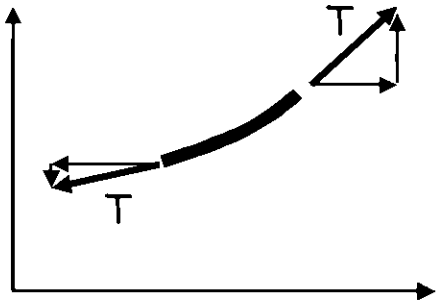


Stretched string



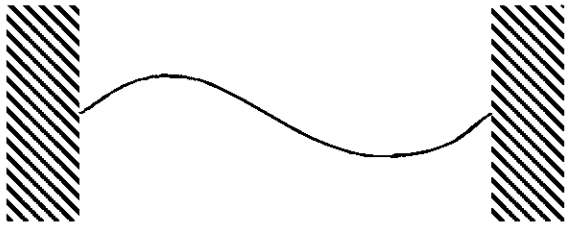
Newton's 2nd

$$\sum F = ma$$

Wave Equation

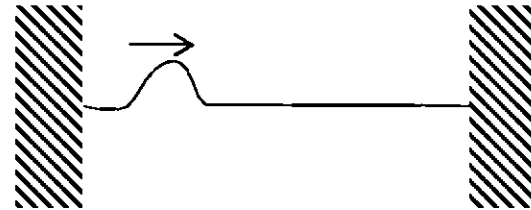
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Standing Waves



$$\sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$$

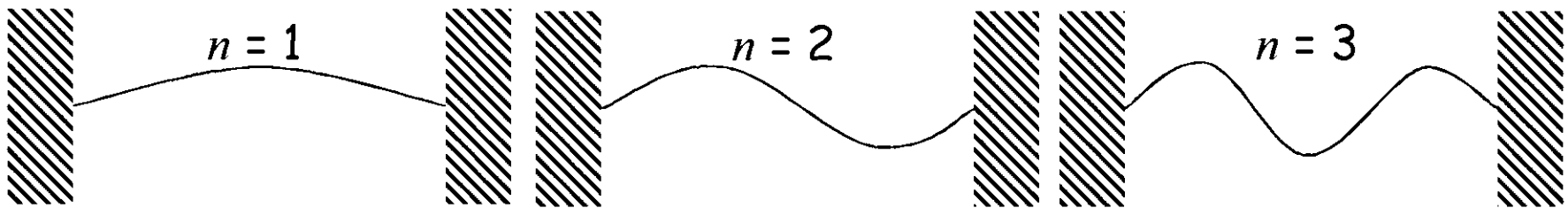
Progressive Waves



$$f(x - vt)$$

Fourier Analysis

Normal Modes: $y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \delta_n)$



Any wave motion on the string can be described by a sum of these modes!



Joseph Fourier, 1768-1830

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \delta_n)$$

One equation, infinity unknowns ...

...thanks for nothin' Joe!

Focus on the shape first! $y(x)$

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

How to find a specific B_n ? Multiply by n^{th} harmonic and integrate.

This reduces series to 1 term!

Mathematically...

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\sin \theta \sin \phi = \frac{1}{2} [\cos(\theta - \phi) - \cos(\theta + \phi)]$$

$$= \sum_{n=1}^{\infty} \frac{B_n}{2} \int_0^L \left[\cos\left(\frac{(n-n^*)\pi x}{L}\right) - \cos\left(\frac{(n+n^*)\pi x}{L}\right) \right] dx$$

$$= \sum_{n=1}^{\infty} \frac{B_n}{2} \left[\frac{L}{(n-n^*)\pi} \sin\left(\frac{(n-n^*)\pi x}{L}\right) - \frac{L}{(n+n^*)\pi} \sin\left(\frac{(n+n^*)\pi x}{L}\right) \right]_0^L$$

$$= \sum_{n=1}^{\infty} \frac{B_n}{2} \left[\underbrace{\frac{L}{(n-n^*)\pi}}_{\substack{n \neq n^* \\ n = n^*}} \underbrace{\sin((n-n^*)\pi)}_{=0} - \underbrace{\frac{L}{(n+n^*)\pi}}_{\substack{n \neq n^* \\ n = n^*}} \underbrace{\sin((n+n^*)\pi)}_{=0} - 0 + 0 \right]$$

$n \neq n^*$

$= 0$

$= 0$

...all terms zero!

$n = n^*$

$= \frac{0}{0}$

$= \frac{0}{x}$

...uh oh....

L'Hopital's Rule:

If $f(c)=0$ and $g(c)=0$ then $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$

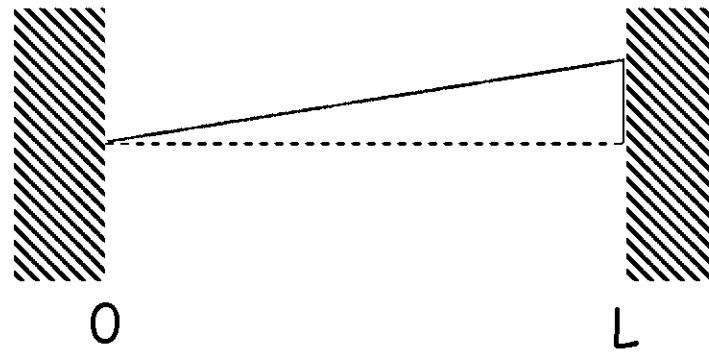
$$= \frac{B_{n^*} L \pi \cos(0)}{2\pi \cdot 1}$$

$$= \frac{B_{n^*} L}{2}$$

$$\int_0^L y(x) \sin\left(\frac{n^* \pi x}{L}\right) dx = \frac{B_{n^*} L}{2}$$

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n \pi x}{L}\right) \qquad B_n = \frac{2}{L} \int_0^L y(x) \sin\left(\frac{n \pi x}{L}\right) dx$$

Any shape $y(x)$ between 0 and L



$$B_n = \frac{2}{L} \int_0^L cx \sin\left(\frac{n\pi x}{L}\right) dx \quad c = \text{slope}$$

Integrate by parts:

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

$$B_n = \frac{2c}{L} \left[-x \frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \Big|_0^L + \frac{L}{n\pi} \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx \right]$$

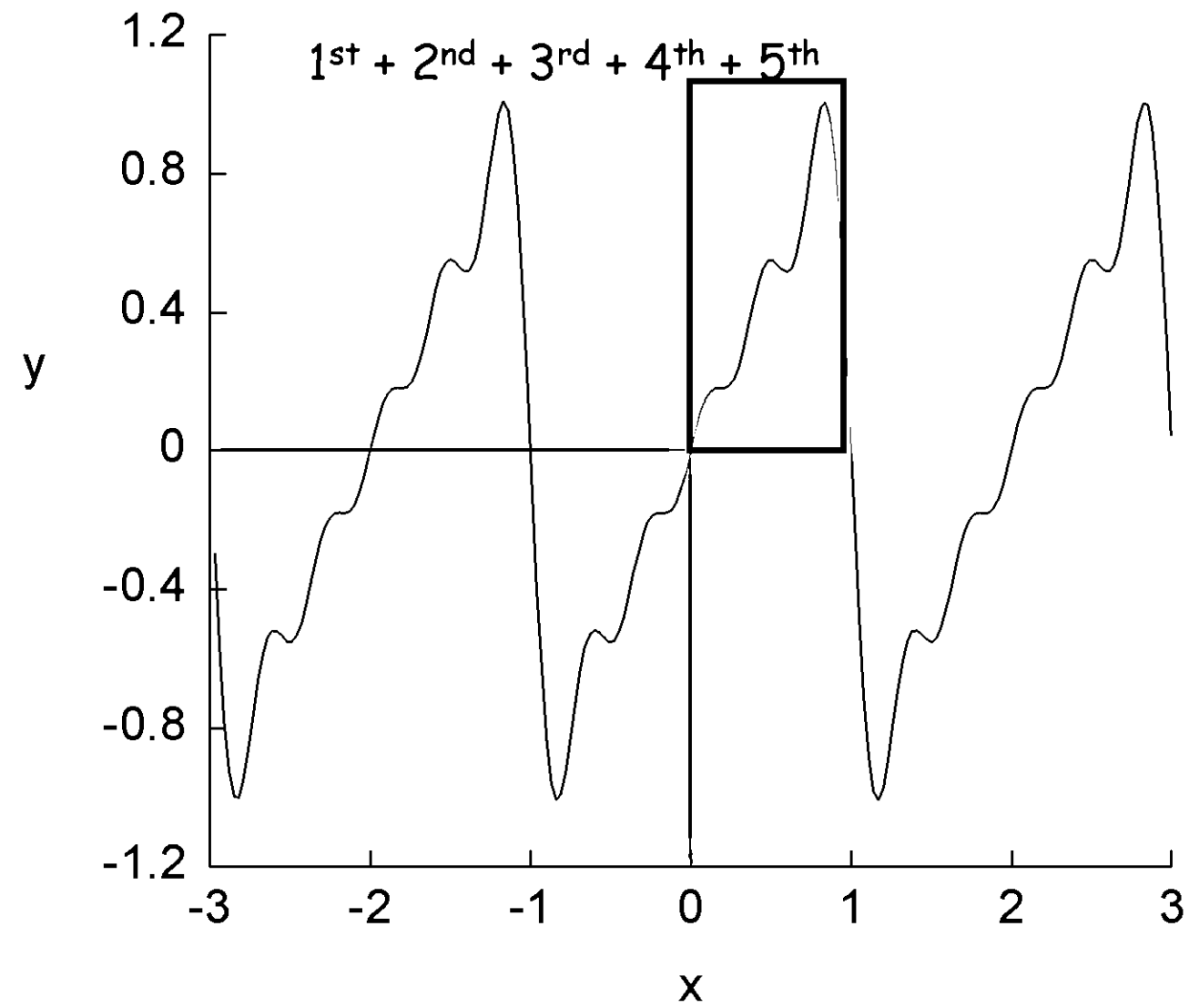
$$B_n = \frac{2c}{L} \left[-x \frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \Big|_0^L + \frac{L}{n\pi} \int_0^L \cos\left(\frac{n\pi x}{L}\right) \right]$$

$$B_n = \frac{2c}{n\pi} \left[-x \cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \right]_0^L$$

$$B_n = \frac{2c}{n\pi} [-L \cos(n\pi) + 0 + 0 - 0]$$

$$B_n = -\frac{2cL}{n\pi} \cos(n\pi)$$

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$



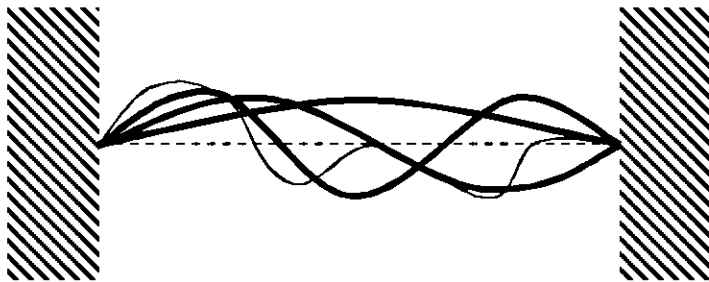
Les Foibles de Fourier en France

(French's Fourier Foibles)

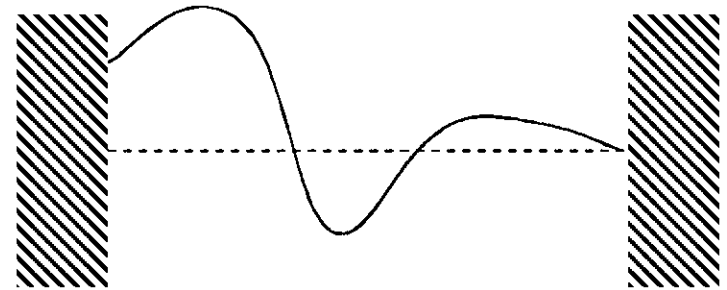
$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L} \int_0^L y(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Good choice if boundaries are at 0:



Not so good for other shapes..



More General: function periodic on interval $x = -L$ to $x = L$.

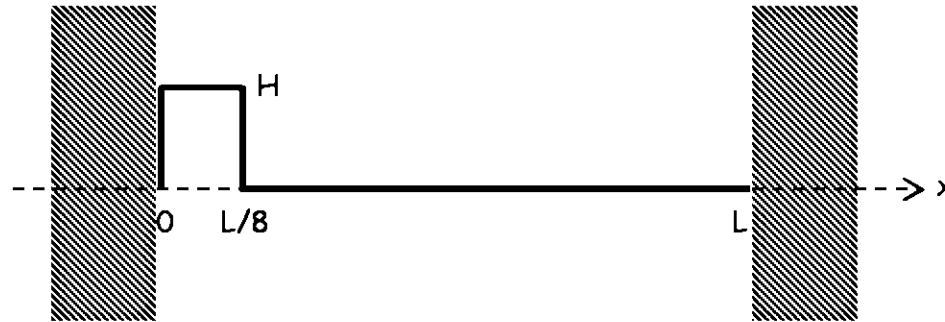
$$y(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{L} \int_{-L}^L y(x) dx$$

$$A_n = \frac{1}{L} \int_{-L}^L y(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{1}{L} \int_{-L}^L y(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Something hard:



$$A_0 = \frac{1}{L} \int_0^{L/8} H dx + \frac{1}{L} \int_{L/8}^L 0 dx$$

$$A_0 = \frac{H}{8}$$

$$A_n = \frac{1}{L} \int_0^{L/8} H \cos\left(\frac{n\pi x}{L}\right) dx + \frac{1}{L} \int_{L/8}^L 0 \cos\left(\frac{n\pi x}{L}\right) dx$$

$$A_n = \frac{H}{L} \left(\frac{L}{n\pi}\right) \sin\left(\frac{n\pi x}{L}\right) \Big|_0^{L/8}$$

$$A_n = \frac{H}{n\pi} \sin\left(\frac{n\pi}{8}\right)$$

$$B_n = \frac{1}{L} \int_0^{L/8} H \sin\left(\frac{n\pi x}{L}\right) dx + 0$$

$$B_n = -\frac{H}{L} \left(\frac{L}{n\pi}\right) \cos\left(\frac{n\pi x}{L}\right) \Big|_0^{L/8}$$

$$B_n = \frac{H}{n\pi} \left(1 - \cos\left(\frac{n\pi}{8}\right)\right)$$

$$y(x) = \frac{H}{8} + \sum_{n=1}^{\infty} \frac{H}{n\pi} \sin\left(\frac{n\pi}{8}\right) \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \frac{H}{n\pi} \left(1 - \cos\left(\frac{n\pi}{8}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

Any well behaved repetitive function can be described as an infinite sum of sinusoids with variable amplitudes (a Fourier Series). On a stretched string these correspond to the normal modes. Fourier analysis can describe arbitrary string shapes as well as progressive waves and pulses.