Exam #: ________________________

Printed Name: ________________________

Signature: ________________________

PHYSICS DEPARTMENT

UNIVERSITY OF OREGON

Unified Graduate Examination

Part III

Quantum Mechanics

Tuesday, September 29, 2015, 10:00 to 12:40

The examination booklet is numbered in the upper right-hand corner of the cover page. Print and then sign your name in the spaces provided on the cover page. For identification purposes, be sure to submit this page together with your answers when the exam is finished.

There are four questions, each beginning on a new page. Read all four questions before attempting any answer. You may answer as many questions as you wish, however, only your top three scores will be used in the evaluation of your performance for a Ph.D. pass in this area, or your top two scores for a master's pass. (These scores will be added to your aggregate if taking the exam "under the old rules".)

Begin each answer on the same page as the question, but continue on additional blank pages if necessary. Write only on one side of each page. Each page should contain work related to only one problem. When you start a new problem, start a new page. Place both the exam number and the question number on all pages you wish to have graded. You are encouraged to use the constants on the following page, where appropriate, to help you solve the problems.

If you need to leave your seat, wait until everyone else is seated before approaching the proctor.

Calculators may be used only for arithmetic and will be provided. **Personal calculators are not allowed**. Dictionaries may be used if they have been approved by the proctor before the examination begins. **Electronic dictionaries are not allowed. No other papers or books may be used.**

When you have finished, come to the front of the room. For each problem, put the pages in order and staple them together. Then put all problems in numerical order and place them in the envelope provided. Finally, hand the envelope to the proctor.

Constants

Electron charge (e)	1.60×10^{-19} C
Electron rest mass (m_e)	9.11×10^{-31} kg ($0.511\,\mathrm{MeV/c^2}$)
Proton rest mass (m_p)	1.673×10^{-27} kg ($938\,\mathrm{MeV/c^2}$)
Neutron rest mass (m_n)	1.675×10^{-27} kg ($940\,\mathrm{MeV/c^2}$)
Atomic mass unit (AMU)	1.66×10^{-27} kg
Atomic weight of a hydrogen atom	1 AMU
Atomic weight of a nitrogen atom	14 AMU
Atomic weight of an oxygen atom	16 AMU
Planck's constant (h)	6.63×10^{-34} J·s
Bohr Magneton (μ_B)	9.27×10^{-24} J/T
Speed of light in vacuum (c)	3.00×10^{8} m/s
Boltzmann's constant (k_B)	1.38×10^{-23} J/K
Gravitational constant (G)	$6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2}$
Permeability of free space (μ_0)	$4\pi \times 10^{-7}$ H/m
Permittivity of free space (ϵ_0)	8.85×10^{-12} F/m
Mass of earth (M_E)	5.98×10^{24} kg
Equatorial radius of earth (R_E)	6.38×10^{6} m
Mass of Sun (M_S)	1.99×10^{30} kg
Radius of Sun (R_S)	6.96×10^{8} m
Classical electron radius (r_0)	2.82×10^{-15} m
Density of water	1.0 kg/liter
Density of ice	0.917 kg/liter
Specific heat of water	4180 J/(kg K)
Specific heat of ice	2050 J/(kg K)
Heat of fusion of water	334 kJ/kg
Heat of vaporization of water	2260 kJ/kg
Specific heat of oxygen (c_V)	21.1 J/mole·K
Specific heat of oxygen (c_P)	29.4 J/mole·K
Gravitational acceleration on Earth (g)	$9.8\,\mathrm{m/s^2}$
1 atmosphere	1.01×10^{5} Pa

Problem 1

a. (6 points) Set an upper bound on the ground state energy of a particle of mass m moving in a spherically symmetric attractive potential

$$V(r) = -\frac{g}{r^\alpha}, \quad (\alpha, g > 0)$$

using a variational wave function $\Psi(\vec{r}) = Ce^{-kr}$ with k as your variational parameter. You may use here

$$\int_0^\infty \mathrm{d}t \; e^{-t} t^{x-1} = \Gamma(x).$$

b. (4 points) Your answer to Part a goes to negative infinity for some range of values of α. What is that range? What does this imply for the true ground state energy? Why does this happen?

Problem 2

Consider a particle of charge $-q_2$ and mass m_2 orbiting a particle of positive charge $+q_1$ and mass m_1. Here, q_1 and q_2 are positive.

a. (2 points) Write down the time-independent Schrodinger equation for the negatively charged particle relative to the center-of-mass. You do not have to write out the Laplacian.

b. (6 points) Write down a formula for the ground state energy of this system of two particles. You can use the value of the hydrogen atom's ground state,

$$E_H \approx -\frac{e^4}{2(4\pi\epsilon_0\hbar)^2} m_{electron} \approx -13.6\text{eV}$$

where e is the proton charge, to normalize your formula.

c. (2 points) When one compares the ratio of the ground state energies of the hydrogen (H) atom and a helium ion (He^+) one finds that $E_{He}/E_H \approx 4.0016$. (Note: a Helium nucleus has $q_1 = 2e$ and $m_1 \approx 4m_{proton}$). Explain this nonintegral value with your formula from Part b.

Problem 3

Suppose the spin-spin interaction of two spin-1/2 systems A and B is described by the following Hamiltonian:

$$H = J\vec{S}_A \cdot \vec{S}_B.$$

a. (7 points) Find the eigenstates of this Hamiltonian and the corresponding energies. Hint: introduce the total spin vector $\vec{S} = \vec{S}_A + \vec{S}_B$.

b. (3 points) Generalize the problem to two spin-j systems and find the eigenenergies for that case.

Problem 4

Consider the 21 cm line of hydrogen. This radiation arises from a transition between two hyperfine levels in the 1s state of hydrogen. In one, the spins of the electron and proton are in the singlet state. In the other, they are in a triplet state. The spontaneous decay rate of the upper hyperfine ground state is A = 2.9×10^{-15} s^{-1}. The transition between the upper and lower hyperfine ground states is a magnetic dipole transition.

a. (6 points) Give two reasons this cannot be an electric dipole transition.

b. (1 point) Imagine for the moment that this *is* an electric dipole transition and use Fermi's Golden Rule for dipole transitions,

$$A' = \frac{\omega_\circ^3 |p|^2}{3\pi\epsilon_\circ \hbar c^3},$$

where $\omega_\circ$ is the transition frequency, $\epsilon_\circ$ is the vacuum permittivity, $\hbar$ is the Planck constant, c is the speed of light, and $|p|$ s the magnitude of the electric dipole moment, to calculate the spontaneous decay rate. Calculate the spontaneous decay rate $A^{'}$ using the approximation $|p| \approx ea$, where e is the electron charge, a is the Bohr radius. Note that this simply requires numerically evaluating the given expression for $A^{'}$. Your answer need only be correct to within an order of magnitude.

c. (3 points) We can estimate the decay rate for the upper hyperfine state as it compares to the $A^{'}$ calculated in Part b by comparing a typical *magnetic* dipole moment to $|p|$. (i) What would you substitute in place of $|p| = ea$ in the equation for the spontaneous decay rate? (ii) Make that substitution and estimate the spontaneous decay rate of the upper hyperfine state.