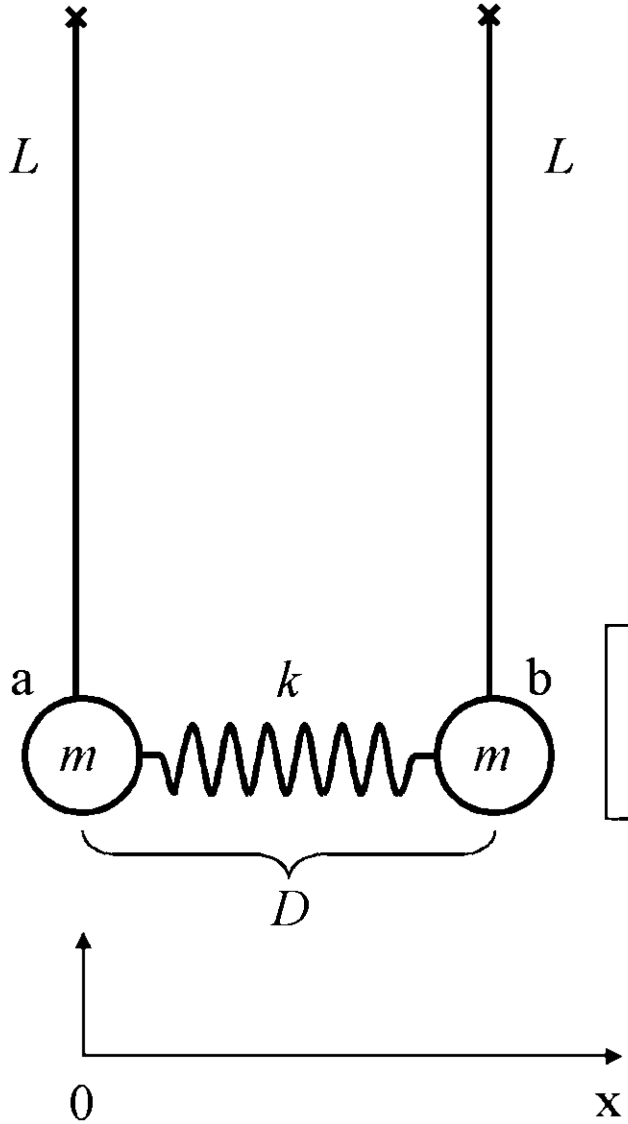
6. Coupled Oscillators



$$\Sigma F_a = m\ddot{x}_a$$

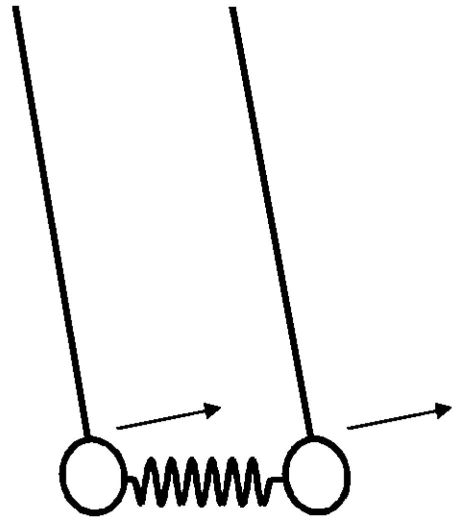
$$-\frac{mg}{L}x_a - k\left(x_a - \left(x_b - D\right)\right) = m\ddot{x}_a$$

$$\Sigma F_b = m\ddot{\left(x_b - D\right)}$$

$$-\frac{mg}{L}\left(x_b - D\right) - k\left(\left(x_b - D\right) - x_a\right) = m\ddot{\left(x_b - D\right)}$$

coupled equations of motion

Normal Modes : motions where all bodies move at the same frequency.



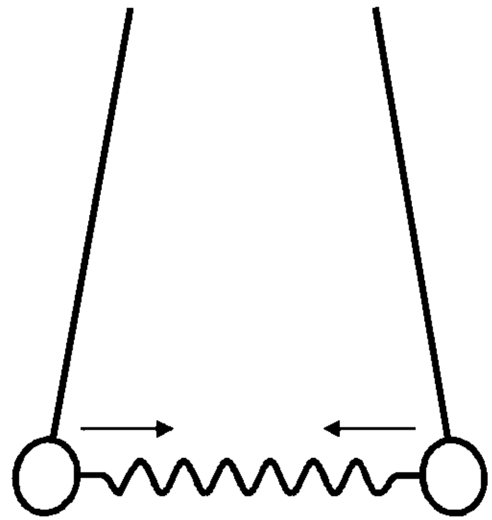
Sum:

$$-\frac{mg}{L}x_a - k\left(x_a - (x_b - D)\right) = m\ddot{x}_a$$

$$+\quad -\frac{mg}{L}(x_b - D) - k\left((x_b - D) - x_a\right) = m\ddot{(x_b - D)}$$

$$-\frac{mg}{L}(x_a + x_b - D) \qquad -0 \qquad = m\ddot{(x_a + x_b - D)}$$

Normal Coordinate 1: $q_{a+b} = (x_a + x_b - D)$ Normal Frequency 1: $\omega_{a+b} = \sqrt{\frac{g}{L}}$



Difference: $$-\frac{mg}{L}x_a \quad - \quad k\left(x_a - (x_b - D)\right) \quad = \quad m\ddot{x}_a$$

$$- \qquad -\frac{mg}{L}(x_b - D) - k\left((x_b - D) - x_a\right) \quad = \quad m\ddot{(x_b - D)}$$

$$-\frac{mg}{L}(x_a - x_b + D) - k\left(2x_a - 2(x_b - D)\right) = m\left(\ddot{x}_a - \ddot{x}_b + \ddot{D}\right)$$

$$-\frac{g}{L}(x_a - x_b + D) - \frac{2k}{m}(x_a - x_b + D) = \left(\ddot{x}_a - \ddot{x}_b + \ddot{D}\right)$$

$$-\left(\frac{g}{L} + \frac{2k}{m}\right)(x_a - x_b + D) = \left(\ddot{x}_a - \ddot{x}_b + \ddot{D}\right)$$

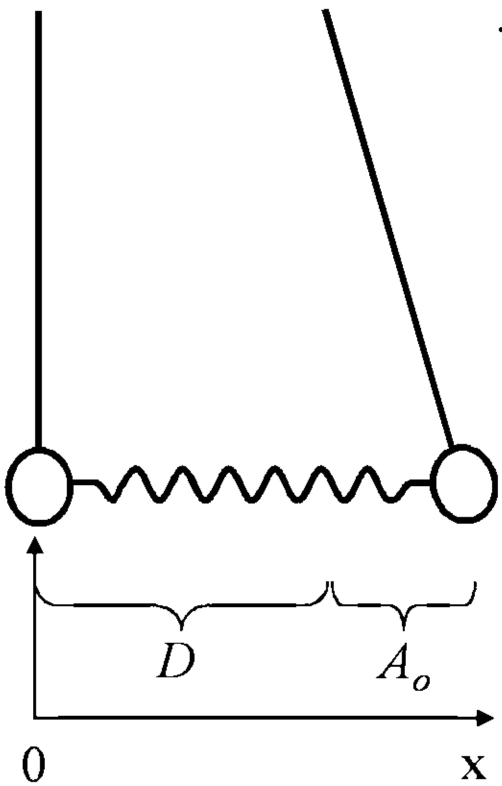
Normal Coordinate 2: $q_{a-b} = (x_a - x_b + D)$

Normal Frequency 2: $\omega_{a-b} = \sqrt{\frac{g}{L} + \frac{2k}{m}}$

Normal Mode Solutions :

$$q_{a+b} = A_{a+b}\cos\left(\sqrt{\frac{g}{L}}t + \phi_{a+b}\right) \qquad q_{a-b} = A_{a-b}\cos\left(\sqrt{\left(\frac{g}{L} + \frac{2k}{m}\right)}t + \phi_{a-b}\right)$$

"Abnormal" motion:



Initial Conditions:

$$q_{a+b} = A_0 \qquad q_{a-b} = -A_0 \qquad \dot{q}_{a+b} = 0 \quad \dot{q}_{a-b} = 0$$

$$-\sqrt{\frac{g}{L}}A_{a+b}\sin(\phi_{a+b}) = 0$$

$$\phi_{a+b} = 0$$

$$A_{a+b}\cos(0) = A_0$$

$$A_{a+b} = A_0$$

similarly…

$$q_{a+b} = A_0 \cos(\omega_{a+b} t)$$

$$q_{a-b} = -A_0 \cos(\omega_{a-b} t)$$

Switch back to x_a and x_b:

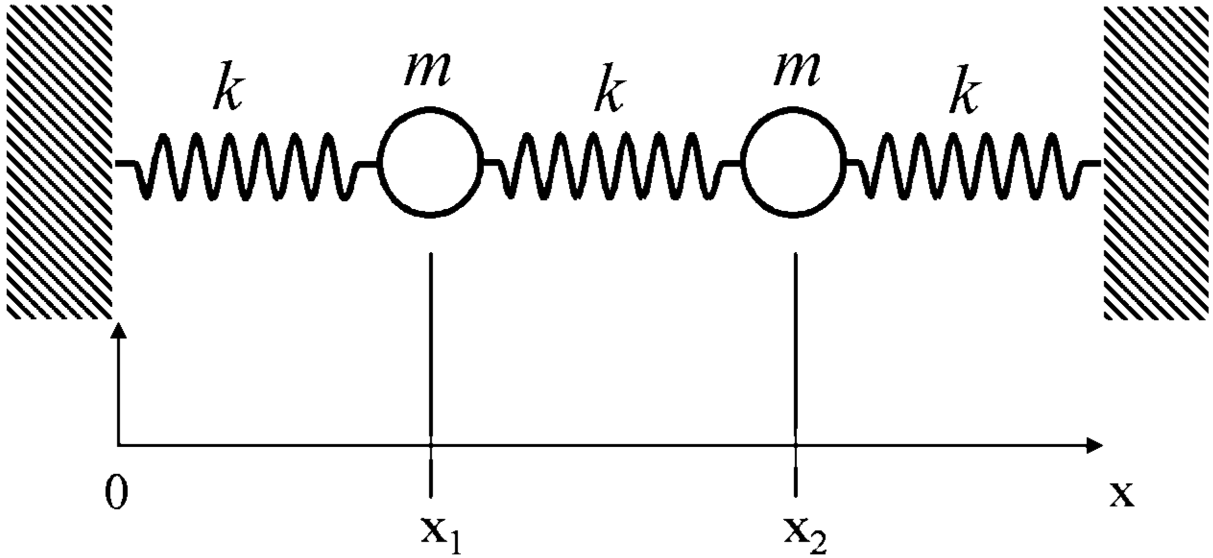
$$x_a = \frac{q_{a+b} + q_{a-b}}{2}$$

$$x_b = \frac{q_{a+b} - q_{a-b}}{2} + D$$

$$x_a = \frac{A_0}{2}\left[\cos(\omega_{a+b} t) - \cos(\omega_{a-b} t)\right]$$

$$x_a = -A_0\left[\sin\left(\frac{\omega_{a-b} - \omega_{a+b}}{2} t\right)\sin\left(\frac{\omega_{a-b} + \omega_{a+b}}{2} t\right)\right]$$

x_b also beats



Equations of motion
$$\begin{cases} m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) \\ m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) \end{cases}$$

Look for Normal Modes!
$$\begin{cases} x_1(t) = A_1 e^{j\omega t} \\ x_2(t) = A_2 e^{j\omega t} \end{cases}$$

Plug in....
$$\begin{cases} -\omega^2 m A_1 e^{j\omega t} = -2kA_1 e^{j\omega t} + kA_2 e^{j\omega t} \\ -\omega^2 m A_2 e^{j\omega t} = -2kA_2 e^{j\omega t} + kA_1 e^{j\omega t} \end{cases}$$

Simplify and group by A's

$$\begin{cases} \left(\omega^2 m - 2k\right)A_1 + kA_2 = 0 \\ kA_1 + \left(\omega^2 m - 2k\right)A_2 = 0 \end{cases}$$

Matrix representation

$$\begin{bmatrix} \omega^2 m - 2k & k \\ k & \omega^2 m - 2k \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

True if the Determinant is 0:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

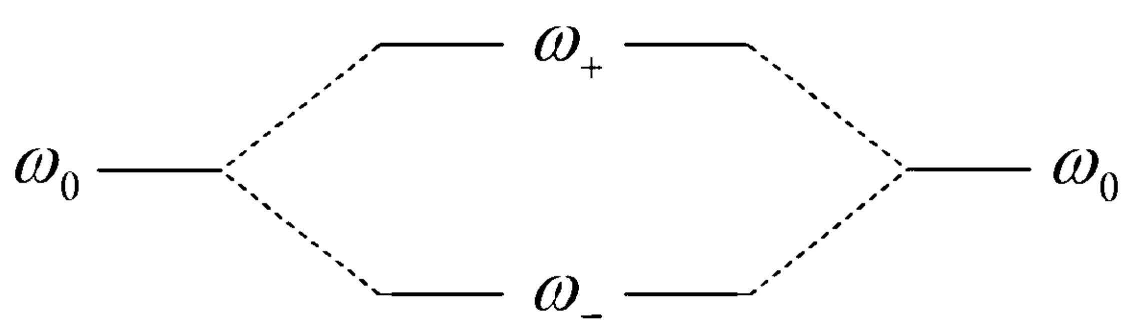
$$\left(\omega^2 m - 2k\right)^2 - k^2 = 0$$

$$\left(\omega^2 m - 2k\right) = \pm k$$

$$\omega^2 = \frac{2k \pm k}{m}$$

$$\boxed{\omega_- = \sqrt{\frac{k}{m}} \qquad \omega_+ = \sqrt{\frac{3k}{m}}}$$

Normal Mode frequencies



Larger N: www.falstad.com/coupled

Oscillators which exchange energy are coupled, and their equations of motion are coupled. Their complex motion can be described simply in terms of normal modes in which all bodies oscillate with the same frequency.