Continua Continued

More Useful: find **Normal Mode** solutions:

$$y(x,t) = f(x)\cos(\omega t)$$

$$\frac{\partial^2 f}{\partial x^2}\cos(\omega t) = -\frac{1}{v^2} f(x)\omega^2 \cos(\omega t)$$

good solution if… $$\frac{d^2 f}{dx^2} = -\frac{\omega^2}{v^2} f(x)$$

therefore if… $$f(x) = A\sin\left(\frac{\omega}{v}x + \phi\right)$$ … normal modes have a sinusoidal shape.

$$y(x,t) = A\sin\left(\frac{\omega}{v}x + \phi\right)\cos(\omega t)$$

Set $\phi = \pi/2$ for the moment and apply product-to-sum:

$$y(x,t) = A\cos\left(\frac{\omega}{v}x\right)\cos(\omega t)$$

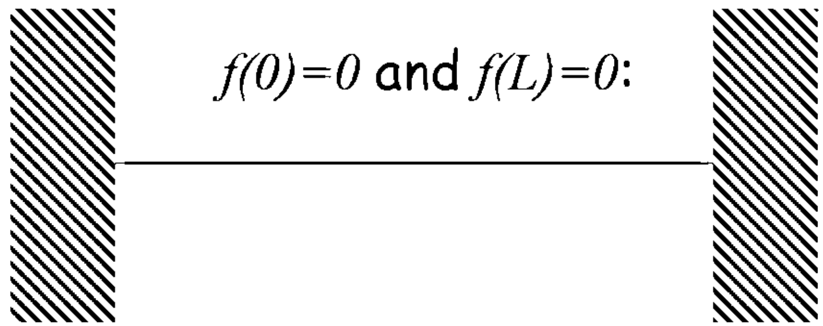
$$y(x,t) = \frac{A}{2}\left[\cos\left(\frac{\omega}{v}x - \omega t\right) + \cos\left(\frac{\omega}{v}x + \omega t\right)\right]$$

Rearrange to $f(x\text{-}vt)$

$$y(x,t) = \frac{A}{2}\left[\cos\left(\frac{\omega}{v}(x - vt)\right) + \cos\left(\frac{\omega}{v}(x + vt)\right)\right]$$

Normal mode = two counter-propagating traveling waves

Infinite strings give infinite solutions. Try a bound string:



L, μ, T

Wave equation still applies.

Normal mode solutions still apply.

But now we can apply boundary conditions

$$f(0) = A\sin(\phi) = 0$$

$$\phi = 0$$

$$f(L) = A\sin\left(\frac{\omega}{v}L\right) = 0$$

$$\frac{\omega L}{v} = n\pi \qquad n = 1, 2, 3, 4, \ldots$$

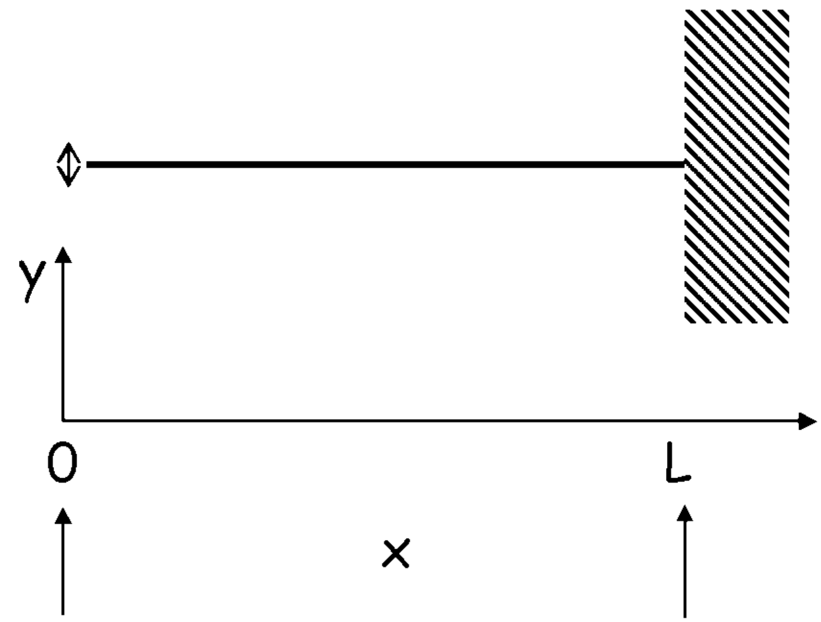
Boundary conditions reveal specific normal mode frequencies!
(for the clamped string)

$$y_n(x,t) = A_n \sin\left(\frac{\omega_n}{v}x\right)\cos(\omega_n t) \qquad \omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

These are the normal mode solutions, NOT the solution to some driving force or initial condition!!

If we shake one end….

With these Boundary conditions:



$$y(0,t) = B\cos(\omega t) \qquad y(L,t) = 0$$

$$y(x,t) = f(x)\cos(\omega t)$$ … is this a good solution?

$$f(x) = A\sin\left(\frac{\omega}{v}x + \phi\right)$$

Boundary conditions reveal normal mode wavelengths & frequencies!
(for the clamped string)

Apply boundary condition:

$$y(L,t) = A\sin\left(\frac{\omega}{v}L + \phi\right)\cos(\omega t) = 0$$

Sine is zero if:

$$\frac{\omega}{v}L + \phi = n\pi \qquad n = 1, 2, 3, 4, \ldots$$

$$\phi = n\pi - \frac{\omega}{v}L$$

Apply other boundary condition:

$$y(0,t) = A\sin(\phi)\cos(\omega t) = B\cos(\omega t)$$

$$A = \frac{B}{\sin(\phi)}$$

Wavenumber: $k = \dfrac{2\pi}{\lambda}$

French is the only book still in print that defines k = 1/λ !!!

The equation of motion for a stretched string (a continuum) is the wave equation. It has traveling wave and normal mode solutions (which are really the same thing).