Exam #: ______________________________

Printed Name: ______________________________

Signature: ______________________________

PHYSICS DEPARTMENT
UNIVERSITY OF OREGON
Ph.D. Qualifying Examination, Part III
Wednesday, April 2, 2003, 1:00 p.m. to 5:00 p.m.

The examination pages are numbered in the upper left-hand corner of each page. Print and then sign your name in the spaces provided on this page. For identification purposes, be sure to submit this page together with your answers when the exam is finished. Be sure to place both the exam number and the question number on any additional pages you wish to have graded.

There are six equally weighted questions, each beginning on a new page. Read all six questions before attempting any answers.

Begin each answer on the same page as the question, but continue on additional blank pages if necessary. Write only on one side of each page. Each page should contain work related to only one problem. If you need extra space for another problem, start a new page.

If you need to leave your seat, wait until everyone else is seated before approaching the proctor.

Calculators may be used only for arithmetic. **Calculators with stored equations or text are not allowed.** Dictionaries may be used if they have been approved by the proctor before the examination begins. **No other papers or books may be used.**

When you have finished the exam, come to the front of the room and hand your examination paper to the proctor; first put all problems in numerical order and staple them together.

Please make sure you follow all instructions carefully. If you fail to follow instructions, or to hand your exam paper in on time, an appropriate number of points may be subtracted from your final score.

Constants

Electron charge (e)	1.60×10^{-19} C
Electron rest mass (m_e)	9.11×10^{-31} kg (0.511 MeV/c^2)
Proton rest mass (m_p)	1.673×10^{-27} kg (938 MeV/c^2)
Neutron rest mass (m_n)	1.675×10^{-27} kg (940 MeV/c^2)
W^+ rest mass (m_W)	80.4 GeV/c^2
Planck's constant (h)	6.63×10^{-34} J-s
Speed of light in vacuum (c)	3.00×10^{8} m/s
Boltzmann's constant (k_B)	1.38×10^{-23} J/K
Stefan-Boltzmann constant (σ)	5.67×10^{-8} J/(m^2-s-K^4)
Gravitational constant (G)	6.67×10^{-11} N-m^2/kg^2
Permeability of free space ($\mu_\circ$)	$4\pi \times 10^{-7}$ H/m
Permittivity of free space ($\epsilon_\circ$)	8.85×10^{-12} F/m
Mass of Earth (M_E)	5.98×10^{24} kg
Equatorial radius of Earth (R_E)	6.38×10^{6} m
Radius of Sun (R_S)	6.96×10^{8} m
Mass of Sun (M_S)	1.99×10^{30} kg
Temperature of surface of the Sun (T_S)	$5,800$ K
Earth-Sun distance (R_{ES})	1.50×10^{11} m
Gravitational acceleration on Earth (g)	9.8 m/s^2
atomic mass unit	1.7×10^{-27} kg

Stirling's Formula

$$\ln(x!) = x\,\ln(x) - x - \ln\sqrt{2\pi x} + \mathcal{O}(1/x) \qquad (1)$$

Larmor Radiation Formula

$$P = (\frac{2}{3c^3})q^2 a^2 \qquad (2)$$

Integrals

$$\int_{-\infty}^{\infty} dx\; x^{2n}\exp(-ax^2) = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n a^n}\sqrt{\frac{\pi}{a}} \qquad (3)$$

$$\int_0^{\infty} \frac{dx}{x}\; x^n \exp(-x) = \Gamma(n) \qquad (4)$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) \qquad (5)$$

$$\text{if Re(a)} > 0 \text{ and Im(b)} = 0 \text{ then } \int_{-\infty}^{\infty} e^{-ay^2} e^{iby} dy = e^{-\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}} \qquad (6)$$

Problem 1

For one dimensional potential problems:

a. Show that for any normalized $|\psi\rangle$, $\langle\psi|H|\psi\rangle \geq E_0$, where E_0 is the lowest energy eigenvalue (ground state energy). Hint: expand the state $|\psi\rangle$ in the orthonormal eigenstate basis and consider the sum.

b. Given this result, state whether in one dimension every potential of the form:

$$V(|x| \ \rightarrow \ \infty) \ \rightarrow \ 0,$$

$$V(x) < 0$$

has a bound state. (Hint: use the trial wavefunction $\psi(x) \sim e^{-ax^2}$.)

Problem 2

Consider the simple harmonic oscillator described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

a. Compute the expectation values $\langle\, n \mid x^2 \mid n \,\rangle$ and $\langle\, n \mid x^3 \mid n \,\rangle$, where $\mid n \,\rangle$ is the n-th energy eigenstate.

b. Give the ground state wavefunction of the harmonic oscillator in *momentum* space. Make sure your wavefunction is properly normalized.

Integral: if $Re(a) > 0$ and $Im(b) = 0$ then $\int_{-\infty}^{\infty} e^{-ay^2} e^{iby} dy = e^{-\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}$

Problem 3

Consider the following operators on a Hilbert space:

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

a. What are the possible values one can obtain if L_z is measured?

b. Take the state with L_z eigenvalue $+1$. In this state, what are $\langle L_x \rangle$ and $\langle L_x^2 \rangle$?

c. The eigenstates of L_x in the L_z basis are:

$$|L_x = +1\rangle = (1/2\ ,\ 1/\sqrt{2}\ ,\ 1/2)$$

$$|L_x = 0\rangle = (-1/\sqrt{2}\ ,\ 0\ ,\ 1/\sqrt{2})$$

$$|L_x = -1\rangle = (1/2\ ,\ -1/\sqrt{2}\ ,\ 1/2)$$

If the particle is in the $L_z = -1$ state and L_x is measured, what are the possible outcomes and their probabilities?

d. Consider the state

$$|\psi\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}.$$

If L_z^2 is measured to be $+1$, what is the state after measurement? How probable was the result? If L_z is measured, what are the outcomes and respective probabilities? Does the state change?

Problem 4

Consider a system of N noninteracting particles moving in one dimension x (i.e., on a line), confined by a potential. The Hamiltonian of the system then is

$$H(p_1, \ldots, p_N, x_1, \ldots, x_N) = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \alpha \left(\frac{x_i}{L} \right)^4 \right)$$

where p_i are the momenta and x_i the positions of the particles labeled by $i = 1 \ldots N$. The constants determining the strength of the potential are an energy $\alpha > 0$ and a length L. Assume that the system is in thermal equilibrium at temperature T and that classical statistical mechanics is applicable.

a. Show that the classical partition function Q_N in the canonical ensemble can be written as the product

$$Q = Q_N^{ideal} f_N(L^4 kT/\alpha)$$

where Q_N^{ideal} is the partition function of the **ideal gas** with N particles and $f_N(L^4 kT/\alpha)$ is a function of T and the confinement strength.

b. From (a), you can obtain the Helmholtz free energy A. From A, calculate the entropy S_N of the N-particle system by differentiation, keeping the confining potential fixed:

$$S_N = -\left(\frac{\partial A}{\partial T} \right)_L .$$

Show that S_N is the sum of the ideal-gas value and a correction term due to the external potential:

$$S_N = S_N^{ideal} + S_N^{ext}$$

Problem 5

Consider a system of 3 spinless, noninteracting Bosons with 3 nondegenerate energy levels ε_1, ε_2, ε_3. Let ε_1 be the ground state.

a. Make a table of all possible states of the system, by listing for each state the occupation numbers N_i $(i = 1, 2, 3)$ and the corresponding total energy E.

b. Write down the canonical partition function of the system at temperature T.

c. Find the average number $\langle N_1 \rangle$ of particles in the ground state at temperature T.

d. Now assume $\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon$, and $\varepsilon_3 = 2\varepsilon$ with $\varepsilon/(kT) > 0$. What will $\langle N_1 \rangle$ be in the limits $T \to 0$ and $T \to \infty$?

Problem 6

Consider a system containing a large number of independent, noninteracting molecules. Each molecule is shaped like an equilateral triangle, with spins located at each corner. The spin of the i-th atom may adopt only the two values $S_i = \pm 1$. The atoms interact (1) with an external magnetic field through their magnetic dipole moment of magnitude μ, and (2) with each other through an exchange interaction J, $U_{ij} = -JS_iS_j$.

a. Write an expression for the total energy of the spins in one such molecule, in a magnetic field H.

b. Find the partition function for the spin system of a single molecule in field H.

c. Find the free energy associated with the spin system of a single molecule at temperature T and field H.

d. Find an expression for the average spin $\langle \sigma \rangle$ on a single molecule at temperature T and field H.

e. Comment on the average value of $\langle \sigma \rangle$ at low temperature $T \ll |J|/k_B$, as a function of the magnetic field.