Exam #: ______________________

Printed Name: ______________________

Signature: ______________________

PHYSICS DEPARTMENT

UNIVERSITY OF OREGON

Unified Graduate Examination

PART II

Tuesday, March 31, 2015, 13:00 to 17:00

The examination papers are numbered in the upper right-hand corner of each page. Print and then sign your name in the spaces provided on this page. For identification purposes, be sure to submit this page together with your answers when the exam is finished. Be sure to place both the exam number and the question number on any additional pages you wish to have graded.

You are encouraged to use the constants on the following page, where appropriate, to help you solve the problems.

There are six equally weighted questions, each beginning on a new page. Read all six questions before attempting any answers.

Begin each answer on the same page as the question, but continue on additional blank pages if necessary. Write only on one side of each page. Each page should contain work related to only one problem. When you start a new problem, start a new page.

If you need to leave your seat, wait until everyone else is seated before approaching the proctor.

Calculators may be used only for arithmetic and will be provided. **Personal calculators are not allowed**. Dictionaries may be used if they have been approved by the proctor before the examination begins. **Electronic dictionaries are not allowed. No other papers or books may be used.**

When you have finished, come to the front of the room. For each problem, put the pages in order and staple them together. Then put all problems in numerical order and place them in the envelope provided. Finally, hand the envelope to the proctor.

Please make sure you follow all instructions carefully. If you fail to follow instructions, or to hand in your exam paper on time, an appropriate number of points may be subtracted from your final score.

Constants

Electron charge (e)	1.60×10^{-19} C
Electron rest mass (m_e)	9.11×10^{-31} kg (0.511 MeV/c^2)
Proton rest mass (m_p)	1.673×10^{-27} kg (938 MeV/c^2)
Neutron rest mass (m_n)	1.675×10^{-27} kg (940 MeV/c^2)
Atomic mass unit (AMU)	1.66×10^{-27} kg
Atomic weight of a hydrogen atom	1 AMU
Atomic weight of a nitrogen atom	14 AMU
Atomic weight of an oxygen atom	16 AMU
Planck's constant (h)	6.63×10^{-34} J·s
Speed of light in vacuum (c)	3.00×10^{8} m/s
Boltzmann's constant (k_B)	1.38×10^{-23} J/K
Gravitational constant (G)	6.67×10^{-11}N·m^2/kg^2
Permeability of free space (μ_0)	$4\pi \times 10^{-7}$ H/m
Permittivity of free space (ϵ_0)	8.85×10^{-12} F/m
Mass of earth (M_E)	5.98×10^{24} kg
Equatorial radius of earth (R_E)	6.38×10^{6} m
Mass of Sun (M_S)	1.99×10^{30} kg
Radius of Sun (R_S)	6.96×10^{8} m
Classical electron radius (r_0)	2.82×10^{-15} m
Density of water	1.0 kg/liter
Density of ice	0.917 kg/liter
Specific heat of water	4180 J/(kg K)
Specific heat of ice	2050 J/(kg K)
Heat of fusion of water	334 kJ/kg
Heat of vaporization of water	2260 kJ/kg
Specific heat of oxygen (c_V)	21.1 J/mole·K
Specific heat of oxygen (c_P)	29.4 J/mole·K
Gravitational acceleration on Earth (g)	9.8 m/s^2
1 atmosphere	1.01×10^{5} Pa

Problem 1

Consider two quantum systems, denoted by 1 and 2, each of which has just two energy eigenstates, $|a\rangle_{1,2}$ with zero energy, and $|b\rangle_{1,2}$ with energy $E_b > 0$.

There is a constant interaction between the two systems, represented by the operator

$$\hat{V} = g|a\rangle_1 \otimes |b\rangle_2 \langle b|_1 \otimes \langle a|_2 + g|b\rangle_1 \otimes |a\rangle_2 \langle a|_1 \otimes \langle b|_2,$$

where g is real.

a. (3 points) Write the total Hamiltonian in matrix form, and label the rows and columns.

b. (3 points) At time $t = 0$, the combined system is in the state $|a\rangle_1 \otimes |b\rangle_2$. Independent of the math, state in words how you think the system will behave for later times.

c. (4 points) For the same case as in (b), solve mathematically for the subsequent time evolution of the combined state.

Problem 2

Given a particle with mass m in an infinite potential well, with walls at $x = 0$ and $x = a$.

a. (4 points) Derive the energy wave functions $\psi_n(x)$ and the corresponding eigenenergies E_n.

b. (3 points) Suppose the particle is in the ground state, and we move the walls very slowly to $x = -a/2$ and $x = 3a/2$. What is the amount of work done on the system?

c. (3 points) Again suppose the particle is in the ground state, and now we move the walls suddenly to $x = -a/2$ and $x = 3a/2$. What is the probability of finding the particle in the new ground state?

Problem 3

a. (2.5 points) Given a Hamiltonian H with a discrete spectrum, and any normalized (trial) state $|\psi\rangle$, show that the expectation value $\langle\psi|H|\psi\rangle$ is greater than or equal to the ground-state energy.

b. (2.5 points) Consider the following trial wave function (in spherical coordinates) for the electron in a hydrogen atom:

$$\psi(r, \theta, \phi) = C \exp(-(r/r_\circ)^2),$$

where $r_\circ$ is a variable parameter and C is a normalization constant. Write down the expression for the expectation value of the Hamiltonian, in the form of appropriate integrals involving $\psi(r, \theta, \phi)$. (You do not need to evaluate the integrals.)

c. (2 points) What condition determines which value of $r_\circ$ for the trial function in part b, gives the best bound on the ground-state energy?

d. (3 points) Explain why each of the following trial wave functions will *not* give an estimate of the ground-state energy that is as accurate as the one from $\psi(r, \theta, \phi)$ given in part b:

i. $\psi_a(r, \theta, \phi) = C_a \cos\phi \sin\theta \exp(-(r/r_\circ)^2)$

ii. $\psi_b(r, \theta, \phi) = C_b/(r^4 + r_\circ^4)$

iii. $\psi_c(r, \theta, \phi) = C_c(r - r_\circ)^2 \exp(-(r/r_\circ)^2)$.

Problem 4

Consider a one-dimensional quantum mechanical harmonic oscillator with frequency ω in thermal contact with a heat bath held at a temperature T.

a. (2 points) Show that the canonical partition function is given by

$$Z = \frac{1}{2\sinh\left(\beta\hbar\omega/2\right)}$$

where $\beta = 1/k_B T$ and k_B is the Boltzmann constant.

b. (2 points) Calculate a closed form for the internal energy $U = \langle E \rangle$ directly from the weighted average of quantized eigenenergies. You may wish to put this in the form a hyperbolic trigonometric function for part (c).

c. (2 points) Check your result in (b) by calculating the internal energy from the partition function using $U = -\partial \ln(Z)/\partial\beta$.

d. (2 points) Find the classical ($\hbar \to 0$) and low temperature ($T \to 0$) limits of U, and interpret your results.

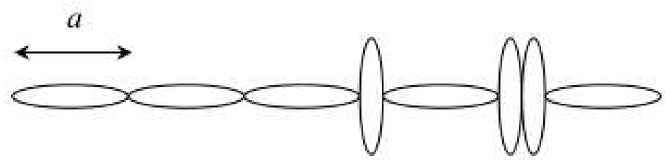
e. (2 points) Calculate and sketch the heat capacity C as a function T.

Note:

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ for } 0 \leq x \leq 1$$

Problem 5

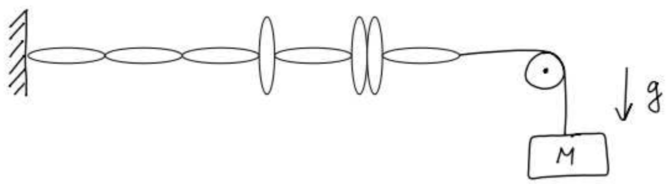
A simple model of a rubber band (polymer) is a one-dimensional chain consisting of N ($N \gg 1$) linked segments, as shown in the diagram. Each segment has two possible states: horizontal with length a_1, and vertical contributing length a_2. The segments are linked such that they cannot come apart. The chain is in thermal equilibrium at temperature T.



a. (2 points) If there is no energy difference between the two states, what is the average length, L_0, of the chain?

b. (2 points) If the energy difference between the contracted and elongated state is $\Delta E = E_{elg} - E_{cont}$, what is the average length, L_0, of the chain?

c. (3 points) The chain is now fixed at one end and a weight is hung from the other end, supplying a force $F = Mg$ as shown below. Determine the average length L of the chain as a function of T and F.



d. (3 points) In the high temperature limit, the polymer acts like a Hookean spring. Calculate the constant of proportionality between L and F. Does the rubber band get stiffer or more flexible as temperature increases? Does the rubber band get stiffer or more flexible as the number of segments increases?

Problem 6

Consider the atmosphere of a planet whose mass is M and radius is R. The planet has uniform mass density. The atmosphere is composed of N non-interacting monatomic gas molecules, each of mass m, at an equilibrium temperature T.

a. (2 points) Calculate the isotropic distribution of molecule *speeds* in this atmosphere, $P(s)$, where $s^2 = \mathbf{v} \cdot \mathbf{v}$ and $\mathbf{v}$ is the particle velocity. (Hint: *speed* is always ≥ 0, and the result is not Gaussian.) You may use that

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a.}}. \tag{1}$$

b. (2 points) Calculate the escape velocity, v_e, for the planet.

c. (2 points) Assuming the particle velocities are isotropic and that the escape velocity is measured normal to the planet's surface, how many molecules, N_e, will escape from the atmosphere, as a function of N, $P(s)$, and the escape velocity v_e, assuming that the temperature of the atmosphere remains constant during this process? You may leave your answer in integral form.

d. (2 point) What is the temperature, T_h, of the 'hotter' molecules that escaped, as a function of N, N_e, m, v_e, and $P(s)$? You may leave your answer in integral form. For this question, define $1.5k_B T_h$ as the average kinetic energy of the escaped molecules, where k_B is the Boltzmann constant.

e. (2 points) Find an expression for the final equilibrium temperature T_f of the planet's atmosphere after all of the 'hotter' particles have escaped, in terms of T, N, N_e, and T_h. (Note: you can solve this without the answers to any other part of the question.)