Exam #: ____________________

Printed Name: _______________

Signature: __________________

PHYSICS DEPARTMENT

UNIVERSITY OF OREGON

Ph.D. Qualifying Examination, PART III
Quantum Mechanics and Statistical Mechanics

Wednesday, March 31, 2004, 1:00 p.m. to 5:00 p.m.

The examination papers are numbered in the upper right-hand corner of each page. Print and then sign your name in the spaces provided on this page. For identification purposes, be sure to submit this page together with your answers when the exam is finished. Be sure to place both the exam number and the question number on any additional pages you wish to have graded.

There are six equally weighted questions, each beginning on a new page. Read all six questions before attempting any answers.

Begin each answer on the same page as the question, but continue on additional blank pages if necessary. Write only on one side of each page. Each page should contain work related to only one problem. If you need extra space for another problem, start a new page.

If you need to leave the seat, wait until everyone else is seated before approaching the proctor.

Calculators may be used only for arithmetic. Calculators with stored equations or text are not allowed. Dictionaries may be used if they have been approved by the proctor before the examination begins. No other papers or books may be used.

When you have finished, come to the front of the room and hand your examination paper to the proctor; first put all problems in numerical order and staple them together.

Please make sure you follow all instructions carefully. If you fail to follow instructions, or to hand your exam paper in on time, an appropriate number of points may be subtracted from your final score.

Constants

Electron charge (e)	1.60×10^{-19} C
Electron rest mass (m_e)	9.11×10^{-31} kg (0.511 MeV/c^2)
Proton rest mass (m_p)	1.673×10^{-27} kg (938 MeV/c^2)
Neutron rest mass (m_n)	1.675×10^{-27} kg (940 MeV/c^2)
W+ rest mass (m_W)	80.4 Gev/c^2
Planck's constant (h)	6.63×10^{-34} J·s
Speed of light in vacuum (c)	3.00×10^{8} m/x
Boltzmann's constant (k_B)	1.38×10^{-23} J/K
Gravitational constant (G)	6.67×10^{-11} Nm^2/kg^2
Permeability of free space (μ_0)	$4\pi \times 10^{-7}$ H/m
Permittivity of free space (ε_0)	8.85×10^{-12} F/m
Mass of earth (M_E)	5.98×10^{24} kg
Equatorial radius of earth (R_E)	6.38×10^{6} m
Radius of sun (R_S)	6.96×10^{8} m
Temperature of surface of sun (T_S)	5.8×10^{3} K
Earth-sun distance (R_{ES})	1.5×10^{11} m
Density of iron at low temperature (ρ_{Fe})	7.88×10^{3} kg/m^3
Classical electron radius (r_0)	2.82×10^{-15} m
Gravitational acceleration on earth (g)	9.8 m/s^2
Atomic mass unit	$1.7\text{x}10^{-27}$ kg
Specific heat of oxygen (C_V)	21.1 J/mole·K
Specific heat of oxygen (C_P)	29.4 J/mole·K
Avogadro's number	$6.02\text{x}10^{23}$ atoms/mole
Gas constant (R)	8.31 J/mole·K

Stirling's formula:

$\ln(x!) \cong x\ln(x) - x - \ln(\sqrt{2\pi x})$ (the last term may often be neglected)

Additional information

S = k log W
1/T = ∂S /∂E

Problem 1

The energy levels of a quantum oscillator with frequency ν are given by,

$$\varepsilon = (n + \tfrac{1}{2})\, h\nu \qquad\qquad n = 0, 1, 2, 3, \ldots$$

Consider a system of N almost independent oscillators with total energy,

$$E = \tfrac{1}{2} Nh\nu + Mh\nu. \quad \text{(energy } Mh\nu \text{ above the ground state).}$$

a) Find the thermodynamic weight, that is, the total number of possible quantum states for this energy.

b) Assuming that the system is in thermal equilibrium, show that the probability that a given oscillator is in the n-th excited state, in the limit $N \gg 1$, and $M \gg n$, is

$$[M/(M+N)]^n\, N/(M+N)$$

c) Assuming that $N \gg 1$, $M \gg 1$ and simplifying with Stirling's formula, show that the relation between the temperature T and the energy E of this system is

$$E = N\{\tfrac{1}{2} h\nu + h\nu/(e^{h\nu/kT} - 1)\}.$$

Problem 2

We wish to consider the properties of a simplified model of a white dwarf star. In this model, the star consists of a dense electron gas of uniform density. We can ignore the presence of the ionized (carbon and oxygen) nuclei. Use the following typical values for the temperature (T) and the number density (n) of the electron gas: $T=10^7$ K and $n=10^{36}$ m^{-3}.

a) Consider the electron gas to be a fully degenerate Fermi gas. That is, let the distribution of occupied states be described by a Fermi distribution in the limit $T \to 0$. Calculate the density of states in a volume V, assuming spherical symmetry, for N electrons. Integrate this over momentum to the Fermi momentum p_F to show that

   $n = N/V = 8\pi(p_F /h)^3 /3.$

b) Assuming the electrons are non-relativistic, calculate the Fermi energy E_F and temperature $T_F = E_F/k$. Use this to evaluate the assumption of low temperature.

c) Use the result above to calculate the pressure P of the electron gas. Show that

   $P \propto n^{5/3}.$

d) Show that in the extreme relativistic limit, the pressure of the electron gas is $n^{4/3}$.

   Note: The difference in the dependence on n in parts c) and d) will determine if the star remains a white dwarf or collapses to a neutron star.

Problem 3

A simple model of an n-type semiconductor consists of n_D electrons distributed among N_D donor levels which are located at energy E_D below the bottom of the continuum. The donor levels are far apart and do not interact whether or not they are occupied.

a) What is the entropy of the n_D electrons in the N_D donor levels? Assume that the donor levels are each occupied by at most one electron, of either spin.

b) What is the associated (Helmholtz) free energy, F, of the electrons in the donor levels? Express this energy relative to the energy of the bottom of the continuum.

c) In terms of N_D, n_D, E_D, and temperature T, derive an expression for the chemical potential of the electrons in the donor levels. It will be useful to employ Stirling's approximation.

Problem 4

Consider a particle of charge q and mass m moving in a one dimensional harmonic potential along the x direction. If one applies an additional external uniform electric field E along the same direction,

a) write down the Hamiltonian of the system.

b) Find all the eigenenergies of the system.

c) Find the eigenwave functions of the system given the eigenwave functions of the harmonic oscillator $\{\psi_n(x)\}$.

Problem 5

Let $|\psi_m^{(j)}>$ be an eigenfunction of total angular momentum and its z-component defined by

$$J^2|\psi_m^{(j)}> = \hbar^2 j(j+1)|\psi_m^{(j)}>$$

$$J_z|\psi_m^{(j)}> = \hbar m|\psi_m^{(j)}>$$

a) Calculate the 3x3 matrix J_x in the basis of the three states $|\psi_m^{(1)}>$.

b) Show that $(J_x/\hbar)^3 = (J_x/\hbar)$.

c) Show that the rotation operator for a rotation around the x-axis by angle ϕ for j = 1 states can be written as

$$R_x(\phi) = 1 + \frac{1}{2}(\cos\phi - 1)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \frac{i}{\hbar} J_x \sin\phi$$

d) Rotate the state $|\psi_1^{(1)}>$ by $\pi/4$ around the x-axis. Express the result as

$$R_x(\pi/4)\,|\psi_1^{(1)}> = \sum_m C_m|\psi_m^{(1)}>$$

and determine C_m.

Problem 6

Consider a particle of mass m moving in a central potential given by

$$V(r) = -\frac{C}{r^{\alpha}} \qquad (C > 0 \text{ and } \alpha > 0)$$

a) If the ground-state wave function of the particle has a mean radius $\langle r \rangle = r_0$, estimate the typical linear momentum of this state using the Heisenberg Uncertainty Principle.

b) Estimate the energy $E(r_0)$ of the ground state.

c) Minimize $E(r_0)$ as a function of r_0 to obtain an estimate of r_0. Express your answer in terms of $\hbar$, m, C, and α.

d) Find the critical value of $\alpha = \alpha_c$ such that if $\alpha > \alpha_c$, no minimum exists. What happens in this case and why?