The solutions to the following exercises should be submitted to Xiao Liu’s office at 2033 Duncan Hall by 4 pm Wednesday, 12 December 2018. If he is not in, slip your work under the door.

You may collaborate and discuss with other Rice students about non-pledged problems, but you must write your own solutions. Solutions should either be typed (in Latex) or written very neatly.

You may not consult any solution manuals, online forums, course materials from other courses, including courses at either Rice or other schools, or any other sources that may be viewed as outside assistance.

Where applicable, the following exercises use the notation from Evans’ textbook.

Notation:

- $U \subset \mathbb{R}^n$ is a bounded open set with smooth boundary;
- $f$ is a function in $L^2(U)$.

**Problem 1**

Prove the existence and uniqueness of solution $u \in H^2_0(U)$ for the following problem

$$\int_{U} \Delta u \Delta v \, dx = \int_{U} f v \, dx,$$

for all $v \in H^2_0(U)$.

**Problem 2**

a) Derive the weak formulation (in $H^1(U)$) of Poisson’s equation with Robin boundary condition

$$\begin{cases}
-\Delta u = f & \text{in } U \\
u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U.
\end{cases}$$

b) Prove the existence and uniqueness of the weak solution of Part a).

**Problem 3**

Consider the following two equations:

1. $-\Delta u + c(x)u = 0,$
2. $-\text{div}(a(x)Dv) = 0,$

where $c(x)$ is a coefficient function, and $a(x)$ is a positive coefficient function.

a) Show that if $u, w$ solve (1) and $w > 0$, then $u/w$ solves (2) for $a = w^2$.

b) Show that if $v$ solves (2), then $va^{1/2}$ solves (1) for some $c$. 

Problem 4

This problem is pledged. You cannot discuss this problem with anyone other than the instructor, and you should not consult with any sources other than the textbook and your own notes.

a) Show that \( w = e^{i\sigma x_1} \) solves \( -\Delta w = \sigma^2 w \) in \( \mathbb{R}^n \).

b) Show that \( \tilde{w} = \frac{e^{i\sigma |x|}}{4\pi |x|} \) solves \( -\Delta w = \sigma^2 w \) in \( \mathbb{R}^3 \setminus \{0\} \).

c) The Sommerfeld radiation condition is
\[
\lim_{|x| \to \infty} (Dw \cdot x - i\sigma w |x|) = 0.
\]
Show that \( w \) from Part a) does NOT satisfy this condition, but that \( \tilde{w} \) from Part b) does.

Problem 5

Let \( U = \{x \in \mathbb{R}^3 : |x| < L\} \). Use separation of variables to solve the wave equation for radial functions in \( U \times (0, \infty) \) with Dirichlet boundary conditions. That is, solve
\[
\begin{align*}
\partial^2_t u(r, t) - \partial^2_r u(r, t) - 2r^{-1} \partial_r u(r, t) &= 0, \quad 0 < r < L, t > 0 \\
u(r, 0) &= f(r), \quad \partial_t u(r, 0) = g(r), \quad u(L, t) = 0
\end{align*}
\]

Problem 6

Recall that we showed for the wave equation that the total energy was constant in \( t \) by differentiating the energy function in \( t \). An alternate proof can be given using the Fourier transform. Consider the Cauchy problem for the free-space wave equation in \( n \) spatial dimensions:
\[
\begin{align*}
\partial_t u - \Delta u &= 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad u \in H^1(\mathbb{R}^n), \quad g \in L^2(\mathbb{R}^n).
\end{align*}
\]

By using the Fourier transform solution formula
\[
\hat{u}(\xi, t) = \hat{f}(\xi) \cos(|\xi| t) + \xi(\xi) \frac{\sin(|\xi| t)}{|\xi|},
\]
compute directly that the total energy is constant for \( t \geq 0 \); that is, show
\[
\int_{\mathbb{R}^n} |u_t(x, t)|^2 + |D_x u(x, t)|^2 \, dx = \int_{\mathbb{R}^n} |g(x)|^2 + |D f(x)|^2 \, dx.
\]