Exam #: ____________________

Printed Name: ____________________

Signature: ____________________

PHYSICS DEPARTMENT

UNIVERSITY OF OREGON

Unified Graduate Examination

Part IV

Statistical Mechanics

Friday, September 23, 2016, 13:30 to 16:10

The examination booklet is numbered in the upper right-hand corner of the cover page. Print and then sign your name in the spaces provided on the cover page. For identification purposes, be sure to submit this page together with your answers when the exam is finished.

There are four questions, each beginning on a new page. Read all four questions before attempting any answer. You may answer as many questions as you wish, however, only your top three scores will be used in the evaluation of your performance for a Ph.D. pass in this area, or your top two scores for a master's pass.

Begin each answer on the same page as the question, but continue on additional blank pages if necessary. Write only on one side of each page. Each page should contain work related to only one problem. When you start a new problem, start a new page. Place both the exam number and the question number on all pages you wish to have graded. You are encouraged to use the constants on the following page, where appropriate, to help you solve the problems.

If you need to leave your seat, wait until everyone else is seated before approaching the proctor.

Calculators may be used only for arithmetic and will be provided. **Personal calculators are not allowed**. Dictionaries may be used if they have been approved by the proctor before the examination begins. **Electronic dictionaries are not allowed. No other papers or books may be used.**

When you have finished, come to the front of the room. For each problem, put the pages in order and paper clip them together. Then put all problems in numerical order and place them in the envelope provided. Finally, hand the envelope to the proctor.

Constants

Electron charge (e)	1.60×10^{-19} C
Electron rest mass (m_e)	9.11×10^{-31} kg (0.511 MeV/c^2)
Proton rest mass (m_p)	1.673×10^{-27} kg (938 MeV/c^2)
Neutron rest mass (m_n)	1.675×10^{-27} kg (940 MeV/c^2)
Atomic mass unit (AMU)	1.66×10^{-27} kg
Atomic weight of a hydrogen atom	1 AMU
Atomic weight of a nitrogen atom	14 AMU
Atomic weight of an oxygen atom	16 AMU
Planck's constant (h)	6.63×10^{-34} J·s
Bohr Magneton (μ_B)	9.27×10^{-28} J/G
Speed of light in vacuum (c)	3.00×10^{8} m/s
Boltzmann's constant (k_B)	1.38×10^{-23} J/K
Gravitational constant (G)	6.67×10^{-11}N·m^2/kg^2
Permeability of free space (μ_0)	$4\pi \times 10^{-7}$ H/m
Permittivity of free space (ϵ_0)	8.85×10^{-12} F/m
Mass of earth (M_E)	5.98×10^{24} kg
Equatorial radius of earth (R_E)	6.38×10^{6} m
Mass of Sun (M_S)	1.99×10^{30} kg
Radius of Sun (R_S)	6.96×10^{8} m
Classical electron radius (r_0)	2.82×10^{-15} m
Density of water	1.0 kg/liter
Density of ice	0.917 kg/liter
Specific heat of water	4180 J/(kg K)
Specific heat of ice	2050 J/(kg K)
Heat of fusion of water	334 kJ/kg
Heat of vaporization of water	2260 kJ/kg
Specific heat of oxygen (c_V)	21.1 J/mole·K
Specific heat of oxygen (c_P)	29.4 J/mole·K
Gravitational acceleration on Earth (g)	9.8 m/s^2
1 atmosphere	1.01×10^{5} Pa

4.1 Two-state model

A collection of N non-interacting, distinguishable two-level systems is in contact with a heat bath at temperature T. The individual energy levels are at zero for the ground state and $\varepsilon > 0$ for the excited state.

(a) **(3 points)** If n is the number of systems in the excited state, write the canonical partition function Z for this model as a sum over n.

(b) **(1 point)** Show that the sum from part **(a)** evaluates to

$$Z = \left(1 + \exp\left(-\frac{\varepsilon}{k_B T}\right)\right)^N$$

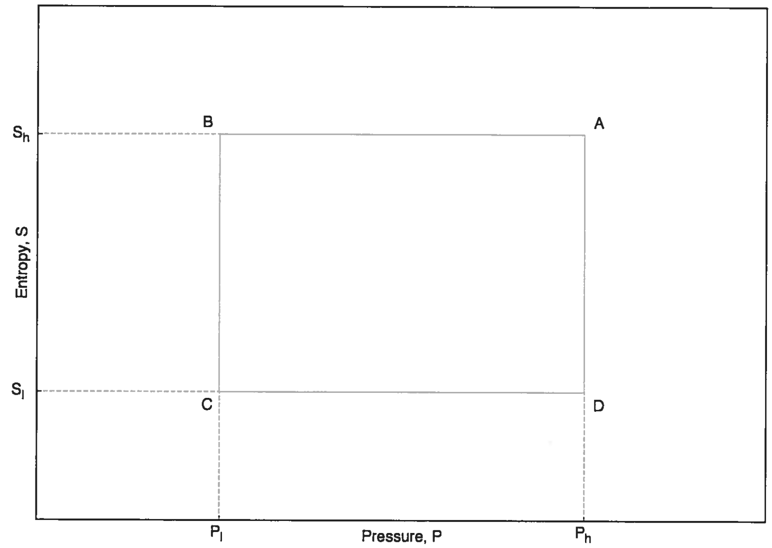
(c) **(2 points)** Calculate the Helmholtz free energy F for this model.

(d) **(3 points)** Calculate the entropy S as a function of temperature and determine the limiting value of S as $T \to 0$. Is the result consistent with the third law of thermodynamics?

(e) **(1 point)** Does $\lim_{T\to\infty} S$ give a result consistent with the definition of the entropy in terms of the number of accessible states?

Problem 4.2

The Brayton engine cycle is shown in an entropy-pressure (S-P) plot. The cycle runs from $D \rightarrow A \rightarrow B \rightarrow C \rightarrow D$ (see below). During step D $\rightarrow$ A, the gas heats at constant pressure with absorption of heat Q_{DA}. During step B $\rightarrow$ C, the gas cools at constant pressure with expulsion of heat Q_{BC}. Let W_{ABCD} represent the net work performed by the gas over the cycle. Define the efficiency of the engine as the ratio $\eta = W_{\mathrm{ABCD}}/Q_{\mathrm{DA}}$. Let the working substance be an ideal gas so that the heat capacities, C_{P} and C_{V} are independent of temperature.



a. (3 points) Plot the Brayton cycle in a pressure-volume (P-V) diagram.

b. (7 points) Calculate the efficiency η of the engine in terms of the specific heats C_{P}, C_{V}, and the ratio of the pressures $\mathrm{P_l/P_h}$.

Problem 4.3

Consider the following two microcanonical systems:

1) A system with 5 identical fermions has singly-occupied energy levels spaced by ϵ and excess energy of 3ϵ above the 5-particle ground state. Assume the lowest energy state is 0ϵ.

(a) (1 point) What is the entropy of this system?

(b) (1 point) What is the highest energy a particle in this system can have?

(c) (1 point) What is the mean particle energy?

(d) (2 points) What are the occupation numbers of the energy levels, $\bar{n}_F(\epsilon)$?

2) A system with identical 5 bosons has energy levels spaced by ϵ and excess energy of 3ϵ above the 5-particle ground state. Assume the lowest energy state is 0ϵ.

(e) (1 point) What is the entropy of this system?

(f) (1 point) What is the highest energy a particle in this system can have?

(g) (1 point) What is the mean particle energy?

(h) (2 points) What are the occupation numbers of the energy levels, $\bar{n}_E(\epsilon)$?

Problem 4.4

The Sackur-Tetrode expression for the entropy of an ideal gas of N identical atoms/molecules of mass m at temperature T in a volume V can be written as a sum of four terms:

$$\begin{aligned} S/k_B &= N \ln V \\ &\quad + \frac{3}{2} N \ln(2\pi e m T) \\ &\quad - \ln N! \\ &\quad - 3N \ln h, \end{aligned}$$

where h is Planck's constant. For each of the four terms sketch a derivation in terms of the number of microstates the particles can occupy. Hint: the last two terms represent corrections to the first two.

Note we assume here all quantities are expressed by their numerical values in SI units such that the arguments of the logarithms are dimensionless.