Exam #: ________________________

Printed Name: ________________________

Signature: ________________________

PHYSICS DEPARTMENT

UNIVERSITY OF OREGON

Unified Graduate Examination

Part IV

Statistical Mechanics

Wednesday, June 21, 2017, 13:30 to 16:10

The examination booklet is numbered in the upper right-hand corner of the cover page. Print and then sign your name in the spaces provided on the cover page. For identification purposes, be sure to submit this page together with your answers when the exam is finished.

There are four questions, each beginning on a new page. Read all four questions before attempting any answer. You may answer as many questions as you wish, however, only your top three scores will be used in the evaluation of your performance for a Ph.D. pass in this area, or your top two scores for a master's pass.

Begin each answer on the same page as the question, but continue on additional blank pages if necessary. Write only on one side of each page. Each page should contain work related to only one problem. When you start a new problem, start a new page. Place both the exam number and the question number on all pages you wish to have graded. You are encouraged to use the constants on the following page, where appropriate, to help you solve the problems.

If you need to leave your seat, wait until everyone else is seated before approaching the proctor.

Calculators may be used only for arithmetic and will be provided. **Personal calculators are not allowed**. Dictionaries may be used if they have been approved by the proctor before the examination begins. **Electronic dictionaries are not allowed. No other papers or books may be used.**

When you have finished, come to the front of the room. For each problem, put the pages in order and paper clip them together. Then put all problems in numerical order and place them in the envelope provided. Finally, hand the envelope to the proctor.

Constants

Electron charge (e)	1.60×10^{-19} C
Electron rest mass (m_e)	9.11×10^{-31} kg (0.511 MeV/c^2)
Proton rest mass (m_p)	1.673×10^{-27} kg (938 MeV/c^2)
Neutron rest mass (m_n)	1.675×10^{-27} kg (940 MeV/c^2)
Atomic mass unit (AMU)	1.66×10^{-27} kg
Atomic weight of a hydrogen atom	1 AMU
Atomic weight of a nitrogen atom	14 AMU
Atomic weight of an oxygen atom	16 AMU
Planck's constant (h)	6.63×10^{-34} J·s
Bohr Magneton (μ_B)	9.27×10^{-28} J/G
Speed of light in vacuum (c)	3.00×10^{8} m/s
Boltzmann's constant (k_B)	1.38×10^{-23} J/K
Gravitational constant (G)	6.67×10^{-11} N·m^2/kg^2
Permeability of free space (μ_0)	$4\pi \times 10^{-7}$ H/m
Permittivity of free space (ϵ_0)	8.85×10^{-12} F/m
Mass of earth (M_E)	5.98×10^{24} kg
Equatorial radius of earth (R_E)	6.38×10^{6} m
Mass of Sun (M_S)	1.99×10^{30} kg
Radius of Sun (R_S)	6.96×10^{8} m
Classical electron radius (r_0)	2.82×10^{-15} m
Density of water	1.0 kg/liter
Density of ice	0.917 kg/liter
Specific heat of water	4180 J/(kg K)
Specific heat of ice	2050 J/(kg K)
Heat of fusion of water	334 kJ/kg
Heat of vaporization of water	2260 kJ/kg
Specific heat of oxygen (c_V)	21.1 J/mole·K
Specific heat of oxygen (c_P)	29.4 J/mole·K
Gravitational acceleration on Earth (g)	9.8 m/s^2
1 atmosphere	1.01×10^{5} Pa

Problem 4.1

Consider a spherically symmetric gas cloud in otherwise empty space. The only forces acting on the gas are pressure and the cloud's own gravity. The gas molecules have mass m and a number density $n(r)$ that depends on the distance r from the cloud's center. Assume that the gas is ideal and in thermal equilibrium at temperature T.

(a) [**2 points**] Consider a thin spherical shell of gas with radius r and thickness $\mathrm{d}r$. Explain in words what forces, acting in what directions, the shell experiences if it is in mechanical equilibrium.

(b) [**5 points**] Now write down expressions for these forces, and derive either an integro-differential equation, or an equivalent differential equation, for $n(r)$.

(c) [**1 point**] Suppose the density is a power-law function of r: $n(r) = \text{const} \times r^{\alpha}$. Find the value of α that satisfies the equation you derived.

(d) [**2 point**] Find the mass contained in a sphere of radius R. Using your result in (c), comment on its implications for a cloud with a finite number of particles.

Problem 4.2

Consider a system of N non-interacting spin $\frac{1}{2}$ particles. Each particle has a magnetic moment μ which can point either parallel or antiparallel to an external field H. All energies and degrees of freedom other than those of the spins can be neglected.

a) [1 point] What is the allowed range of total energies E for this system?

Now suppose the system of spins as a whole is in equilibrium at a temperature T.

b) [1 point] What is the partition function when $N = 1$?

c) [2 points] What is the partition function when $N > 1$?

d) [3 points] Find the average energy $\bar{E}$ as a function of the temperature T of the spins.

Now suppose we fix the total energy E of the spins, rather than their temperature T.

e) [3 points] For $N \gg 1$, calculate and plot the equilibrium temperature $T(E)$ as a function of E over the *entire* range of allowed energies E you found in part (a). Comment on any unusual features of your result.

Problem 4.3

Consider a gas of N free electrons (mass m) in two dimensions, confined to a square of area A.

(a) Assume vanishing probability density on the boundary, and energy E large enough such that the spectrum can be considered continuous. Calculate the density of states, $\mathcal{D}(E)$ (including spin) of the electron gas. Hint: Given a large M, the number of integers n_x, n_y with $n_x^2 + n_y^2 \leq M^2$ is approximately equal to the area of a circle of radius M.

[6 points]

(b) Using the result of (a), obtain an expression for the Fermi energy E_F in terms of N and A.

[2 points]

(c) At temperature $T = 0$, obtain an expression for the average energy per particle in terms of E_F only.

[2 points]

Problem 4.4

A box of volume V contains N diatomic ideal gas molecules at temperature T, each with total mass m.

a) (4 points) Estimate the rate per unit area at which gas molecules collide with the container boundary. Clearly state your assumptions in making this estimate.

b) (3 points) Assuming each molecule is composed of two identical point masses connected by a spring of equilibrium length R with force constant $m\omega^2$, what is the classical heat capacity of this gas? Clearly explain the origin of each contribution to the heat capacity. In which regime is the classical approximation valid?

c) (3 points) For the same molecules, now assume that the internal energy states are quantized, and thus certain degrees of freedom can be 'frozen out' below a certain transition temperature. Estimate the two transition temperatures at which the heat capacity will increase if starting from $T = 0$, and indicate which degrees of freedom 'unfreeze' at these temperatures.