CAAM 454/554: Numerical Analysis II

Homework 1, January, 2019

Due: By 8pm, February 6, 2019

- Unless specified, the problems should be solved by CAAM 454 and by CAAM 554 students.
- All MATLAB functions mentioned in this homework assignment can be found on the CAAM454/554 homepage (http://maartendehoop.rice.edu/caam-454/), or come with MATLAB. You can use the MATLAB codes posted on the CAAM454/554 web-page. Turn in all MATLAB code that you have written/modified and turn in all output generated by your MATLAB functions/scripts. MATLAB functions/scripts must be commented, output must be formatted nicely, and plots must be labeled.

Total points: CAAM 454 - 120 points; CAAM 554 - 190 points.

Problem 1 (20+10 = 30 points)

i. Let $\alpha > 0$, $\gamma \geq 0$, $\beta \in \mathbb{R}$ and consider the matrix

$$
A = \begin{pmatrix}
2\alpha + h^2 \gamma & -(\alpha - \frac{h}{2} \beta) \\
-(\alpha + \frac{h}{2} \beta) & 2\alpha + h^2 \gamma & -(\alpha - \frac{h}{2} \beta) \\
& \ddots & \ddots & \ddots \\
& & -(\alpha + \frac{h}{2} \beta) & 2\alpha + h^2 \gamma & -(\alpha - \frac{h}{2} \beta) \\
& & & -\alpha & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
& & & & \ddots & \ddots & \ddots \\
& & & & & -\alpha & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
& & & & & & -\alpha & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
& & & & & & & -\alpha & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
\end{pmatrix}.
$$

Show that if $h < 2\alpha/|\beta|$, the entries $a_{ij}$ of the tridiagonal matrix in (1) satisfy

$$
\sum_{j \neq i} |a_{ij}| \leq |a_{ii}| \quad \text{for } i = 1, \ldots, n,
$$

with “$<$” for $i = 1$ and $i = n$.

ii. Let $\alpha > 0$, $\gamma \geq 0$, $\beta \geq 0$ and consider the matrix

$$
A = \begin{pmatrix}
2\alpha + h\beta + h^2 \gamma & -\alpha \\
-(\alpha + h \beta) & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
& \ddots & \ddots & \ddots \\
& & -(\alpha + h \beta) & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
& & & -\alpha & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
& & & & \ddots & \ddots & \ddots \\
& & & & & -\alpha & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
& & & & & & -\alpha & 2\alpha + h\beta + h^2 \gamma & -\alpha \\
\end{pmatrix}.
$$
Show that for any $h > 0$ the entries $a_{ij}$ of the tridiagonal matrix in (2) satisfy

$$\sum_{j \neq i} |a_{ij}| \leq |a_{ii}| \quad \text{for } i = 1, \ldots, n,$$

with “<” for $i = 1$ and $i = n$.

**Problem 2** \((10+20 = 30 \text{ points})\) Let

$$A = D - E - F,$$

where $D$, $-E$ and $-F$ are given as in the notes. Given $\omega \in \mathbb{R}$, the relaxed Jacobi method is given by

$$x^{(k+1)} = (1 - \omega)x^{(k)} + \omega D^{-1}([E + F]x^{(k)} + b) = ((1 - \omega)I + \omega D^{-1}[E + F])x^{(k)} + \omega D^{-1}b.$$

Let $G_J = D^{-1}[E + F]$ be the iteration matrix of the Jacobi method and let

$$G_{J,\omega} = (1 - \omega)I + \omega D^{-1}[E + F]$$

be the iteration matrix of the relaxed Jacobi method.

i. \((10 \text{ points})\) Show that if $G_J$ has eigenvalues $\lambda_i$, $i = 1, \ldots, n$, then $G_{J,\omega}$ has eigenvalues $1 - \omega + \omega \lambda_i$, $i = 1, \ldots, n$

ii. \((20 \text{ points})\) Show that if all eigenvalues of $G_J$ are real and ordered such that $\lambda_1 \geq \ldots \geq \lambda_n$ and if $\lambda_1 < 1$, then the spectral radius of $G_{J,\omega}$ is minimal for

$$\omega_{\text{opt}} = \frac{2}{2 - \lambda_1 - \lambda_n}.$$
Problem 3 (10 points) Let \( A \in \mathbb{R}^{n \times n} \) be symmetric positive definite, \( b \in \mathbb{R}^n \), and let \( x^\ast \) be the solution of \( Ax = b \). Show that the iteration
\[
x^{(k+1)} = x^{(k)} - \omega (Ax^{(k)} - b)
\]
converges to the solution \( x^\ast \) of \( Ax = b \) for any \( x^{(0)} \) if and only if
\[
\omega \in \left( 0, \frac{2}{\|A\|_2} \right).
\]

Note: If we define \( q(x) = \frac{1}{2} x^T Ax - b^T x \), then \( \nabla q(x) = Ax - b \) and the iteration \( x^{(k+1)} = x^{(k)} - \omega (Ax^{(k)} - b) = x^{(k)} - \omega \nabla q(x^{(k)}) \) is the steepest descent method for the minimization of \( q \). In the context of solving symmetric positive definite linear systems, this iteration is also known as the Richardson iteration.

Problem 4 (50/70 points) Consider the linear system
\[
\begin{pmatrix}
A & B^T \\
B & D
\end{pmatrix}
\begin{pmatrix}
y \\
z
\end{pmatrix}
= \begin{pmatrix}
e \\
f
\end{pmatrix}
\]
(3)
with nonsingular symmetric matrix \( A \in \mathbb{R}^{n \times n} \), symmetric matrix \( D \in \mathbb{R}^{m \times m} \), and matrix \( B \in \mathbb{R}^{m \times n} \).

i. (15 points) Show that (3) is equivalent to the Schur complement system
\[
(D - BA^{-1}B^T)z = f - BA^{-1}e.
\]
Conclude that \( S = D - BA^{-1}B^T \in \mathbb{R}^{m \times m} \) is nonsingular if and only if \( K \in \mathbb{R}^{(m+n) \times (m+n)} \) is nonsingular.

ii. (5 points) Show that
\[
\begin{pmatrix}
I & 0 \\
-BA^{-1} & I
\end{pmatrix}
\begin{pmatrix}
A & B^T \\
B & D
\end{pmatrix}
\begin{pmatrix}
I & -A^{-1}B^T \\
0 & I
\end{pmatrix}
= \begin{pmatrix}
A & 0 \\
0 & S
\end{pmatrix}

= L
\]

= K
\]

= L^T


iii. (20 points) Show that \( K \) is positive definite if and only if \( A, D, \) and \( S \) are positive definite.

iv. (10 points) Assume that \( K \) is positive definite (in particular \( A, D, \) and \( S \) are positive definite).

Show that the block (forward) Gauss-Seidel method
\[
\begin{pmatrix}
y^{(k+1)} \\
z^{(k+1)}
\end{pmatrix}
= \begin{pmatrix}
A & 0 \\
B & D
\end{pmatrix}^{-1}
\begin{pmatrix}
0 & -B^T \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
y^{(k)} \\
z^{(k)}
\end{pmatrix}
+ \begin{pmatrix}
A & 0 \\
B & D
\end{pmatrix}^{-1}
\begin{pmatrix}
e \\
f
\end{pmatrix}
\]
is equivalent to the iteration
\[
z^{(k+1)} = D^{-1}BA^{-1}B^Tz^{(k)} + D^{-1}(f - BA^{-1}e).
\]
v. (CAAM 554, 20 points) Again assume that $K$ is positive definite (in particular $A$, $D$, and $S$ are positive definite).

Show that the eigenvalues of the iteration matrix $D^{-1}BA^{-1}B^T$ are contained in $[0, 1)$.

**Problem 5 (CAAM 554, 20+10+5+15 = 50 points)** Let $A, M \in \mathbb{R}^{n \times n}$ be symmetric positive definite and consider the iteration

$$x^{(k+1)} = (I - M^{-1}A)x^{(k)} + M^{-1}b.$$ 

i. Show that if $2M - A$ is positive definite, then all eigenvalues of $I - M^{-1}A$ are contained in the interval $(-1, 1)$. In particular, $\rho(I - M^{-1}A) < 1$.

ii. Show that if $M - A$ is positive semidefinite, then all eigenvalues of $I - M^{-1}A$ are contained in the interval $[0, 1)$. In particular, $\rho(I - M^{-1}A) < 1$.

iii. Let $A = D - E - F$ where $D$, $-E$ and $-F = -E^T$ are given as in the notes, and let $M = D$.

Show that if $2D - A$ is positive definite, the Jacobi method converges.

iv. Let $A = D - E - F$ where $D$, $-E$ and $-F = -E^T$ are given as in the notes, and let

$$M = (D - E)D^{-1}(D - E^T).$$

Show that $\rho(I - M^{-1}A) < 1$.

This iteration matrix corresponds to the so-called symmetric Gauss-Seidel method.