Exam #: ______________________

Printed Name: ______________________

Signature: ______________________

PHYSICS DEPARTMENT

UNIVERSITY OF OREGON

Unified Graduate Examination

Part III

Quantum Mechanics

Friday, September 23, 2016, 10:00 to 12:40

The examination booklet is numbered in the upper right-hand corner of the cover page. Print and then sign your name in the spaces provided on the cover page. For identification purposes, be sure to submit this page together with your answers when the exam is finished.

There are four questions, each beginning on a new page. Read all four questions before attempting any answer. You may answer as many questions as you wish, however, only your top three scores will be used in the evaluation of your performance for a Ph.D. pass in this area, or your top two scores for a master's pass.

Begin each answer on the same page as the question, but continue on additional blank pages if necessary. Write only on one side of each page. Each page should contain work related to only one problem. When you start a new problem, start a new page. Place both the exam number and the question number on all pages you wish to have graded. You are encouraged to use the constants on the following page, where appropriate, to help you solve the problems.

If you need to leave your seat, wait until everyone else is seated before approaching the proctor.

Calculators may be used only for arithmetic and will be provided. **Personal calculators are not allowed**. Dictionaries may be used if they have been approved by the proctor before the examination begins. **Electronic dictionaries are not allowed. No other papers or books may be used.**

When you have finished, come to the front of the room. For each problem, put the pages in order and paper clip them together. Then put all problems in numerical order and place them in the envelope provided. Finally, hand the envelope to the proctor.

Vector Operators in Spherical Coordinates:

$$
\begin{aligned}
\nabla f &= \hat{r}\frac{\partial}{\partial r}f + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}f + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}f \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2 A_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\sin\theta A_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi \\
\nabla \times \mathbf{A} &= \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\sin\theta A_\phi - \frac{\partial}{\partial \phi}A_\theta\right) + \hat{\theta}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}rA_\phi\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial}{\partial r}rA_\theta - \frac{\partial}{\partial \theta}A_r\right)
\end{aligned}
$$

Constants

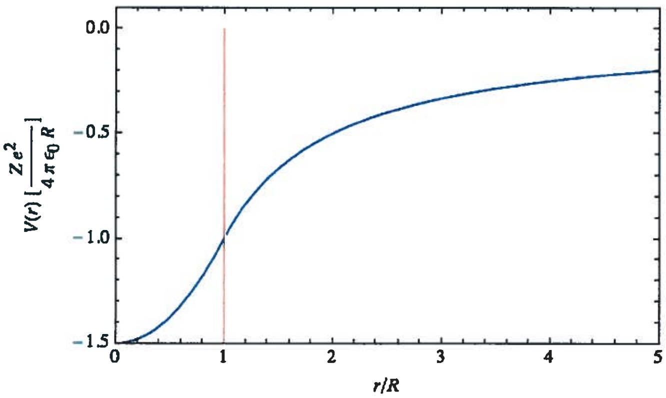
Electron charge (e)	1.60×10^{-19} C
Electron rest mass (m_e)	9.11×10^{-31} kg (0.511 MeV/c^2)
Proton rest mass (m_p)	1.673×10^{-27} kg (938 MeV/c^2)
Neutron rest mass (m_n)	1.675×10^{-27} kg (940 MeV/c^2)
Atomic mass unit (AMU)	1.66×10^{-27} kg
Atomic weight of a hydrogen atom	1 AMU
Atomic weight of a nitrogen atom	14 AMU
Atomic weight of an oxygen atom	16 AMU
Planck's constant (h)	6.63×10^{-34} J·s
Bohr Magneton (μ_B)	9.27×10^{-28} J/G
Speed of light in vacuum (c)	3.00×10^{8} m/s
Boltzmann's constant (k_B)	1.38×10^{-23} J/K
Gravitational constant (G)	6.67×10^{-11}N·m^2/kg^2
Permeability of free space (μ_0)	$4\pi \times 10^{-7}$ H/m
Permittivity of free space (ϵ_0)	8.85×10^{-12} F/m
Mass of earth (M_E)	5.98×10^{24} kg
Equatorial radius of earth (R_E)	6.38×10^{6} m
Mass of Sun (M_S)	1.99×10^{30} kg
Radius of Sun (R_S)	6.96×10^{8} m
Classical electron radius (r_0)	2.82×10^{-15} m
Density of water	1.0 kg/liter
Density of ice	0.917 kg/liter
Specific heat of water	4180 J/(kg K)
Specific heat of ice	2050 J/(kg K)
Heat of fusion of water	334 kJ/kg
Heat of vaporization of water	2260 kJ/kg
Specific heat of oxygen (c_V)	21.1 J/mole·K
Specific heat of oxygen (c_P)	29.4 J/mole·K
Gravitational acceleration on Earth (g)	9.8 m/s^2
1 atmosphere	1.01×10^{5} Pa

Problem 3.1

Assume that a single electron is orbiting a large nucleus of radius R and charge Ze (where e is the positive elementary charge). Then the Coulomb potential has to be modified inside the nuclear radius. We choose the model potential energy

$$V(r) = \begin{cases} -\frac{Ze^2}{4\pi\epsilon_0 r} & (r \geq R) \\ \frac{Ze^2}{8\pi\epsilon_0 R}\left(\left(\frac{r}{R}\right)^2 - 3\right) & (r < R) \end{cases}$$

which is continuous at $r = R$. A plot of $V(r)$ in units of $\frac{Ze^2}{4\pi\epsilon_0 R}$ is shown below; it describes the potential of a uniform sphere with constant charge density:



(a) (3 points) Write the Hamiltonian with potential energy $V(r)$ as a sum $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$, where $\hat{H}^{(0)}$ is the hydrogen Hamiltonian with a nuclear charge Ze concentrated at the origin, and identify what $\hat{H}^{(1)}$ has to be.

(b) (4 points) Treat $\hat{H}^{(1)}$ as a perturbation and write down the integral for the first-order energy shift $E_0^{(1)}$ of the hydrogenic ground state ψ_0 (i.e., the eigenfunction with $\hat{H}^{(0)}\psi_0 = E_0^{(0)}\psi_0$ where $E_0^{(0)}$ is the unperturbed ground-state energy).

(c) (3 points) Show that the first-order energy shift is proportional to R^2, the square of the nuclear radius. To simplify the integral, you are allowed to use the approximation $R \ll a_0/Z$ where a_0 is the Bohr radius. It's not required to calculate numerical prefactors of R^2. The important observation is that although R is the physically small quantity in this problem, the first-order energy shift is not of first order in R.

Problem 3.2

A deuteron consists of a proton and a neutron a weakly bound system with no excited bound states. Scattering measurements show that the separation of the proton and neutron is roughly 1.5 fm and mass measurements show that the binding energy of the deuteron is $E_b \approx 2.23$ MeV. Model the deuteron as one particle with reduced mass μ moving in a spherically symmetric potential of form

$$V(r) = \begin{cases} -V_\circ \ , & r \leq a \\ 0 \ , & r > 0 \end{cases} \tag{1}$$

where r is the distance between the neutron and proton, $V_\circ > 0$ is a constant, and a is the characteristic length scale of the potential.

a. (2 points) Argue that the ground state of the deuteron must have zero angular momentum. You need not perform a detailed calculation.

b. (3 points) Starting from the Schrödinger equation for the relative coordinate r, show that the time-independent radial Schrödinger equation describing the relative motion of the proton and neutron without angular momentum can be written

$$\left(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + V(r) - E\right) u(r) = 0 \tag{2}$$

for wave functions of the form

$$\Psi(r) = \frac{u(r)}{r}. \tag{3}$$

c. (2 points) Find the general solution for the radial ground state wave function $u(r)$ in regions $r < a$ and $r > a$. You may leave normalization constants undetermined for this part of the problem.

d. (3 points) Show that from the general solution, you get a relation between E_b, $V_\circ$, a, μ and $\hbar$ of the form

$$k_1 \cot\, k_1 a = -k_2 \tag{4}$$

for the bound states of the deuteron. In the above, k_1 and k_2 are functions of E_b, $V_\circ$, $\hbar$ and μ.

Problem 3.3

Consider a 1D harmonic oscillator with an anharmonic perturbation such that the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x^4.$$

You may find it useful to recall that the raising and lower operators are $a_{\pm} = \sqrt{\frac{m\omega}{2\hbar}}\left(x \mp \frac{i}{m\omega}p\right)$.

(a) (2 points) What are the energy eigenvalues when $\lambda = 0$?

(b) (4 points) What is the shift in the ground state energy to $O(\lambda)$? If this is to be considered a small perturbation, what is the dimensionless parameter?

(c) (4 points) Using a Gaussian trial wave function of the form $\psi(x) = \left(\frac{m\omega\alpha}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}\alpha x^2}$, make a variational estimate of the ground state energy for $\lambda \neq 0$ using α as the variational parameter. You may leave your answer as the solution to an equation for α.

Problem 3.4

A particle of mass m with energy $E > 0$ is incident from the left on a one-dimensional barrier that is approximated by a delta function potential,

$$V(x) = \lambda\delta(x).$$

(a) (1.5 points) What is the form of the wave function to the left of $x = 0$? What is the form of the wave function to the right of $x = 0$? Call these wave functions $\psi_L(x)$ and $\psi_R(x)$, respectively. Your answer should still contain *two* unknown parameters, t and r, describing transmission and reflection of the incident wave. Ignore normalization of the wave function.

(b) (2.5 points) What boundary conditions relate Ψ_L to Ψ_R? (Hint: one of the conditions involves the first derivative of the wave function. Determine this latter condition by integrating the time-independent Schrödinger equation from $x = -\epsilon$ to $x = +\epsilon$, and then taking the limit $\epsilon \to 0$ from above.)

(c) (4 points) Write down the equations determining r and t, and solve them. Use the result to calculate the probability that the particle will be reflected.

(d) (2 points) For both signs of λ, how do the quantum results differ from the final results of a classical particle scattering off a delta-function potential?