

Exam #: \_\_\_\_\_

Printed Name: \_\_\_\_\_

Signature: \_\_\_\_\_

PHYSICS DEPARTMENT  
UNIVERSITY OF OREGON  
Unified Graduate Examination

PART I

Monday, April 2, 2012, 12:30 to 16:30

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## Constants

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Heat of fusion of water	334 kJ/kg
Specific heat of oxygen ( $c_V$ )	21.1 J/mole·K
Specific heat of oxygen ( $c_P$ )	29.4 J/mole·K
Gravitational acceleration on Earth ( $g$ )	$9.8 \text{ m/s}^2$
1 atmosphere	$1.01 \times 10^5 \text{ Pa}$

## Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

## Trigonometric identities

$$1 - \cos \theta = 2 \sin^2(\theta/2)$$

$$1 + \cos \theta = 2 \cos^2(\theta/2)$$

$$\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)$$

## Stirling's formula

$$\log(N!) \approx N \log N - N$$

for large  $N$ , with an error that grows only like  $\log N$ .

### Problem 1

Consider the circular motion of a particle of mass  $m$  in a plane [polar coordinates  $(r, \theta)$ ] subjected to a central force derived from the potential

$$V(r) = -\frac{\kappa}{r^n},$$

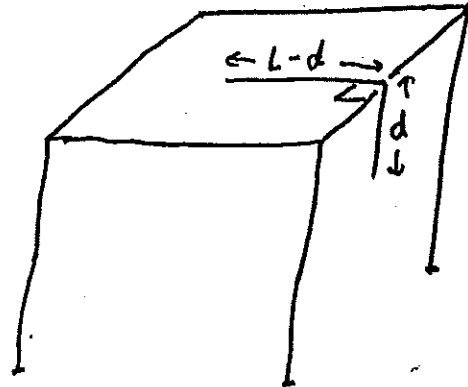
where  $\kappa$  and  $n$  are positive constants.

- (a) Write down the equations of motion for  $r(t), \theta(t)$ .
- (b) For what values of  $n$  are there stable circular orbits?
- (c) Suppose that there are  $d$  spatial dimensions. Determine what  $n$  is for the gravitational force of a point source and thus find what values of  $d$  allow for gravitationally stable circular orbits. (Hint: use Gauss' law applied to gravity in  $d$  spatial dimensions.)

## Problem 2

A rope of length  $L$  and mass  $m$  is stretched out on a large horizontal frictionless table (all dimensions greater than  $L$ ). A length  $d < L$  of rope hangs over one edge (see diagram). The rope is constrained to have a right-angle bend at the table edge by frictionless pulleys (not shown). The rope is released and falls with negligible friction.

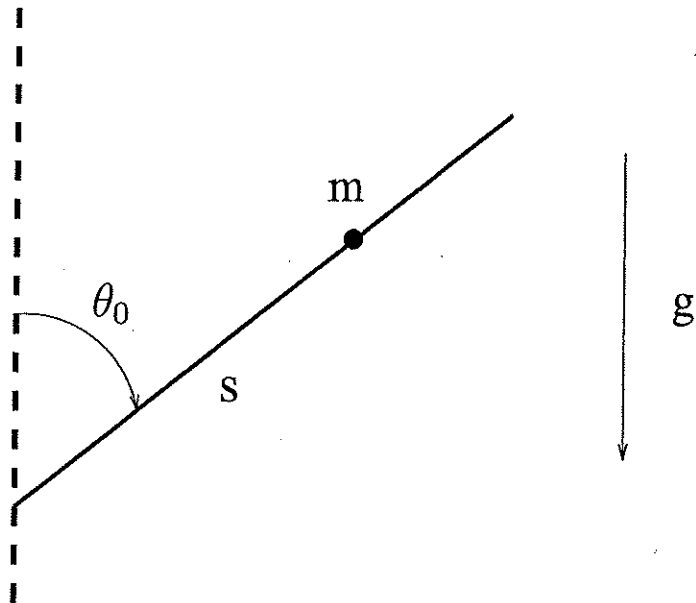
- (a) How long does it take for the rope to fall off the table?
- (b) What is the speed of the rope as it leaves the table?



### Problem 3

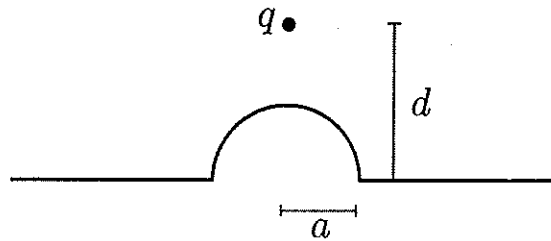
A bead of mass  $m$  slides without friction along a straight wire that is rotating with constant angular velocity  $\omega$  about a vertical axis. The wire makes a fixed angle  $\theta_0$  with the rotational axis. Gravity acts downward.

- Construct the Lagrangian for the bead using as generalized coordinate the distance  $s$  measured along the wire from the point of intersection with the rotation axis.
- Obtain the equation of motion for  $s$  and use it to find the condition for an equilibrium circular orbit of radius  $s_0 \sin \theta_0$ . Is the equilibrium stable or unstable? Explain.
- What is the magnitude of the force on the bead in the  $\hat{\phi}$  direction (where  $\phi$  is the azimuthal angle, with  $\dot{\phi} = \omega$ ). Express your general result in terms of  $s$  and the other variables given above. When the bead is in equilibrium (cf. part b), what is the force in the  $\hat{\phi}$  direction?



#### Problem 4

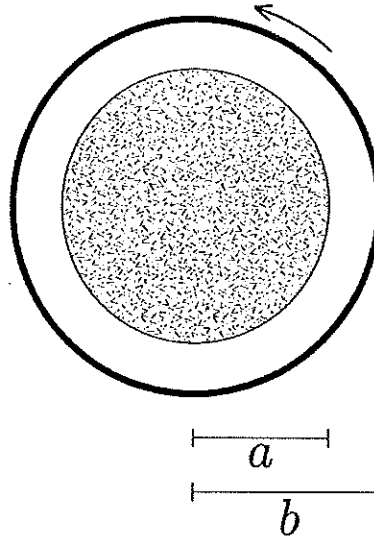
Consider an infinitely large conducting plate at potential  $V = 0$  that has a hemispherical bulge of radius  $a$  centered about the origin. (See the figure for a cross-sectional view). A charge  $q$  is placed above the center of the bulge at a distance  $d$  from the plane. Use the method of image charges to find the force on the charge  $q$ .



Hint: First assume that there is only the hemispherical bulge (no plane), and show that placing a charge  $Q = -qa/d$  in a suitable position leaves the surface of the hemisphere at a constant potential  $V = 0$ . Then show that placing one or more additional charges in suitable positions leaves the plane plus the hemisphere at zero potential.

### Problem 5

A positive line charge  $\rho$  is glued onto the rim of a wheel of radius  $b$ . The wheel is lying in a horizontal plane and is suspended so that it is free to rotate. In the central region, out to a radius  $a$ , there is a time dependent, spatially uniform magnetic field  $\mathbf{B}(t)$ , pointing up. (See the view from the top in the figure. The region with magnetic field is shaded. The spokes, not shown in the figure, are made of nonconducting, non-magnetic materials.) At time  $t = 0$  the wheel is at rest, and  $\mathbf{B}(t = 0) = \mathbf{B}_0$ . The  $\mathbf{B}$  field is then gradually turned off, so that at a later time  $t = t_f$ ,  $\mathbf{B}(t = t_f) = 0$ .



- Write down the expression for the force acting on an infinitesimal line element of the rim while the magnetic field is being turned off.
- Write down the expression for the total torque acting on the wheel as a result of all such forces exerted on the rim.
- What are the magnitude and direction of the total angular momentum transmitted to the wheel during this process, and do they depend on how fast the magnetic field is turned off? Express your answer in terms of some or all of  $B_0$ ,  $\rho$ ,  $t_f$ ,  $a$ , and  $b$ .
- Where does the angular momentum come from?

### Problem 6

This problem discusses how, in classical physics, one can estimate the radius of an electron.

- (a) According to special relativity, how is the total energy,  $W_m$ , of an electron at rest related to its rest mass  $m_e$  ?
- (b) In classical electrostatics, how is the total energy,  $W_f$ , of the electric field in a volume  $V$  related to the electric field strength,  $E(\mathbf{r})$ , inside the volume? (Assume SI units.)
- (c) A total charge,  $e$ , is distributed uniformly over the surface of an infinitely thin sphere of radius  $R$ . Calculate the electric field,  $E(\mathbf{r})$ , both inside and outside the sphere.
- (d) Evaluate the energy of the electric field generated by the charged sphere in (c), using the expression in (b), when the volume  $V$  extends over all space.
- (e) Assuming that the electron is an infinitely thin sphere of radius  $R$ , determine  $R$  by setting  $W_f = W_m$ . Express the value of  $R$  in terms of  $m_e$ , the elementary electron charge  $e$ , and other constants of nature.



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## Stirling's formula

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### Problem 1

Compute numerical values for the following (squared) Clebsch–Gordan coefficients. The notation convention for the coefficients is  $\langle j_1, m_1; j_2, m_2 | j_3, m_3 \rangle$ .

(a)  $|\langle 1, 1; 1, 0 | 3, 1 \rangle|^2$ .

(b)  $|\langle 1, 1; 1, 0 | 1, -1 \rangle|^2$ .

(c)  $|\langle 0, 0; 1, 0 | 1, 0 \rangle|^2$ .

(d)  $|\langle 1, 0; 1, 0 | 0, 0 \rangle|^2$ .

## Problem 2

Consider a particle with mass  $m$  in the attractive delta-function potential in one dimension,

$$V(x) = -\beta \delta(x)$$

with  $\beta > 0$  a constant. Consider the following family of normalized trial wavefunctions,  $\psi_a(x) \equiv \langle x | \psi_a \rangle$ , of the (triangular) form

$$\psi_a(x) = \begin{cases} \sqrt{\frac{3}{2a^3}} (a - x) & \text{for } 0 \leq x \leq a \\ \sqrt{\frac{3}{2a^3}} (x + a) & \text{for } -a \leq x \leq 0 \\ 0 & \text{for } |x| > a \end{cases}$$

You may use units with  $\hbar = 1$  if you like.

- Calculate the expectation value of the potential energy in the states  $|\psi_a\rangle$ , i.e., calculate  $\langle \psi_a | \hat{V} | \psi_a \rangle$ . (Here  $\hat{V}$  denotes the operator associated with the potential  $V$  given above.)
- Calculate the expectation value of the kinetic energy in the states  $|\psi_a\rangle$ , i.e., calculate  $\langle \psi_a | \hat{T} | \psi_a \rangle$ . Hint: be careful calculating this, as the derivative of  $\psi_a(x)$  is not continuous at  $x = a$ , nor at  $x = -a$ , nor at  $x = 0$ . You may want to sketch the function  $\psi_a(x)$ .
- Would you expect  $\langle \psi_a | \hat{T} + \hat{V} | \psi_a \rangle$  to be bigger or smaller than the ground state energy for the Hamiltonian  $\hat{H} = \hat{T} + \hat{V}$ ?
- Using the variational method, estimate the ground-state energy by varying  $a$ .
- The exact ground state energy is  $E_{\text{gs}} = -m\beta^2/(2\hbar^2)$ . Is your bound in part (c) consistent with your answer to part (d)?

### Problem 3

A crude model potential for a molecule can be written

$$U(r) = -2U_0 \left[ \frac{a}{r} - \frac{1}{2} \left( \frac{a}{r} \right)^2 \right].$$

Here  $U_0 > 0$  is a characteristic energy,  $a$  is a characteristic length, and  $r = |\vec{x}|$  is the distance from the potential center. You may use units with  $\hbar = 1$  if you like.

- (a) If we write the wave function in spherical polar coordinates in the form

$$\psi(r, \theta, \phi) = R(r)Y_l^m(\theta, \phi),$$

what form does the Schrödinger equation  $H|\psi\rangle = E|\psi\rangle$  take as a differential equation for  $R(r)$ ? What is meant by the centrifugal barrier term in this equation?

- (b) Sketch and briefly discuss both  $U(r)$  and the effective potential  $U_{\text{eff}}(r)$  that is obtained in the radial equation once the centrifugal barrier is taken into account. Compare  $U_{\text{eff}}$  with the effective potential  $V_{\text{eff}}$  that results from a Coulomb potential,  $V(r) = -e^2/r$ , plus the centrifugal barrier.
- (c) Recall that the energy spectrum for a particle with mass  $m$  in a Coulomb potential is

$$E_{n_r, \ell}^{(c)} = \frac{-E_R}{(n_r + \ell + 1)^2},$$

where  $E_R = me^4/2\hbar^2$ ,  $n_r = 0, 1, \dots$  is the radial quantum number, and  $\ell = 0, 1, 2, \dots$  is the angular momentum quantum number. Find the energy spectrum for the potential  $U(r)$ . Write the energy levels as a function of a radial quantum number  $n_r$  and the angular momentum quantum number  $\ell$ . Please use the parameter  $\gamma$  where

$$\gamma^2 = 2U_0ma^2/\hbar^2.$$

- (d) Is the degeneracy of the energy levels for  $U(r)$  the same as in the case of a Coulomb potential? If not, what is the difference?

#### Problem 4

A gas consisting of an enormous number of indistinguishable, non-interacting, spin 0 particles is in equilibrium with an atomic trap in which each of the particles can be in one of two single particle states, one with energy  $-E_1 - \epsilon$ , the other with energy  $-E_1$ . (Here  $E_1$  and  $\epsilon$  are both  $> 0$ .) The mean number of particles in the trap is  $N$ . Find the temperature  $T$  at which there are, on average, twice as many particles in the ground state as in the excited state.

### Problem 5

- (a) When a small amount  $\Delta Q$  of heat is transferred to a body at temperature  $T$ , its entropy changes by an amount  $\Delta S$ . What is the relation between  $\Delta S$ ,  $\Delta Q$ , and  $T$ ?
- (b) A block of material with a temperature independent heat capacity  $C = 500 \text{ J/K}$  is initially at a temperature  $T_0 = 600 \text{ K}$ . It is then placed in a large lake at a temperature  $T_L = 300 \text{ K}$  and allowed to come into thermal equilibrium with the lake. Find the change in entropy of the block, the lake, and the total system. (Give numerical answers).
- (c) An identical block at an initial temperature  $T_0 = 600 \text{ K}$  is placed in a large geothermally heated pool at  $T_P = 373 \text{ K}$  and allowed to come to thermal equilibrium with the pool. The block is then placed in the lake at  $T_L = 300 \text{ K}$  and allowed to come into thermal equilibrium with the lake. Find the change in entropy of the block, the pool, the lake, and the total system. (Give numerical answers).
- (d) Explain how the total entropy change involved in cooling the block from  $600 \text{ K}$  to  $300 \text{ K}$  could be made as small as one wished.

## Problem 6

A surface contains a number  $B \gg 1$  of sites to which the molecules of a gas in contact with the surface can bind. Each site can be occupied by only one particle (or none). The energy of a particle bound to a site is  $-\epsilon$ . (Here  $\epsilon > 0$ .)

- (a) Will all the sites be occupied at a non-zero temperature? Justify your answer, in words.
- (b) Suppose  $N$  particles bind to the surface. How many ways can they be arranged? What is the entropy  $S$  associated with this? Simplify your expression for  $S$  in the limit that  $N$ ,  $B$ , and  $B - N$  are all  $\gg 1$ .
- (c) What is the energy of the surface when  $N \gg 1$  particles are on it? What is its Helmholtz free energy  $F$ ? Finally, what is the chemical potential  $\mu$  for particles on the surface? Express your answers in terms of the occupied fraction of sites:  $f = N/B$ ,  $B$ , and  $\epsilon$ .
- (d) If the adsorbed particles are in thermal equilibrium with the particles in the (non-degenerate, ideal) gas, what is  $f$ ? Express your answer in terms of  $\epsilon$ , the temperature  $T$ , the number density  $n$  of particles in the ideal gas, and fundamental constants.

Hint: The chemical potential of a non-degenerate ideal gas is

$$\mu = k_B T \ln \left( n \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{3/2} \right)$$