$\Pi_2 = \Pi + p_1 x - e(x)$$

The model is solved for the *subgame perfect* equilibrium. That is we solve for the last stage first, then solve for the first stage given how the last stage depends on the first. Consider first the choice stage.

Firm 1 maximizes

$$\underset{x}{Max}\ \Pi_1 = rx - c(x) - p_2 x - \alpha_1 (p_1 - p_2)^2$$

with first order condition

$$r = c'(x) + p_2$$

notice that this implies

Firm 2 is passive in the choice stage.$x = x(p_2)$ with $x'(p_2) < 0$.

Next consider the announcement stage. Notice that for any p_2 that firm 1 expects firm 2 to announce it simply chooses

$$p_1 = p_2$$

Now firm 2 knows that p_2 has an indirect effect on its own profits via $x(p_2)$ so it chooses p_2 to maximize

$$\Pi_2 = \Pi + p_1 x(p_2) - e(x(p_2))$$

so the first order condition is

$$[p_1 - e'(x)]\, x'(p_2) = 0$$

Since $x'(p_2) \neq 0 \Rightarrow p_1 = e'(x)$ since $p_1 = p_2 \Rightarrow r = c'(x) + p_2 = c'(x) + p_1 = c'(x) + e'(x)$ which is the condition for efficiency.

Remark 35 *The intuition here is that firm 1 always wishes to match firm 2's announcement. Firm 2 can manipulate x via its announcement p_2, hence it will always choose p_2 so that $p_1 = e'(x)$.*

Problems with Varian's mechanism. Given the firms have full information why doesn't firm 2 figure out that firm 1 always plays $p_1 = p_2$ and manipulate its own profits via

$$\Pi_2 = \Pi + p_2 x(p_2) - e(x(p_2))$$

this is a deviation from Nash behavior, but it might be appropriate.

3.3.4 Strategic Matching.

3.4 Solutions to the Common Pool Resource Problem.

3.4.1 Cornes, Mason and Sandler's Oligopoly Solution.

3.4.2 Sharing Schemes.

3.4.3 Common Pool Equities.

3.5 Problems with Private Information.

3.5.1 Bargaining.

3.5.2 The Vickrey Mechanism.

Chapter 4
INFORMATION PROBLEMS.

4.1 Externalities.

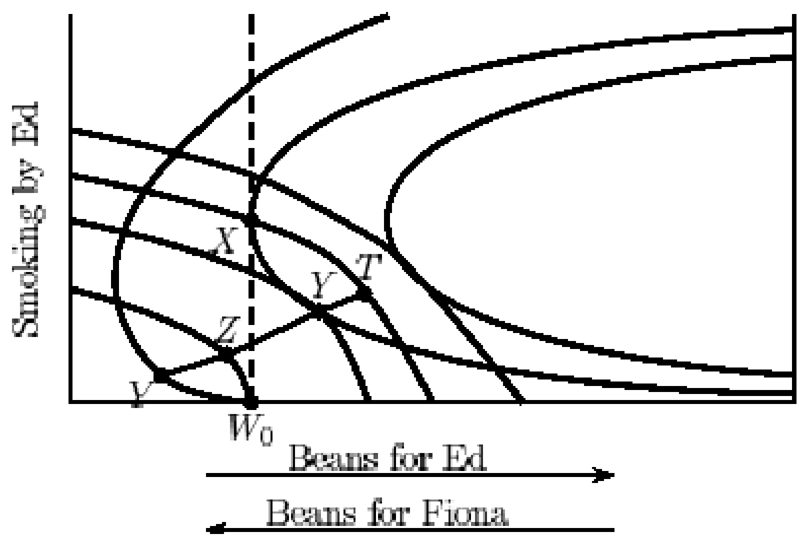
Definition 36 *An externality is an effect that one economic agent has on another over which the effected agent does not fully consent.*

In a sense an externality is a gain or loss that one agent imposes on another. This may be a gain or loss between consumers, firms, regions, countries or some combination of these. Since the imposition of an externality does not typically involve taking fully into account the preferences of the effected party, it is often the case that the external effect is inefficiently supplied. An inefficiency of course implies the possibility of realizing a Pareto improvement for the agents involved if they engage in voluntary trade.

Example: Bergstrom's "Smoking Box".

2 goods, beans and smoke. Ed likes beans and smoking, Fiona also likes beans but suffers a negative externality from Ed's smoking. The initial allocation of beans is given by W_0.In the absence of restrictions on smoking the outcome would be at X.

Figure 5.1: A One-Sided Externality



A shift away from X to any point on the line YT would be a pareto improvement.

4.1.1 Sources of Externalities.

Externalities as Missing Markets.

In the framework of a competitive market economy we might view the presence of an externality as synonymous with the absence of a market. Consider two consumers one of who plays music at high volume, but only derives a small benefit from doing do, the other has a headache and would greatly benefit from quiet. Why doesn't the headache victim pay the music player to turn it off? Why is there no market for this good? We know that if a market for the good were established the first fundamental theorem of welfare economics would apply and the market outcome would be Pareto efficient.

Externalities as the Absence of Property Rights.

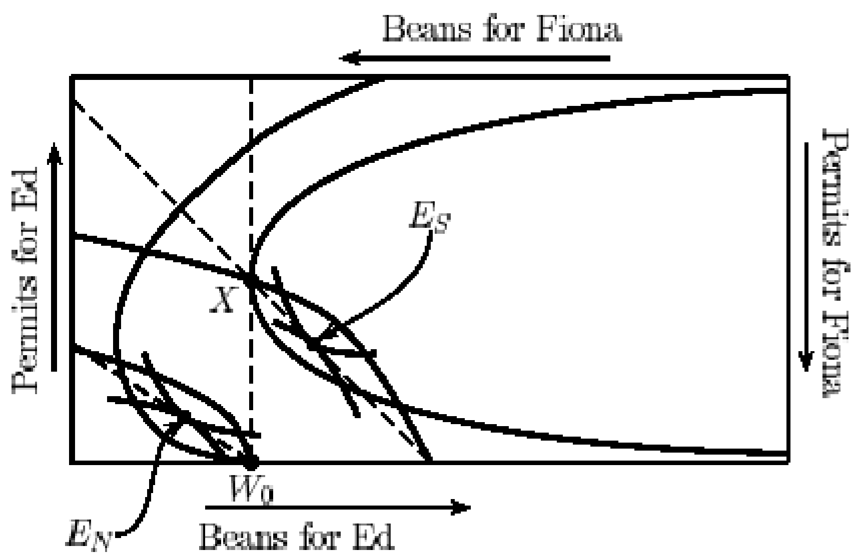
Following from the realization that externalities may be viewed as arising from the absence of markets we necessarily ask why such markets do not exist. property rights over the good in question are not clearly defined, if this is the case it is not clear if anyone has the option of placing the good up for sale. Consider the loud music example, does the headache sufferer have the right to silence of the music lover the right to his music. If the headache sufferer has the rights to silence then the music lover must pay him to listen to his tunes. If the music lover has a right to listen to his tunes, then the headache sufferer must pay him for silence.

The Reciprocal Nature of Externalities.

There is always potential for problems in establishing property rights as externalities are by definition reciprocal in nature. One economic agent benefits when the actions of one agent generate a positive or negative effect on another the initial action benefits the actor, the consequences either positive or negative effect the other agent. Does the first agent have the right to the action, or the second the right to deny the action? Both desire the property rights as they have value. Property rights are wealth!

Bergstrom's "Smoking Box" example revisited. Here smoking permits may be created but the outcome depends on who receives the initial property rights. If Ed receives then we start at X and Fiona must pay him not to smoke. The resultant equilibrium is at E_S. If Fiona receives the property right Ed must pay her for the right to smoke and the resultant equilibrium will be at E_N. Clearly it is better to receive the property rights.

Figure 5.3: A Market for Smoking Permits



The Costs of Establishing or Maintaining Property Rights.

Even if there is agreement that the property rights to a good should belong to a given agent it may still be costly to establish or maintain them. Consider an area of public grazing land, if private property rights are not established each rancher does not consider the effects on other ranchers of his cattle grazing the "commons". Degradation of the commons effects all ranchers and is thus a negative externality. If the commons is divided between all the ranches such that each gets a private ranch, then fences must be built to prevent cattle straying between the areas. But if there are many small ranches building the necessary fences may be too costly and the property right may not be effectively established.

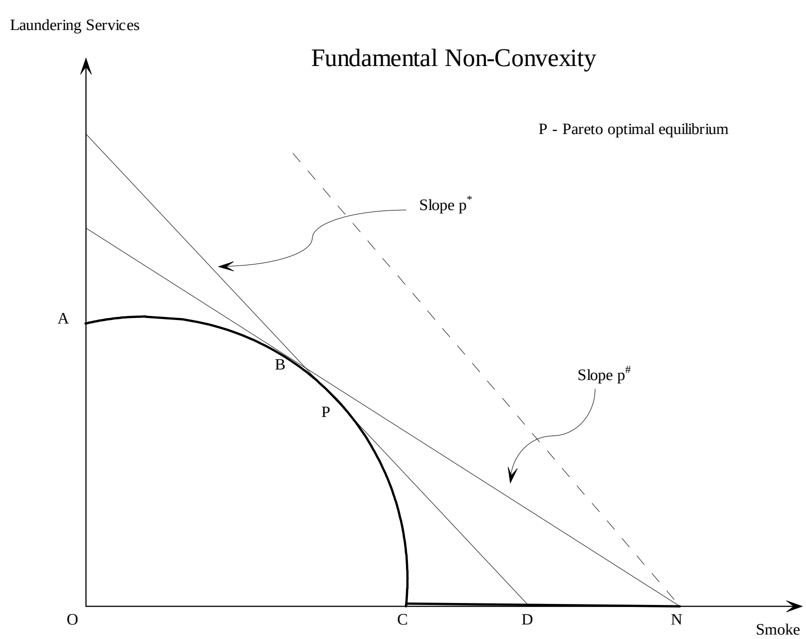
Property Rights may not be Enough: Non-Convexities.

It is possible to show that even if

- property rights are well established
- a competitive market equilibrium potentially exists
- in the competitive equilibrium price ratios between all goods satisfy the marginalist conditions for efficiency

It may be the case that in the presence of an externality some agents will find it in their private best interests to close down some markets. This may be because

1. Fixed costs make the production of some goods unprofitable.
2. Even if production of some goods is profitable, if they have externalities attached to them it might be more profitable to sell off the rights to the externalities and cease production.



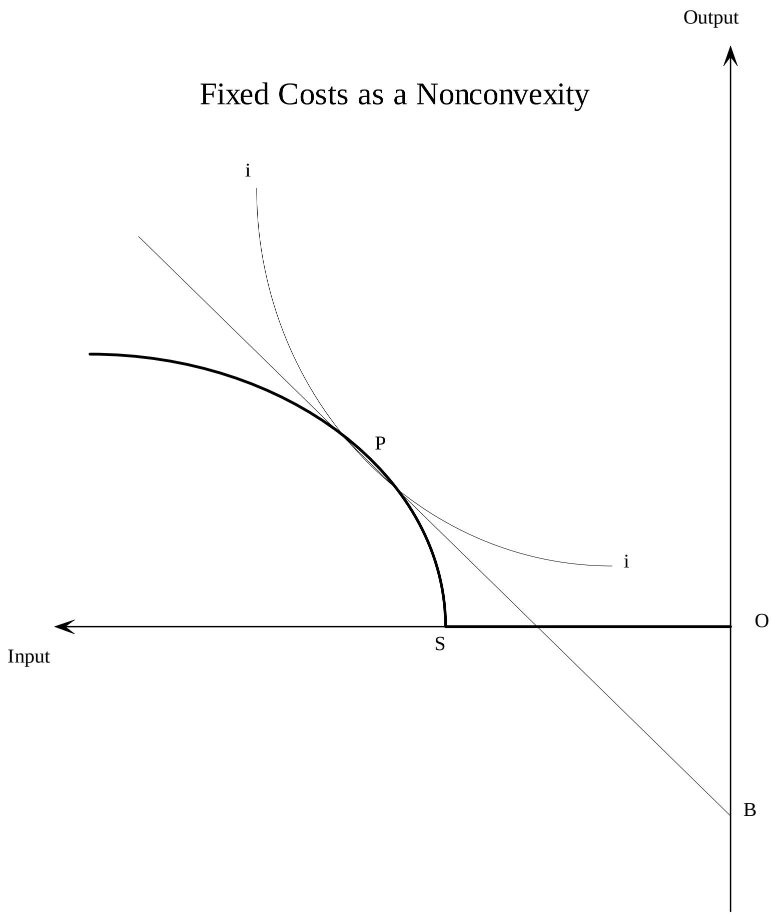
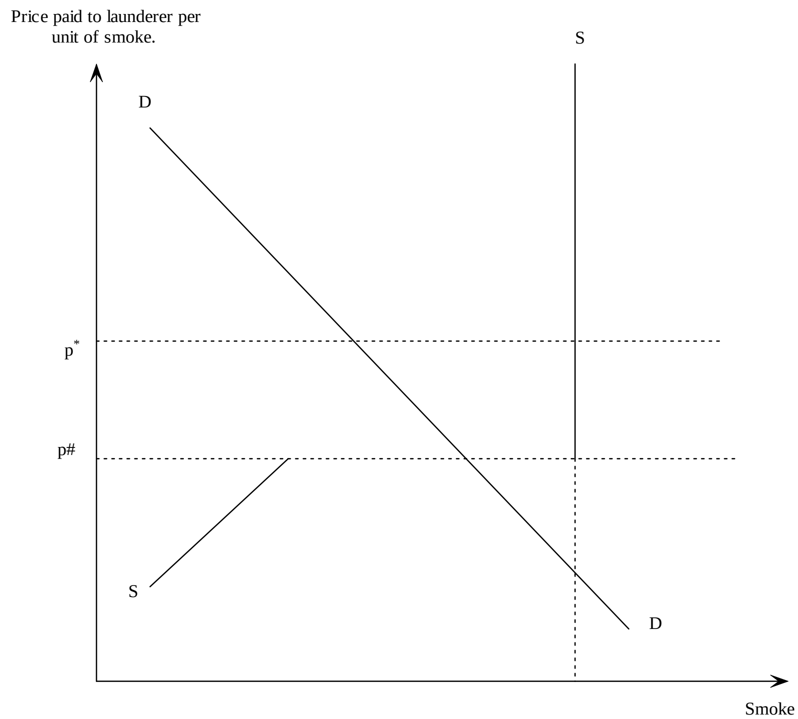


Figure 1



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where Π is a constant and $e(x)$ is an increasing convex function that captures the negative effect of banjo sales on the book producers profits. (Note this is Varian's simple model with the addition of a constant).

In the absence of any corrective measures the banjo manufacturer chooses to

$$\underset{x}{Max} rx - c(x)$$

the first order condition to which defines the privately optimal action $\widetilde{x}$ as satisfying

$$r - c'(\widetilde{x}) = 0$$

Efficiency requires the banjo producer maximizes profit taking into account the full costs of his actions or

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the first order condition to which defines the optimal action x^*as satisfying

$$r - c'(x^*) - e'(x^*) = 0$$

The Pigouvian solution then requires setting a pigouvian tax p such that $p = e'(x^*)$. the banjo manufacturers profit maximization problem becomes

$$\underset{x}{Max} rx - c(x) - px$$

and we immediately get

$$r - c'(x^*) - p = 0 = r - c'(x^*) - e'(x^*)$$

and efficiency is obtained.

Problems with the Pigouvian Solution. To set $p = e'(x^*)$ it is necessary that the government know $e'(x^*)$. This is difficult for the government to know,

1. The government may not be able to observe the effect $e(x)$.
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Remark 37 *The Nash bargaining solution involves the two bargainers maximizing the product of their joint surplus over and above that which they could receive in the absence of agreement. It can be justified via a bargaining process of alternating concessions where each agent concedes to the other until further concessions on their part are more costly than concessions by the other agent.*

1. Property rights allocated to the banjo producer.

Here the outcome (*threat points*) in the absence of agreement is

$$\widetilde{\pi}_1 = r\widetilde{x} - c(\widetilde{x})$$

and

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the Nash bargaining solution involves

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where A is a *side payment* agreed between the two bargainers. The first order conditions are

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$$\left[e(\widetilde{x}) - e(x) - A\right] - \left[rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})\right] = 0 \quad (4.2)$$

from (2) we have

$$\left[e(\widetilde{x}) - e(x) - A\right] = \left[rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})\right] \quad (4.3)$$

substituting (3) into (1) gives

$$\left[r - c'(x) - e'(x)\right]\left[e(\widetilde{x}) - e(x) - A\right] = 0$$

dividing both sides by $\left[e(\widetilde{x}) - e(x) - A\right]$ gives

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hence we have efficiency when the property rights are allocated to the banjo producer.

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If the book producer were allocated the property rights then prior to any bargaining the banjo producer would be denied the right to produce hence the outcome in the absence of agreement is given by

$$\overline{\pi}_1 = 0$$

and

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the Nash bargaining solution now involves

$$\begin{aligned} & \underset{x,B}{Max}(\pi_1 - \overline{\pi}_1)(\pi_2 - \overline{\pi}_2) \\ = & \ [rx - c(x) - B]\,[\Pi - e(x) + B - \Pi] \\ = & \ [rx - c(x) - B]\,[B - e(x)] \end{aligned}$$

notice that the side payment B is now a payment from the banjo producer to the book producer. The first order conditions are

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hence we have efficiency when the property rights are allocated to the book producer.

Problems with Coasian Solutions. As we previously noted the problems with Coasian solutions are twofold.

1. Everyone will want to obtain the initial property rights so establishing them is not a trivial problem.

2. In some cases, perhaps due to non-convexities, the market solution will be a corner solution rather than the interior pareto efficient solution.

Varian's Solution - Compensation Mechanisms.

Suppose we are in situation where a government does not have the information necessary to design pigouvian taxes, nor is it able (perhaps due to political constraints) to adopt a Coasian solution. Varian suggests that if the agents involved in the externality problem have full information concerning its causes and effects then the government may exploit this to achieve efficiency.

The mechanism consists of two stages

Announcement Stage: Each firm simultaneously announces a pigouvian tax. Firm 1's announcement is p_1, firm two announces p_2.

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The model is solved for the *subgame perfect* equilibrium. That is we solve for the last stage first, then solve for the first stage given how the last stage depends on the first. Consider first the choice stage.

Firm 1 maximizes

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notice that this implies

Firm 2 is passive in the choice stage.$x = x(p_2)$ with $x'(p_2) < 0$.

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so the first order condition is

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Since $x'(p_2) \neq 0 \Rightarrow p_1 = e'(x)$ since $p_1 = p_2 \Rightarrow r = c'(x) + p_2 = c'(x) + p_1 = c'(x) + e'(x)$ which is the condition for efficiency.

Remark 38 *The intuition here is that firm 1 always wishes to match firm 2's announcement. Firm 2 can manipulate x via its announcement p_2, hence it will always choose p_2 so that $p_1 = e'(x)$.*

Problems with Varian's mechanism. Given the firms have full information why doesn't firm 2 figure out that firm 1 always plays $p_1 = p_2$ and manipulate its own profits via

$$\Pi_2 = \Pi + p_2 x(p_2) - e(x(p_2))$$

this is a deviation from Nash behavior, but it might be appropriate.

The Ellis - van den Nouweland Mechanism.

Suppose the two firms of our previous example are owned by $N_1$and N_2 shareholders respectively, then in the Nash equilibrium each share earns

$$\frac{1}{N_1}\left[r\widetilde{x} - c(\widetilde{x})\right]$$

and

$$\frac{1}{N_2}\left[\Pi - e(\widetilde{x})\right].$$

Suppose that the property rights system is reformed such the $S = N_1 + N_2$ shares are now acceptable as claims on the profits of either (but only one) firm. Let s_1 be the shares that make claims on firm 1 and $S - s_1$ make claim on firm 2. It follows that the S shares will be allocated such that they earn the same return everywhere, so

$$\left(\frac{1}{s_1}\right)\left[rx - c(x)\right] = \left(\frac{1}{S - s_1}\right)\left[\Pi - e(x)\right]$$

which may be rewritten

$$(S - s_1)\left[rx - c(x)\right] = (s_1)\left[\Pi - e(x)\right]$$

This may be rewritten

$$s_1 = \frac{S\left[rx - c(x)\right]}{\Pi - e(x) + rx - c(x)}$$

Firm 1 now maximizes

$$\underset{x}{Max}\left(\frac{1}{s_1}\right)\left[rx - c(x)\right]$$

subject to

$$s_1 = \frac{S\left[rx - c(x)\right]}{\Pi - e(x) + rx - c(x)}$$

substituting into the objective function from the constraint gives

$$\begin{aligned}
\underset{x}{Max}\left(\frac{1}{s_1}\right)\left[rx - c(x)\right] &= \left(\frac{\Pi - e(x) + rx - c(x)}{S\left[rx - c(x)\right]}\right)\left[rx - c(x)\right] \\
&= \left(\frac{\Pi - e(x) + rx - c(x)}{S}\right)
\end{aligned}$$

the first order condition to which is

$$\frac{1}{S}\left[x - c'(x) - e'(x)\right] = 0 \Rightarrow x - c'(x) - e'(x) = 0$$

and we have efficiency.

Remark 39 *The intuition is that each firm knows profits will be arbitraged, so the only way they can maximize their own profits per share is to maximize total profits, but this requires they internalize the externality.*

Problems with the Ellis van den Nouweland mechanism. It only works if you can set up the share system as described.

4.2 Public Choice.

Public choice is the study of situations that require *collective action* where a good or service provision level need to be jointly agreed upon. Obvious examples include provision of public goods such as legal systems, police, defence, pollution abatement, and the like. The need for collective action stems from the standard *prisoners dilemma* problem..

		Player #1.	
		No-Cooperation	Cooperation
Player #2	No-Cooperation	5,5	7,2
	Cooperation	2,7	6,6

We see immediately that no-cooperation is a *dominant strategy*, hence no-cooperation is both a *dominant strategy equilibrium* and the unique *Nash equilibrium*. There is a need for some form of collective action to achieve cooperation. The way in which societies often determine whether or not to engage in collective action is to put the option up to a vote thus we investigate voting theory. There are essentially two strands to this area, unanimous voting rules and majority rules.

4.2.1 Unanimity Rules.

Unanimity rules require every voter agree on a decision. To understand these rules Mueller's diagram is very useful

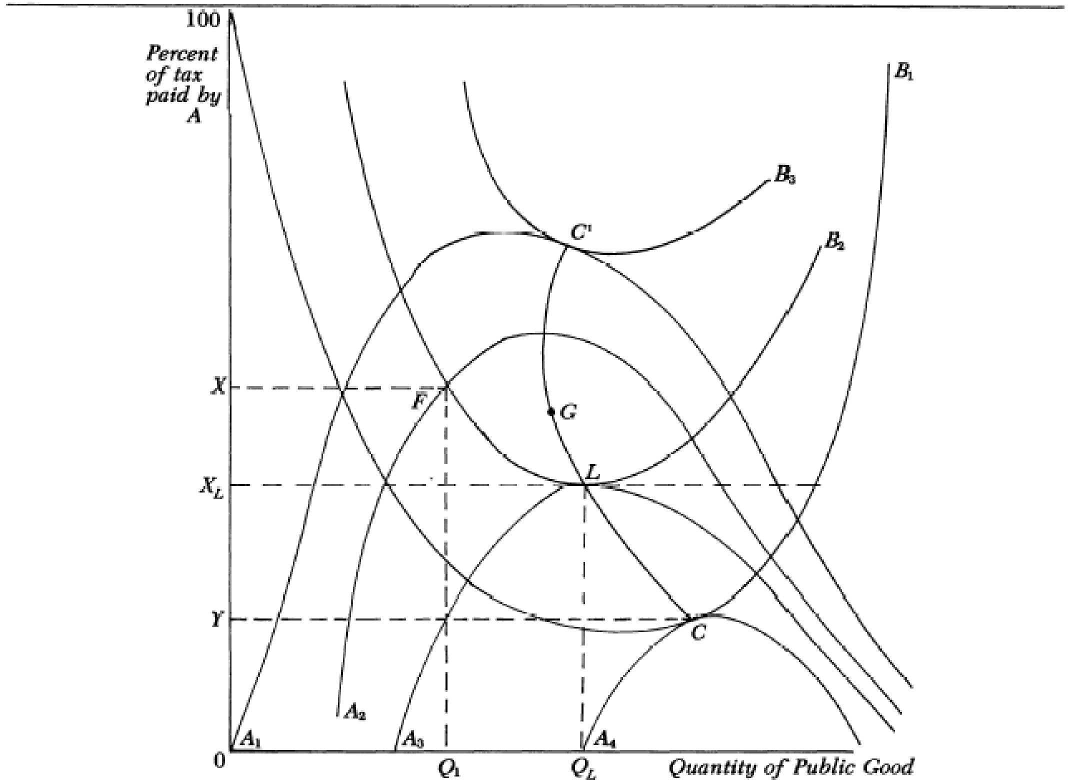


Figure 1.

On the diagram two individuals A and B who vote over the levels of public good provision when faced by different shares of the costs. The vertical axis represents the share of the cost of the provision of a public good borne by individual A, 100-A is the share borne by B, the horizontal axis gives the quantity of the public good.. We begin our analysis by assuming that F is the status quo, this is clearly inefficient at it des not lie on the contract curve CC'. Next consider voting equilibria.

Pairwise Voting.

Suppose that a government were to hold a sequence of votes between the status quo and a proposed alternative. Any point proposed that lies in the lens originating from F would be preferred to F and receive the votes of all the voters. The new point would become the status quo, a new lens would emerge and a new vote held, again the status quo would be unanimously beaten . The process would continue until no new point could be found that would unanimously defeat the status quo. This would be a pairwise voting equilibrium, and would be pareto efficient. Any point on the contract curve within the original lens represents a potential outcome. It follows that where the equilibrium finally occurs lies partially at the discretion of whoever proposes the alternatives. Indeed as we can demonstrate while the outcome is an equilibrium under unanimity, the individuals would possibly still each prefer a different level of public good provision given their tax shares.

Lindahl Voting.

With Lindahl voting the participants are faced with a personalized tax share X, each then votes on the quantity of the public good they desire. If they vote for the same level of provision there is unanimity, if not then a new set of taxes is proposed and they vote again. On the diagram there is a unique Lindahl equilibrium at L

Problems

1. Pairwise voting does not lead to unique outcome.
2. Both pairwise and Lindahl mechanisms might be subject to strategic manipulation.
3. If voting is costly there is a free rider problem, why bother to vote. This gets worse as the number of participants increases. Each individuals vote faces a smaller chance of being crucial and thus it takes less to discourage them from voting.

4.2.2 Majority Voting.

In the real world a large number of different voting procedures are adopted to make decisions, these each involve some form of majority voting. Quite commonly the options are considered pairwise with the winner determined by a simple majority. As can be seen from the following examples this potentially leads to problems

Single and Multiple Peakedness: Pairwise voting with a simple majority.

Consider the following preferences

		Options		
		A	B	C
Players	Fred (Brian)	1	2	3
Rankings	Brian	3	1	2
	Melissa (Brian)	3	2	1

these preferences are single peaked and lead to the simple result that B beats both A and C and is a clear winner. Preferences here are single peaked.

Now suppose

		Options		
		A	B	C
Players	Fred (Brian)	1	2	3
Rankings	Brian	2	3	1
	Melissa (Brian)	3	1	2

Now preferences are not single peaked (Brian is causing trouble again!!). A beats B, C beats A, B beats C!!* This leads to

1. Voting cycles - no clear result obtains.

2. Agenda manipulation - the sequence of voting determines the outcome.

Conclusion - We might want to consider other forms of majority voting. One conclusion that is often reached when problems of this type arise is that the problem is that individuals are unable to express the strength of their preferences. That a candidate receives a lot of second place votes counts for very little. The following voting procedure "solves" this problem.

Borda's Rule.

Each candidate picks up points from their position in the ranking of each voter. If a candidate is ranked last 0 points are received, last but one 1 point is received and so on. Consider the following example

	Rankings			
	bca	acb	cba	abc
Number of individuals	7	7	6	1

Borda scores are as follows

a gets $(7 \cdot 2) + (1 \cdot 2) = 16$

b gets $(7 \cdot 2) + (6 \cdot 1) + (1 \cdot 1) = 21$

c gets $(7 \cdot 1) + (7 \cdot 1) + (6 \cdot 2) = 26$

Hence c wins according to the Borda rule. Notice that under a standard one vote plurality rule the votes are b=7, a=8, and c=6 and a wins!!

A Problem with Borda's Rule - The Independence of Irrelevant Alternatives (IIA). Consider the following

	Number of votes for each set of preferences					
	30	1	29	10	10	1
Preferences	Brian	Brian	Ghandi	Ghandi	Reid	Reid
over	Ghandi	Reid	Brian	Reid	Brian	Ghandi
candidates	Reid	Ghandi	Reid	Brian	Ghandi	Brian

according too the Borda rule

Brian gets $(30.2) + (1.2) + (29.1) + (10.1) = 101$

Ghandi gets $(30.2) + (29.2) + (10.2) + (1.1) = 139$

*Thanks Brian.

Reid gets (1.1) + (10.1) + (10.2) + (1.1) = 32

According to the Borda rule the correct ordering of candidates is Ghandi, Brian, Reid, but if we take a pairwise comparison we get
Brian vs Ghandi - Brian wins 41-40,
Brian vs Reid - Brian wins 60-21.

Problem: *Brian beats them both head to head!.* The key to the problem involves the *Independence of Irrelevant Alternatives* which states that in making comparisons between any two options only those options should matter, everything else should be treated as irrelevant. In our example Reid is considered inferior to both Brian and Ghandi, so why should the choice between them depend in any way on Reid?

The Problem's Bigger that it Might Appear. Not only does the *Independence of Irrelevant Alternatives* cause problems for Borda's rule it causes problems for *any scoring rule.* If a scoring rule involves a sequence of real numbers such that $s_1 > s_2 > .... > s_n$ where higher ranked alternatives receive higher scores. Then we can show that the scoring rule will always give poor results.

Consider our example again but let $s_1 = 8, s_2 = 1, s_3 = 0$. According to this new scoring rule
Brian gets (30.8) + (1.2) + (29.1) + (10.1) = 281
Ghandi gets (30.8) + (29.8) + (10.8) + (1.1) = 553
Reid gets (1.1) + (10.1) + (10.8) + (1.1) = 92.
Again Ghandi beats Brian according to the scoring rule but head to head we get the same as before
Brian vs Ghandi - Brian wins 41-40,
Brian vs Reid - Brian wins 60-21.

Why the Independence of Irrelevant Alternatives?

1. If it is not present the outcome can be manipulated by introducing extraneous alternatives. In our example we see that Reid is a "No Hoper" but under a scoring rule he can change the outcome by entering the contest.

2. From a practical point of view it allows decisions to be made over a restricted range of choices, we don't have to consider every alternative. It is thus quick and cheap.

Arrow's Impossibility Theorem.

Unfortunately Arrow has shown that when there are more than two alternatives available every reasonable decision rule sometimes violates the IIA condition. First lets define a reasonable decision rule by giving some properties that we think one should possess. Suppose we have two individuals Andy and Ghandi, who choose between three options A,B, and C. We require that any ranking of the options A, B, C satisfy the following axioms

1. Completeness: Either $A \succ B$, $B \succ A$, or AIB.
2. Transitivity: If $A \succ B$ and $B \succ C$ then $A \succ C$.
3. If $A \succ B$ for both Andy and Ghandi then the ranking must rank A ahead of B.
4. IIA: If $A \succ B$ and $B \succ C$ then these do not change simply because some new option D appears.
5. The ranking should be derived from the preferences of the individuals.
6. No dictatorship: No one individuals preferences may determine societies preferences.

Suppose now for our two individuals we have the following preferences
Andy: $A \succ_a B \succ_a C$.
Ghandi: $C \succ_g A \succ_g B$.

We shall show that we cannot obtain a social ranking of these options and not violate one of Arrow's axioms.

- $B \succ_a C$ and $C \succ_g B$ it must be the case socially that CIB or we would violate axiom 6 and have dictatorship.
- Since $A \succ_a B$ and $A \succ_g B$ it must be the case by axiom 3 that for a social preference ranking $A \succ B$.
- So applying axiom 2. transitivity we get $A \succ BIC \Rightarrow A \succ C$ but this violates the non-dictatorship axiom 6.

Conclusion: We have to give up an axiom.

Condorcet Winners.

The idea behind Condorcet's approach is that there is a best outcome to any voting situation and that individuals will on average know what that best out come is. Consider the following situation Two candidates, George and Al, run for political office, each promises to build a new road across the country, each claims to be able to organize the project more efficiently that the other. For simplicity we shall assume there are two voters the famous Brian twins, we now that each Brian is able to correctly identify the best candidate 60% of the time. We observe that both Brians vote for George who then obtains a simple majority (50%+1 here is 2, and note that this also implies for Al to win requires both Brians vote for him.). We now ask what is the likelihood that the Brians voted correctly and the correct winner is chosen by majority voting. We do this by first computing conditional probabilities

1. The probability that both Brians choose George given that George is indeed the best candidate.

$$P(\text{Brians 1 and 2 choose George}) = (0.6)\,(0.6) = 0.36$$

2. The probability that both Brians choose Al given that George is indeed the best candidate.

$$P(\text{Brians 1 and 2 choose Al}) = (0.4)\,(0.4) = 0.16$$

So the choice of George is $\frac{0.36}{0.16} = 2.25$ more likely when George is the best choice. This is called the likelihood ratio. Indeed it can be shown that with a large population (many Brians) it is only required that the probability that each individual makes the correct decision be slightly over 50% for the odds of the correct individual winning to be come very large (approach 1 as the size of the population goes to infinity).

Three of More Alternatives. Suppose the objective is to reduce traffic congestion in Eugene, the options are

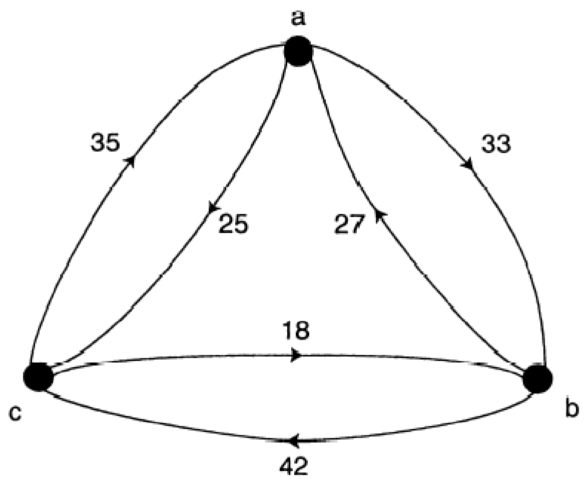
a. Supply more busses.

b. Build more roads.

c. Make it more difficult to obtain a driving licence, by requiring greater testing.

The question then is which of these alternatives is the most effective per dollar. The votes of the voters are given by the following graph.

A Vote Graph



Each vertex on the graph represents a number of votes, for example the arrow from $a \to b$ represents 33 votes for a over b, while b has 27 votes over a. To calculate the pairwise support for the ranking abc we compute $a \to b$ plus $b \to c$ plus $a \to c$ so $33 + 42 + 25 = 100$.

Hence

abc	100	bca	104
acb	76	cab	86
bac	94	cba	80

Using his probabilistic method Condorcet showed that the ranking that is most likely to be correct is the one with maximal pairwise support. In this case bca this solution is known as *Condorcet's rule of three.* To see that this is true let's compute the relative likelihoods that each ranking is correct. Let $p > 1/2$ be the probability the an individual voter chooses correctly.

1. abc - $p^{33}(1-p)^{27} \times p^{42}(1-p)^{18} \times p^{25}(1-p)^{35} = p^{100}(1-p)^{80}$
2. bca - $p^{42}(1-p)^{18} \times p^{35}(1-p)^{25} \times p^{27}(1-p)^{33} = p^{104}(1-p)^{76}$
3. acb - $p^{25}(1-p)^{35} \times p^{18}(1-p)^{42} \times p^{33}(1-p)^{27} = p^{76}(1-p)^{104}$
4. cab - $p^{35}(1-p)^{25} \times p^{33}(1-p)^{27} \times p^{18}(1-p)^{42} = p^{86}(1-p)^{94}$
5. bac - $p^{27}(1-p)^{33} \times p^{25}(1-p)^{35} \times p^{42}(1-p)^{18} = p^{94}(1-p)^{86}$
6. cba - $p^{18}(1-p)^{42} \times p^{27}(1-p)^{33} \times p^{35}(1-p)^{25} = p^{80}(1-p)^{100}$

Since $p > 1/2$ it immediately follows that the ranking is $bca \succ abc \succ bac \succ cab \succ cba \succ acb$.

Condorcet and Arrow's axioms. By definition Condorcet's rule does not satisfy all of Arrow's axioms (that has been shown to be impossible) but how close does it come? Surprisingly quite close. It satisfies a weakened version of the IIA called the local independence of irrelevant alternatives. LIIA. Consider the following

30	1	29	10	10	1
Kevin	Kevin	Ty	Ty	Reid	Reid
Ty	Reid	Kevin	Reid	Kevin	Ty
Reid	Ty	Reid	Kevin	Ty	Kevin

clearly the real contest here is between Kevin and Ty, Reid has far fewer first place votes and many more last place votes. Notice that Kevin and Ty are not separated by any other candidate in the rankings, they are in the same interval. The LIIA asks only that IIA holds within that interval. That is the ranking *within* the interval should be invariant to what happens *outside* that interval. It can be shown that Condorcet's method has this property.

4.3 The Economics of the Family.

4.3.1 The Rotten Kid Theorem.

The rotten kid theorem is an example of a two stage mechanism that has implications way beyond the original interpretation given it by originator Gary Becker.

Theorem 40 *If a family consists of an altruistic head who transfers income to the other family members who are selfish, then each member of the family will seek to maximize family income. That is each will internalize all externalities between them and act efficiently.*

To see how the theory works consider the following. Suppose that each family member i consumes a single consumption good X_i. All family members except the head maximize their own selfish consumption, the head is altruistic and cares only about the utility of the other family members. The heads utility is written

$$U(X_1, ..., X_n)$$

the family members utilities are

$$V\left(X_i\right)\ \ i = 1, .., n$$

and let I_i be the income of family member i The families budget constraint must satisfy (all prices are assumed to be unity)

$$\sum_i X_i = \sum_i I_i$$

If we now assume all goods X_i are normal goods (that is the head consumes more of each as income increases) then each family members utility will be increasing in total family income , and each will have an incentive to maximize total family income. This proves the theorem.

If we substitute Federal Government or European Parliament for altruistic head, and member states for family members we see just how powerful this theorem might be.

Problems with the Rotten Kid Theorem.

Lazy Rotten Kids. Asymmetric Information.

The rotten kid theorem fails to apply when there is asymmetric information. Let each family members income be a function $I_i\left(Y_i\right)$ and let the parent act as before transferring income to the children so as to maximize utility subject the family budget constraint. Then a selfish child will have insufficient incentive to work as she receives only a share of the income she generates but incurs the full disutility of working.

The Parent's Utility Depends on the Children's Utility.

Suppose instead the parent can observe effort, and his utility depend on their utilities rather than their consumption. We can see from the following example that problems again arise. Let the head of household have two children, Bart and Lisa, each child has the utility function

$$U_i = X_i (1 - Y_i)$$

which is maximized subject to

$$X_i = wY_i + T$$

where T is the transfer from the parent, so

$$U_i = (wY_i + T)(1 - Y_i)$$

with FOC

$$\frac{\partial U_i}{\partial Y_i} = w(1 - Y_i) - (wY_i + T) = 0$$

so

$$Y_i = \frac{w - T}{2w}$$

We immediately see that individual effort is decreasing in the transfer the child receives from the parent.

$$\frac{\partial Y_i}{\partial T} = -\frac{1}{2w} < 0$$

Hence the incentive problem is clearly exacerbated by the fact that the parent can observe effort.

The Case of the Controversial Night Light. In this example the general applicability of the Rotten Kid Theorem is examined by introducing a public good into the model. Suppose an altruistic husband gives gifts to a selfish wife, but also likes to read in bed. The wife likes the gifts, but dislikes the night light to husband uses to read by. Now suppose an electrician stops by the house in the husbands absence and offers to discretely disconnect the night light for the wife. According to the RKT the wife should decline the offer since reducing the husbands utility will effectively reduce his "full income" and hence the gifts that he gives to her. Following Bergstrom we shall show that she has the light disconnected!!

Let X_h, X_w be the consumption of a private good consumed by the husband and wife. Let Y be the number of hours the husband reads in bed (the night light is on). The preferences of the husband and wife are given by

$$U_h = X_h(Y + 1)(U_w)^a$$

where $0 < a < 1$ and

$$U_w = X_w e^{-Y}$$

Hence the altruistic husband maximizes

$$U_h = X_h(Y+1)\left(X_w e^{-Y}\right)^a = X_h X_w^a (Y+1) e^{-aY}$$

subject to the constraint

$$X_h + X_w = I$$

forming the Lagrangian we get

$$Max\ X_h X_w^a (Y+1) e^{-aY} + \lambda \left[I - X_h - X_w\right]$$

with FOC

$$\begin{aligned} \frac{\partial \ell}{\partial X_h} &= X_w^a (Y+1) e^{-aY} - \lambda = 0 \\ \frac{\partial \ell}{\partial X_w} &= a X_h X_w^{a-1} (Y+1) e^{-aY} - \lambda = 0 \\ \frac{\partial \ell}{\partial Y} &= X_h X_w^a e^{-aY} - a X_h X_w^a (Y+1) e^{-aY} = 0 \end{aligned}$$

so rearranging and dividing the first FOC by the second yields

$$X_w = aX_h$$

substituting this into the budget constraint yields.

$$X_h + aX_h = I$$

so

$$\begin{aligned} X_h &= \frac{I}{1+a} \\ X_w &= \frac{aI}{1+a} \end{aligned}$$

From the third FOC we have

$$1 - a(Y+1) = 0$$

So

$$Y = \frac{1-a}{a}$$

which is efficient, *but*, since X_w is independent of Y *the wife has the night light turned off.*

4.4 The Theory of Marriage.

Based on two principles.

1. Marriage is voluntary (at least between parents) so it must represent a pareto improvement that can be analyzed using standard preference theory.

2. Since men and women compete for mates a market for marriages can be seen to exist.

4.4.1 The Gains from Marriage.

Consider two individuals Ken and Barbie who must decide whether to marry or not. Their marriage is voluntary so will only take place if they both benefit. Either as singles or a family Ken and Barbie engage in household production, that is they combine their time and consumption goods to produce commodities they consume. For example, time plus a car plus a swimming costume may be combined to produce a day at the beach, or, time plus food may be combined to produce a meal.

Ken and Barbie are Married.

Assume that household commodities may be combined into a single household good denoted Z, which is produced using time t_j, market goods x_i and v is non-labor income.

$$Z = f(x_1, .., x_m, t_1, ..., t_n)$$

A household budget constraint is then written

$$\sum_i p_i x_i = \sum_j w_j l_j + v$$

where w_j is the wage earned by the jth household member and l_j is the labor they sell on the market sector. Each individual time constraint may be written

$$t_j + l_j = T$$

where T is total time (24hrs). Combining the two constraints we get the *full income* constraint.

$$\sum_i p_i x_i + \sum_j w_j t_j = \sum_j w_j T + v = S$$

which is the maximum money achievable by the household. Note that this can be spent on t_j time spent on non-market activities.

We assume that an increase in Z makes no family member worse off, hence each will cooperate to maximize Z. We can now analyze the household optimization problem using the standard tools of consumer theory. The usual conditions equating marginal rates of substitution to price ratios then characterize the optimum. For the Ken and Barbie this translates into

$$\frac{MP_{t_k}}{MP_{t_b}} = \frac{\frac{\partial Z}{\partial t_k}}{\frac{\partial Z}{\partial t_b}} = \frac{w_k}{w_b}$$

$$\frac{MP_{x_i}}{MP_{t_b}} = \frac{p_i}{w_b}$$

If $w_k > w_b$ and $MP_{t_b} \geq MP_{t_k}$ when $t_f = t_b$ then more time would be allocated to the market sector by Ken than Barbie. Barbie would specialize in non-market production ($l_b = 0$) if $\frac{w_k}{w_b}$ or $\frac{MP_{t_b}}{MP_{t_k}}$ are sufficiently large.

Ken and Barbie are Single.

If they are single the problem is the same except for Ken $T_b = 0$, and for Barbie $T_k = 0$. We write their maximal outputs when single as x

$$Z_{k0}, Z_{0b}$$

where time is allocated optimally between market and non-market activities according to the same principles as for the family.

Are Ken and Barbie about to make a Terrible Mistake?

Our favorite dolls decide to marry!! Are they making a terrible mistake. Let's take a look.

Write m_k and m_b as the incomes our two plastic lovers enjoy in marriage, it follows that a necessary condition (but not sufficient ....as we shall see later Barbie and Ken are playing the field) fro them to marry is

$$\begin{aligned} m_k &\geq Z_{k0} \\ m_b &\geq Z_{0b} \end{aligned}$$

If follow that marriage will only occur if

$$m_k + m_b = Z_{mf} \geq Z_{k0} + Z_{0b}$$

one the obvious question is when will this condition hold. When can the two together earn more that two singles. Clearly if the time of one spouse is a perfect substitute for the time of the other then there is no gain from marriage. Each single is half a marriage. I on the other hand Ken and Barbie time contributions are not perfect substitutes for each other then marriage is a good idea. This can easily be seen if

$$Z = f(x_1, .., x_m, t_1, ..., t_n) = x^{\alpha} t_k^{\beta} t_b^{\gamma}$$

here clearly

$$Z_{k0} = Z_{0b} = 0 \; if \; t_k = 0 \text{ or } t_b = 0$$

So when is it more likely that Marriage is a good idea for Ken and Barbie?

1. The greater are the complementaries between the two spouses time contributions. E.g. the more important are children.

2. The greater is property income.

3. The greater are wage rate (typically).

4. The more different are w_b and w_k.

4.4.2 The Marriage Market.

Optimal Sorting.

Obviously individuals don't just marry the first partner for whom $m_k + m_b = Z_{kb} \geq Z_{k0} + Z_{0b}$, this may not be their best option. They wish to find the best partner. Suppose now that both Ken and Barbie have many (n) potential mates. How did they come to choose each other? They considered the following payoff matrix.

	F_1	......	F_n	
M_1	Z_{11}	......	Z_{1n}	Z_{10}
	......	......	......	
M_n	Z_{n1}	.....	Z_{nn}	Z_{n0}
	Z_{01}		Z_{0n}	$\times$

where M =male, F =female. Each individual has $n + 1$ opportunities including remaining single. Further there are $n!$ ways to sort the individuals into pairs. Total output over all marriages for any giving sorting may be written

$$Z^h = \sum_{i \in M,, j \in F} Z_{ij} \; h = 1, ..., n!$$

number a sorting such that total output is maximized and lies on the diagonal and write

$$Z^* = \underset{h}{Max} \; Z^h$$

Now total output is divided between the mates so that

$$m_{ij} + f_{ij} = Z_{ij} \; \forall ij$$

If each mate chooses the partner that maximizes their income then optimal sorting will be pareto efficient. No individual can break away from the sorting and find a different mate and a division of their output such that they both prefer this to their current sorting. Surprisingly this end up maximizing the *total combined output* from all marriages

Example 41 *Suppose there are two males and two females who share the payoff matrix*

	F_1	F_2
M_1	8	4
M_2	9	7

We see that the maximum output from any marriage is $\{F_1, M_2\} = 9$, *but this leaves the other marriage to be* $\{F_2, M_1\} = 4$,*giving overall output* $\{F_1, M_2\} + \{F_2, M_1\} = 4 + 9 = 13$. *This is not an optimal sorting since* $\{F_1, M_1\} + \{F_2, M_2\} = 8 + 7 = 15$. *Suppose* $m_{11} = 3$, $f_{11} = 5$, $m_{22} = 5$, $f_{22} = 2$. *Then* M_2 *and* F_1 *have no incentive to marry since* $m_{22} + f_{11} = 5 + 5 = 10 > 9$, *and neither do* M_1 *and* F_2 *since* $m_{11} + f_{22} = 3 + 2 = 5 > 4$. *Thus the players will choose the optimal sorting.*

This can literally be thought of as a market, where one spouse offers the other a wage f_{ij} and receives the residual profits $m_{ij} = Z_{ij} - f_{ij}$, the spouse with the best match will be able bid highest, thus the sorting will be overall profit maximizing just like any other market.

Assortive Mating.

This involves sorting on a trait, that is do similar or dissimilar individuals mate? Becker's analysis tells us that depends on which maximizes household commodity output.

Assume that males differ in the characteristic A_m, while females differ on the characteristic A_f, and that each trait has a monotonically increasing effect on the value of any marriage. That is

$$\frac{\partial Z_{ij}(A_m, A_f)}{\partial A_m} > 0, \frac{\partial Z_{ij}(A_m, A_f)}{\partial A_f} > 0$$

1. Dissimilar Individuals Marry - If increasing both A_m and A_f adds more to output than the sum of the separate additions (increasing returns to the traits).

2. Similar Individuals Marry - If increasing both A_m and A_f adds less to output than the sum of the separate additions (decreasing returns to the traits).

Mathematically this states that positive or negative assortive mating will occur as

$$\frac{\partial^2 Z_{ij}(A_m, A_f)}{\partial A_m \partial A_f} \gtrless 0$$

Assortive of likes is optimal when the traits are complements and assortive of unlikes is optimal when the traits are substitutes.

Example 42 *Consider the payoff matrix*

	A_1	A_2
A_1	Z_{11}	Z_{12}
A_2	Z_{21}	Z_{22}

, with $A_2 > A_1$

If $Z_{22} - Z_{12} > Z_{21} - Z_{11}$*, and if* $\frac{\partial^2 Z_{ij}\left(A_m, A_f\right)}{\partial A_m \partial A_f} > 0$*, then* $Z_{11} + Z_{22} > Z_{12} + Z_{22}$*, and a positive correlation between* A_m *and* A_f *maximizes total output.*

4.5 Crime and Punishment.

We do not assume that criminals are any different from other members of society, they are criminals because this is where their comparative advantage lies. They are either very good at being criminals or very bad at being anything else. Crime may be analyzed using standard the tools of economic analysis. Criminals supply crime and the criminal justice system demands it!!

4.5.1 A Model of Crime.

Damages.

Let O be the number of criminal offences committed, these offences harm society according to the function $H(O)$ with $H'(O) > 0$ and $H''(O) > 0$ hence the harm from crime increases at an increasing rate. The perpetrators of crimes gain from them according to the function $G(O)$ with $G'(O) > 0$ and $G''(O) < 0$, thus there are positive but diminishing returns to criminal activity. The net damage to society from O offences is given by

$$D(O) \equiv H(O) - G(O)$$

it is assumed that

$$D'(O) \equiv H'(O) - G'(O) > 0 \ \forall O \geq O_a$$

and we have

$$D''(O) \equiv H''(O) - G''(O) > 0$$

so the net damage due to criminal activity is an increasing function of the level of that activity at least once the number of offences exceeds the threshold O_a.

The Cost of Apprehension and Conviction.

Denote as A the level of activity in detecting and prosecuting offenders, this includes both the costs of the police and the judicial system and is written $C(A)$ we assume $C'(A) > 0$. One measure of A is the number of offences convicted, thus if p is the frequency of conviction then

$$A \simeq pO$$

we may thus write

$$\begin{aligned} C_p &= \frac{\partial C(pO)}{\partial p} = C'O > 0 \\ C_O &= \frac{\partial C(pO)}{\partial O} = C'p > 0 \end{aligned}$$

Costs are increasing in both the number of offences and the probability of any offence being convicted. Further

$$\begin{aligned} C_{pp} &= C''O^2 > 0 \\ C_{OO} &= C''p^2 > 0 \\ C_{pO} &= C_{Op} = C''pO + C' > 0 \end{aligned}$$

The Supply of Offences.

The number of offenses committed depends primarily on the probability of conviction and the penalty incurred if convicted or

$$O = O(p, f)$$

with $O_p < 0$, and $O_f < 0$, that is criminals are deterred by higher penalties or being caught and convicted.

The Social Costs of Punishment.

The cost of punishing offenders often effects others, and may have positive or negative effects.

1. Fines may be used to the benefit of others.

2. Imprisonment requires the use of societies resources.

   We thus write the social cost of punishment as

$$f' = bf$$

whether $b \lesseqqgtr 1$ depends on the particular circumstances.

The Social Optimum.

The welfare to society from crime is measured by the loss function which is assumed to be of the linear form

$$L(D, C, bf, O) = D(O) + C(p, O) + bpfO$$

Since for society as a whole p is the frequency of conviction, so pO is the number of convictions. To maximize welfare society can set the fines f and choose the probability of successful conviction (by applying resources). Recall that $O = O(p, f)$ so the first order conditions for a social optimum are

$$\begin{aligned} \frac{\partial L}{\partial f} &= D'O_f + C_O O_f + bpfO_f + bpO = 0 \\ \frac{\partial L}{\partial p} &= D'O_p + C_O O_p + C_p + bpfO_p + bfO = 0 \end{aligned}$$

dividing the expressions by O_f and O_p respectively gives

$$\begin{aligned} D' + C_O + bpf + \frac{bpO}{O_f} &= 0 \\ D' + C_O + \frac{C_p}{O_p} + bpf + \frac{bfO}{O_p} &= 0 \end{aligned}$$

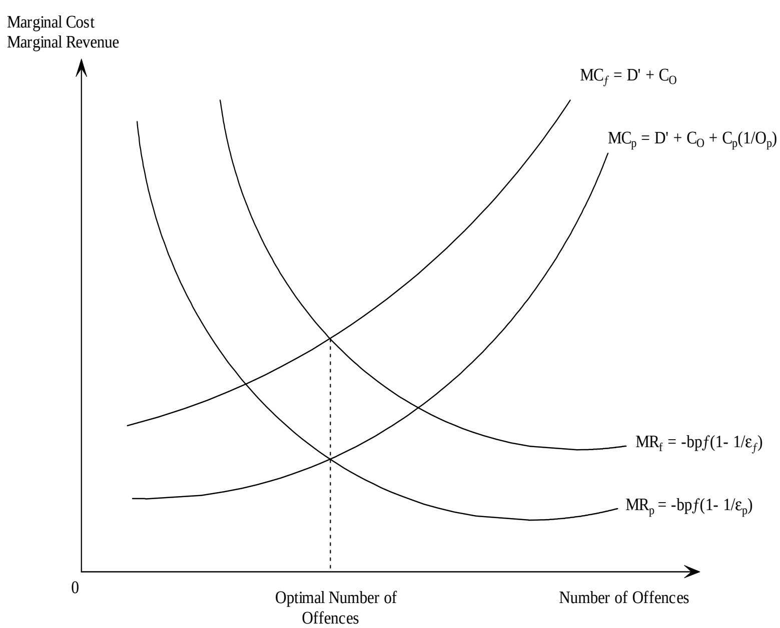
Now if we define the elasticities to commit offences with respect to fines and the probability of conviction as

$$\begin{aligned} \varepsilon_f &= -\frac{f}{O} O_f \\ \varepsilon_p &= -\frac{p}{O} O_p \end{aligned}$$

then we can rewrite the optimality conditions as

$$\begin{aligned} D' + C_O &= -bpf\left(1 - \frac{1}{\varepsilon_f}\right) \\ D' + C_O + \frac{C_p}{O_p} &= -bpf\left(1 - \frac{1}{\varepsilon_p}\right) \end{aligned}$$

the left hand side of each of these expressions represents the marginal cost of using the respective policy variable f or p, the right hand side is the marginal "revenue" (which can be positive or negative dependant on the magnitudes of the elasticities). Taken together these two sets of curves represent the social welfare optimum as seen in the following figure

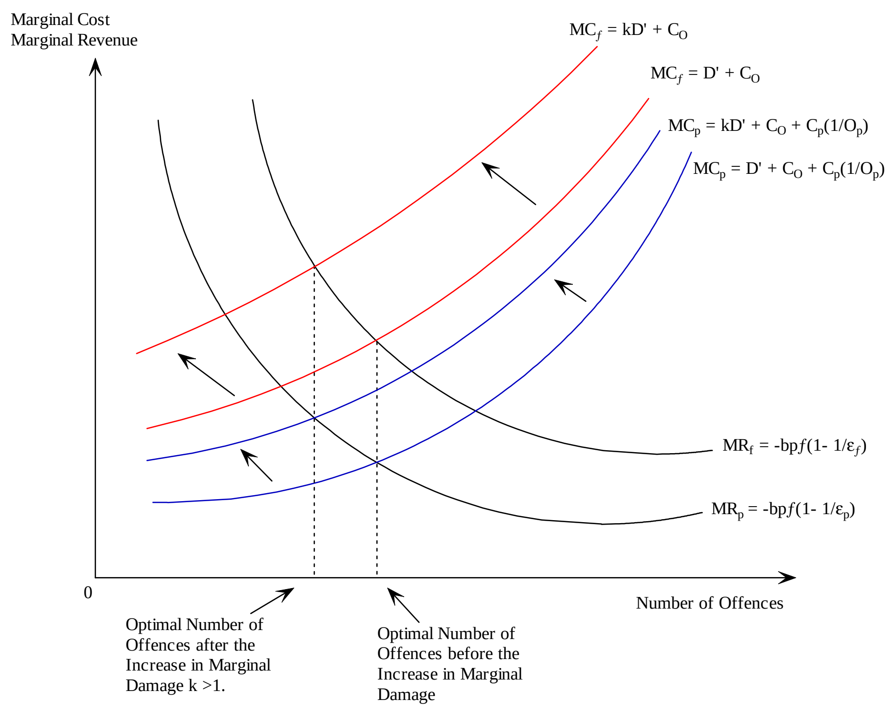


The Socially Optimal Number of Offenses.

Changes in the Optimal Number of Offences.

An Increase in Marginal Damages (D').

Suppose that the marginal damage done to society for any given number of crimes increases. This might be because society becomes less tolerant of crime over time, or perhaps because the type of crimes represented by the term offences changes, murders and rapes replace jay walking and speeding in the aggregate crime statistics. This shifts both marginal cost curves up as represented in the following diagram



A Rise in the Marginal Social Damage of Crime.

Hence we see that

1. There is an increase in optimal size of fines.
2. There is an increase in the optimal frequency of conviction
3. There is an decrease in the optimal number of offences.

An Increase in the Marginal Social Cost of Apprehension and Conviction (C_O).

These have the same effects as an increase in marginal damages. That fines should rise is perhaps obvious. Convictions also rise as the increased cost makes it more important to deter offences.

An Increase in the Marginal Social Cost of Apprehension and Conviction (C_p).

This increases the cost of using p to deter offences, its effects are partially offset by an increase in f hence the optimal number of offences rises.

Cost Increases that Raise both C_O and C_p

Typically have an ambiguous effect on the optimal number of offences. Examples might be

1. Increased salaries for the police.
2. Improved technology for the detection of crime (reduces both components of cost).

A Decrease in the Elasticity $\varepsilon_{f.}$

So that criminal activity becomes less sensitive to the fines used to deter it.

1. Increases the optimal number of offences.
2. Decreases the optimal f.
3. Increases the optimal p but not by enough to offset the effects of the change in f.

A Decrease in the Elasticity $\varepsilon_{p.}$

Criminals are less deterred by the probability of being caught and convicted.

1. Increases the optimal number of offences.
2. Decreases the optimal p.
3. Increases the optimal f but not by enough to offset the effects of the change in p.

A Decrease in both Elasticities

Criminals are simply less easy to deter.

1. Increases the optimal number of offences.
2. Decreases the optimal f.
3. Decreases the optimal p..

An Increase in b the Social Cost of Punishment.

If it is more socially costly to punish offenders then it is desirable to adjust p and f to increase the optimal number of offences. Either p or f or both must fall. It can be shown that the optimal value of p falls and the optimal value of f increases but only enough to have a partially offsetting effect on O.